274 Curves on Surfaces, Lecture 11

Dylan Thurston

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12 Cross-ratio coordinates

Last time we asked for a quiver mutation-equivalent to the default one in Bernhard Keller's applet. As stated on Keller's page, this is $E_8^{(1,1)}$. One way to see this would be to count vertices and then attempt to rule out that the quiver comes from a surface using the classification in terms of blocks.

Last time we also asked for a proof that E_8 cannot be obtained by gluing blocks. First, type IV and V blocks cannot occur because E_8 is acyclic and the cycles in type IV and V cannot be removed by gluing. Type III blocks cannot occur because E_8 does not have a tail of two black vertices. This leaves blocks of type I and II from which it is impossible to construct a trivalent vertex of the type that occurs in E_8 (after some additional case analysis).

Recall that the cross-ratio of four points a, b, c, d is defined as follows: we send $a \to 0, b \to 1, c \to \infty$, and examine where d goes. A straightforward argument shows that

$$d \mapsto \frac{(d-a)(b-c)}{(d-c)(b-a)} = [d, b; a, c]. \tag{1}$$

We will use the negative of the cross ratio τ because positivity properties are important.

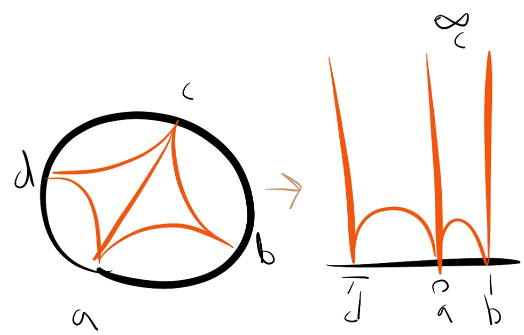

Figure 1: Construction of the cross ratio. (τ should be $-\tau$.)

Some basic properties:

1. $[d, b; a, c]$ is a projective invariant of a, b, c, d; every other such projective invariant is a function of it.

2. $[d, b; a, c] = [b, d; a, c]^{-1} = [d, b; c, a]^{-1}$. As a corollary, $[d, b; a, c] = [b, d; c, a]$, and so there is no need to orient the diagonal edge.

3. The action of S_4 permutes the six possible values of τ generated by $\tau \mapsto \frac{1}{\tau}$ and $\tau \mapsto 1 - \tau$.

$\tau = -[d, b; a, c]$ has other interpretations besides the cross-ratio. It is also a ratio of Euclidean lengths (and phases) $\frac{B \cdot D}{A \cdot C}$ in both the upper half plane and the disk model.

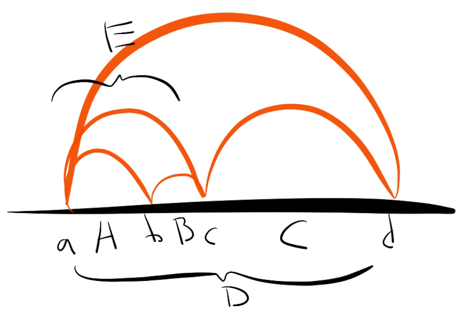

Figure 2: Euclidean distances.

It is also a ratio of λ-lengths

$$\tau(E) = \frac{\lambda(B)\lambda(D)}{\lambda(A)\lambda(C)}. \tag{2}$$

For a more geometric interpretation, we can look at where two angle bisectors intersect on the diagonal. It is reasonable to call these midpoints. They don't intersect at the same point, and the distance between them is the shear $\ln \frac{d}{b}$ if a is sent to the origin.

So in general we have

$$\tau = e^{\text{shear}}. \tag{3}$$

In general, any geometric quantity is some function of the cross-ratio, so for example we also have the following.

Exercise 12.1. *If ℓ is the length between two opposite sides, then $\cosh^2\left(\frac{\ell}{2}\right) = \tau + 1$, whereas for the other pair of opposites sides, $\cosh^2\left(\frac{\ell'}{2}\right) = 1 + \frac{1}{\tau}$.*

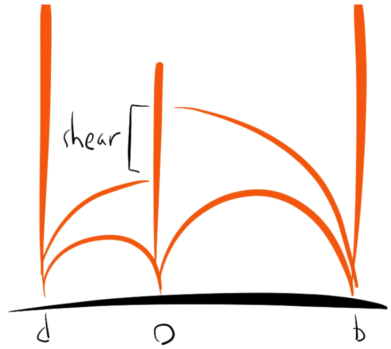

Figure 3: Shear and its relationship to the cross-ratio.

