274 Curves on Surfaces, Lecture 18

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20 Skein theory and the Laurent phenomenon

Skein theory for tagged arcs requires an extra relation beyond the ones we have used so far.

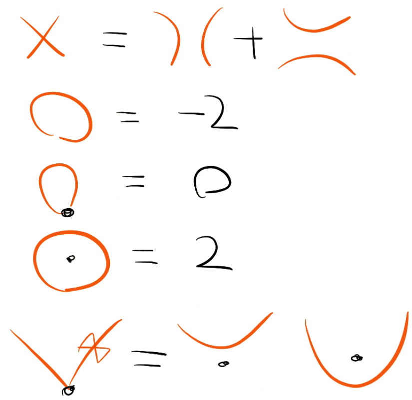

Figure 1: The complete set of relations.

Exercise 20.1. *Check that these reductions satisfy the hypotheses of the diamond lemma.*

We can induce an order on diagrams by considering its complexity, which is a triple $(t(D), c(D), r(D))$ where $t(D)$ is the number of tag mismatches, $c(D)$ is the number of crossings, and $r(D)$ is the number of reducible components (loops bounding disks and arcs bounding monogons), ordering diagrams by lexicographic order on their complexity, and ordering linear combinations by sorting the complexities of the components.

Theorem 20.2. *The skein algebra $Sk(\Sigma)$ of a surface Σ has the following properties:*

1. *It has a basis of simple curves (those with complexity $(0, 0, 0)$).*

2. *Elements are invariant under regular isotopy (RII, RIII, RIIb) and change by -1 under RI.*

3. *Multiplication is given by superimposing curves and is well-defined (independent of how the superposition is done).*

If T is a triangulation of Σ, we have a sequence of inclusions

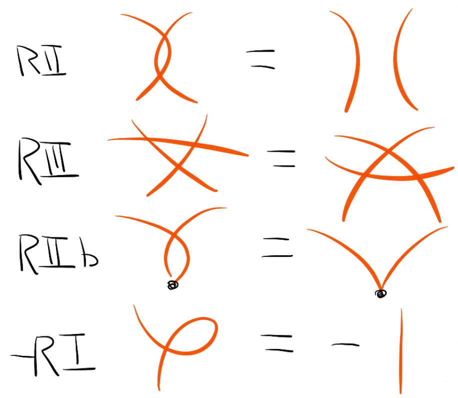

Figure 2: The various Reidemeister moves.

$$\mathbb{Z}[T] \subset A(\Sigma) \subset \mathrm{Sk}(\Sigma) \subset A^+(\Sigma) \subset \mathbb{Z}[T^{\pm 1}] \subset \mathbb{Q}(T) = K(\Sigma) \tag{1}$$

where $\mathbb{Z}[T]$ is the ring of polynomials in the arcs in T, $A(\Sigma)$ is the cluster algebra of Σ, $A^+(\Sigma)$ is the upper cluster algebra (the rational functions of the arcs that are integral Laurent polynomials in any cluster), $\mathbb{Z}[T^{\pm 1}]$ is the ring of integral Laurent polynomials in the arcs in T, and $\mathbb{Q}(T) = K(\Sigma)$ is the fraction field of any of the above.

The inclusion $\mathrm{Sk}(\Sigma) \subset A^+(\Sigma)$ is obtained as follows. As for quantum skeins, we have the following lemma.

Lemma 20.3. *If A is a simple arc and C any curve, then for sufficiently large n, $\langle C \cdot A^n \rangle \in Sk(\Sigma)$ has only terms which are compatible with (do not cross) A.*

It follows that if T is a triangulation consisting of arcs A_i, then we can find a monomial $\prod A_i^{n_i}$ such that $\langle C \cdot \prod A_i^{n_i} \rangle$ is compatible with T, hence is a polynomial in the A_i. Once we know that we can divide by the A_i, it follows that $\langle C \rangle \subset \mathbb{Z}[T^{\pm 1}]$ is a Laurent polynomial in every cluster.

Proof. Without tags, assume that C is simple and that C and A intersect n times. Expanding all of these intersections using the skein relations, we usually obtain two terms with $n-1$ intersections and all of the remaining terms have even fewer intersections. (However, sometimes it is possible to get fewer than $n-1$ intersections.) □

Example Consider an arc C in a triangulated annulus with arcs A_0, A_1. We compute that

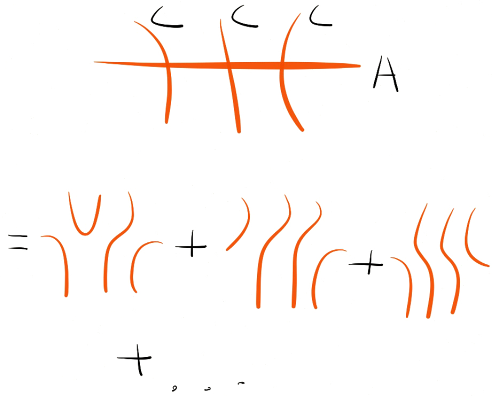

Figure 3: Expanding intersections.