Hint: put the quadrilateral in a more symmetric form.
If θ denotes the angle between the two diagonals, we also have

$$\cos\theta = \frac{x-y}{x+y} = \frac{1-\tau}{1+\tau} \tag{4}$$

or, slightly more nicely,

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{1+\tau}. \tag{5}$$

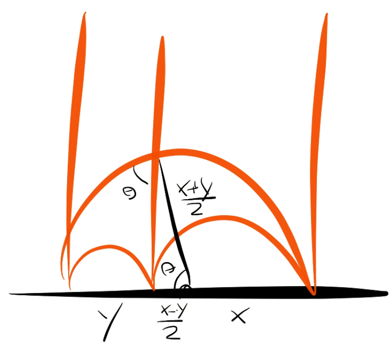

Figure 4: The angle between the two diagonals.

What happens to τ when changing triangulations? Using the interpretation in terms of λ-lengths, we have

$$\tau(E) = \frac{\lambda(B)\lambda(D)}{\lambda(A)\lambda(C)}, \tau(F) = \frac{\lambda(C)\lambda(A)}{\lambda(B)\lambda(D)} \tag{6}$$

so τ gets sent to $\frac{1}{\tau}$.

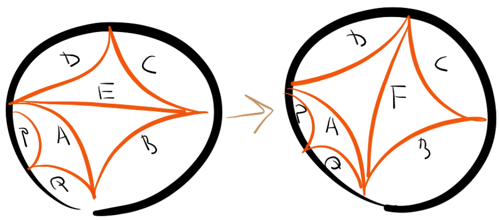

Figure 5: Changing triangulations.

This seems very nice. But when we try to generalize these coordinates to polygons with more sides, shear coordinates depend not only on how an edge changes but on how its neighbors change. So when we add more edges, we get shear coordinates like

$$\tau(A) = \frac{\lambda(Q)\lambda(E)}{\lambda(P)\lambda(F)} \tag{7}$$

which gets transformed to

$$\tau'(A) = \frac{\lambda(Q)\lambda(D)}{\lambda(P)\lambda(F)} = \tau(A)\frac{1}{1+\tau(E)^{-1}}. \tag{8}$$

This factor measures the exponential of the distance between two angle bisectors. Similarly, we find that

$$\tau'(B) = \tau(B)(1+\tau(E)) \tag{9}$$

$$\tau'(C) = \tau(C)\frac{1}{1+\tau(E)^{-1}} \tag{10}$$

$$\tau'(D) = \tau(D)(1+\tau(E)). \tag{11}$$

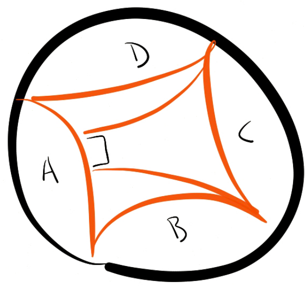

Figure 6: The distance between two angle bisectors.

We have a map from λ-coordinates to τ-coordinates. We know that the former parameterizes decorated Teichmüller space. τ-coordinates do not depend on a choice of horocycle and hence parameterize ordinary Teichmüller space.

But there is someting funny going on. Consider the once-punctured torus.

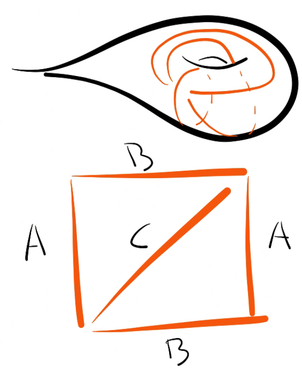

Figure 7: A once-punctured torus.

We parameterized this by three λ-lengths $\lambda(A), \lambda(B), \lambda(C)$ and now we are parameterizing it by three shear coordinates $\tau(A), \tau(B), \tau(C)$ and we are forgetting a horocycle, so this map cannot be surjective. In fact,

$$\tau(A) = \frac{\lambda(B)^2}{\lambda(C)^2}, \tau(B) = \frac{\lambda(C)^2}{\lambda(A)^2}, \tau(C) = \frac{\lambda(A)^2}{\lambda(B)^2} \tag{12}$$

so it follows that $\tau(A)\tau(B)\tau(C) = 1$.

In general, for every puncture, we forget a horocycle, so we should expect a corresponding relation.

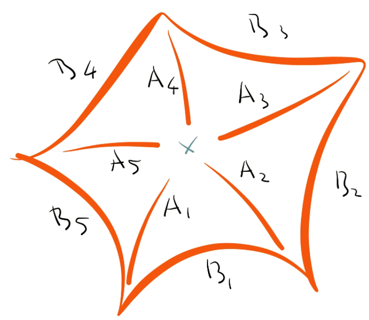

Figure 8: A punctured pentagon.

We get that

$$\prod_i \tau(A_i) = 1 \tag{13}$$

where A_i are the arcs incident to a puncture. What does this mean geometrically, and what do the corresponding τ-coordinates parameterize if we drop this restriction?

Inside each triangle adjacent to a puncture, we can measure the distance between the horocycle and midpoints (we will do this in the upper half-plane with the horocycle at infinity). As we move from triangle to triangle, these distances change by shears, and moving all the way around, the sum of the shears must be 0. The above is the exponential of this relation.

Alternately, on the upper half-plane, shear coordinates are ratios of Euclidean lengths $\tau(A_i) = \frac{x_i}{x_{i+1}}$. The condition that these ratios multiply to 1 is precisely the condition that the two possible lifts of the initial triangle to the upper half-plane have the same width, which in turn expresses the fact that the monodromy around the puncture is a parabolic element (translation).

What if we drop this condition $\prod \tau(A_i) = 1$? Consider a single triangle with two sides identified (alternately, a monogon). There are two midpoints involved, and we do not need to identify the sides so that the midpoints match; in general they can have shear. Correspondingly we get a geometric series of lifts to the upper half-plane which get smaller and smaller.

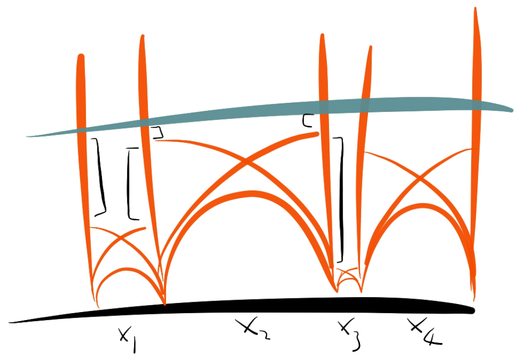

Figure 9: Shears along a horocycle.

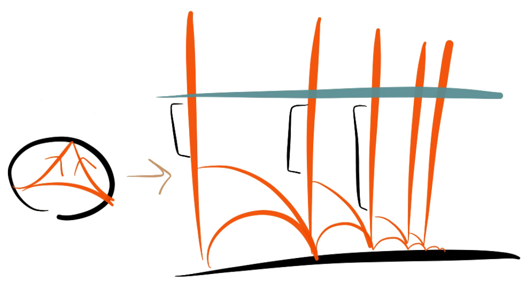

Figure 10: A triangle with two sides identified whose midpoints don't match.

We still get a hyperbolic metric on the punctured monogon, but it is not complete. Trying to draw a horocycle in the upper half-plane will give a spiral which does not close in on itself.

However, we can fix this by completing. In the upper half-plane we do this by adding the limit geodesic of the geometric series of lifts, which appears as extra geodesic boundary.

Exercise 12.2. *What is the length of the new boundary?*

Exercise 12.3. *What determines the direction of spiraling? (There are two things that are spiraling, namely the horocycle and the geodesic arc in the triangulation.)*

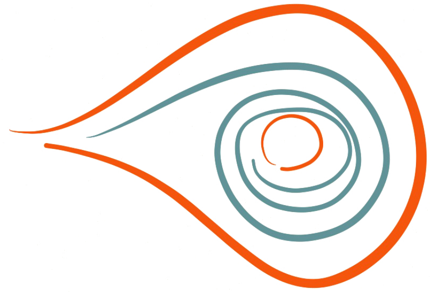

Figure 11: Geodesic boundary and a spiraling horocycle.

Hence the map from λ-coordinates to τ-coordinates has image in a Teichmüller space which includes the possibility of geodesic boundaries where the cusps were. It factors through a Teichmüller space which only describes cusps.