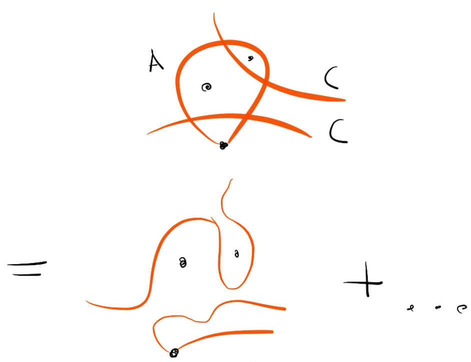

Figure 4: An example in which we get fewer than $n-1$ intersections.

$$\langle C \cdot A_0 \rangle = \langle A_1 \rangle + \langle A_{-1} \rangle \tag{2}$$

where

$$\langle A_1 \cdot A_{-1} \rangle = \langle A_0^2 \rangle + \langle B_1 \cdot B_2 \rangle. \tag{3}$$

Hence as a Laurent polynomial,

$$C = \frac{A_1}{A_0} + \frac{A_0}{A_1} + \frac{B_1 B_2}{A_0 A_1}. \tag{4}$$

This is an example of the following fairly general result.

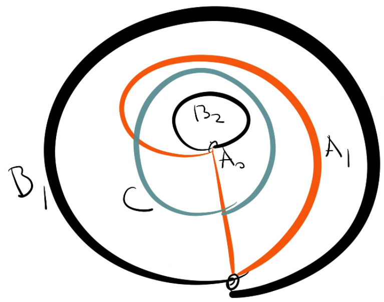

Figure 5: An arc in a triangulated annulus.

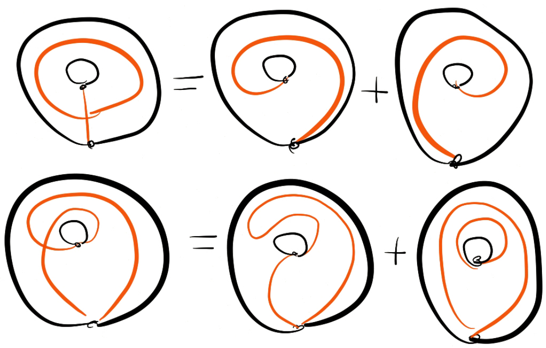

Figure 6: Skein relations in the annulus.

Theorem 20.4. *(Weak positivity) In many cases, any simple curve C expressed as a Laurent polynomial has all positive integer coefficients. These coefficients have combinatorial interpretations.*

Another condition we might ask for is strong positivity, namely that there is a basis x_i of some algebra (over some ordered ring) such that $x_i x_j = \sum n_{ij}^k x_k$ where $n_{ij}^k \geq 0$.

Exercise 20.5. *Find an example in the annulus where strong positivity fails with respect to the simple curves basis. More precisely, consider the curve A_k which is an arc wrapped around k times, and find the first k such that $A_0 A_k$ is not positive.*

It is an interesting problem to find natural strongly positive bases.

Theorem 20.6. *The map $Sk(\Sigma) \to A^+(\Sigma)$ defined above is injective.*

Proof. (Sketch) It is enough to show that multiplication by an arc E is invertible. We can order diagrams by their intersection with E, which has two dominant terms as above given by the left smooth and the right smoothing. We modify the order by approximately a shear; more precisely, we order by $-i(\cdot, A) + i(\cdot, B) - i(\cdot, C) + i(\cdot,)$ where A, B, C, D is a quadrilateral with E as its diagonal, and this picks out one dominant term. Now multiplication by E looks upper triangular. □

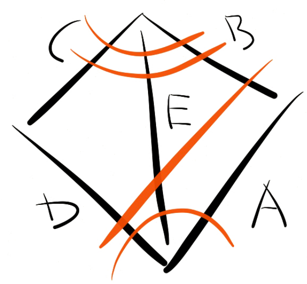

Figure 7: The modification of the order.

Conjecturally we in fact have $\mathrm{Sk}(\Sigma) = A^+(\Sigma)$ unless Σ has one puncture or has tagged arcs and one puncture.