274 Curves on Surfaces, Lecture 19

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21 Strong positivity

Earlier we wrote down a basis (the bangles basis) of the skein algebra that is not strongly positive. Here we will use a different basis, the bracelet basis. It is composed of *bracelet basis curves*, which are diagrams where

1. No two components intersect,
2. No component intersects itself except for multiple covers of a single loop,
3. We do not have two powers of the same simple loop,
4. There are no tag mismatches, disks, punctured disks, or monogons.

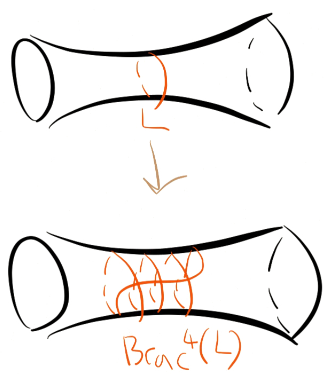

Figure 1: A bracelet basis curve.

Theorem 21.1. *The bracelets basis of $Sk(\Sigma)$ is strongly positive.*

Conjecturally the bracelets basis of $\mathrm{Sk}_q(\Sigma)$ is also strongly positive. There is also another basis, the bands basis, which is not known to be strongly positive for surfaces with punctures. It is strongly positive for the annulus with two or more marked points, and in this case it agrees with Lusztig's dual canonical basis (Lampe).

The Chebyshev polynomials $T_n(z)$ of the first kind are defined by the initial conditions $T_0(z) = 2, T_1(z) = z$, and

$$T_{n+1}(z) = zT_n(z) - T_{n-1}(z) \tag{1}$$

and satisfy

$$T_k(z)T_\ell(z) = T_{k+\ell}(z) + T_{|k-\ell|}(z). \tag{2}$$

They are also uniquely determined by the condition that

$$T_k(e^\ell + e^{-\ell}) = e^{k\ell} + e^{-k\ell}. \tag{3}$$

Lemma 21.2. *For $n \geq 1$, we have $\langle Brac^n(L)\rangle = T_n(\langle L\rangle)$.*

Proof. Check for $n = 1, 2$. For higher values of n apply the skein relation to the outermost crossing. $\square$

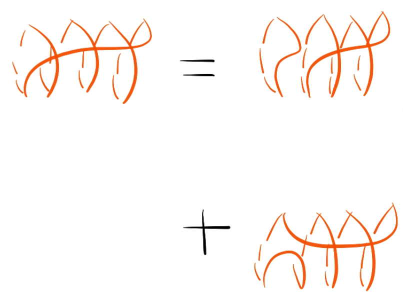

Figure 2: The skein relation applied to a bracelet.

Conjecturally the bracelets basis is an *atomic basis*: for any element of the skein algbra which is not positive in the bracelets basis, there exists a triangulation T with respect to which this element is not a positive Laurent polynomial.

Question from the audience: is the bands basis related to Jones-Wenzl idempotents?

Answer: yes, for $q = 1$. In representation-theoretic terms these correspond to symmeric powers of the defining representation.

For the unpunctured torus, there is a basis parameterized by pairs (k, ℓ) up to the identification $(k, \ell) = (-k, -\ell)$ given by taking

$$T_{(k,\ell)} = \mathrm{Brac}^{\gcd(k,\ell)}(\text{line of slope } \frac{\ell}{k}) \tag{4}$$

and this basis is strongly positive in $\mathrm{Sk}_q(\Sigma)$ (Frohman-Gelca).

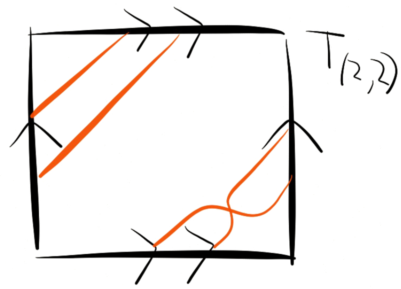

Figure 3: The basis element $T_{(2,2)}$.

Theorem 21.3. *Any diagram D is sign-definite in the bracelets basis in $Sk(\Sigma)$. Any* taut *diagram (immersed with minimal number of self-intersections, no disks, no monogons, no punctured disks) is positive in the bracelets basis.*

Say that a diagram is *positive* if it has no singular 0-gons or singular 1-gons. By this we mean the following. A *segment* of a diagram D is an interval between intersection points or endpoints. A *k-chain* is a sequence of k segments meeting cyclically at endpoints. A 0-*chain* is a loop component of a diagram. For a k-chain C, we can smooth out corners to obtain a loop C^0. A *singular k-gon* is a k-chain C such that C^0 is null-homotopic.

Theorem 21.4. *(Hass-Scott) Any curve diagram can be turned into a taut diagram by Reidemeister reductions (Reidemeister moves that make the diagram less complicated), together with the deletion of boundary loops and unknotted loops.*

This can be done greedily. The idea is to shorten the curves, but this is done combinatorially rather than geometrically.

Corollary 21.5. *A positive curve diagram can be turned into a taut curve diagram using only RIII, RII, RIIb (regular isotopy).*

Proof. By Hass-Scott, we can do this using all of the moves above. We need to show that when doing this we do not use RI, R0 (delete an unknotted loop), or R0b (delete a boundary loop). Before using RI, we have a singular 1-gon, and it suffices to show that going backwards we started out with a singular 1-gon. This follows from an inspection of each move. □

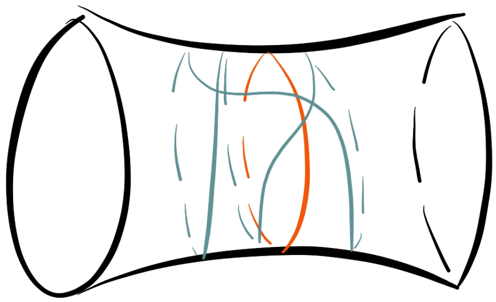

Figure 4: A counterexample.

It is not true that D is taut iff it has no singular 0-gons, 1-gons, or 2-gons. There is a counterexample in an annulus.

The correct salvage is to weaken the notion of singular 2-gon.

Exercise 21.6. *What goes wrong with the going backwards argument for 2-gons?*

We now return to the claim that every taut diagram is positive in the bracelets basis. Say that a *good crossing* of a taut diagram is a crossing where both resolutions are positive.

Lemma 21.7. *Every taut diagram that is not a bracelets basis curve has either a good crossing, two components that are powers of the same loop, or a tag mismatch.*

This is enough to prove the theorem. Let D be a taut diagram. If D is in the bracelets basis, we are done. Otherwise, we are in one of the cases above. In the first case, we resolve the crossing. In the second case, we apply the Chebyshev polynomial identity. The third case is similar to the first.

Question from the audience: is there an inner product with respect to which the bracelets basis is orthogonal?

Answer: that is an interesting question.

Question from the audience: does the bracelets basis admit a monoidal categorification?

Answer: this is not known. Conjecturally there should be a monoidal abelian category $C(\Sigma)$ whose simple objects correspond to the bracelets basis and whose monoidal product reproduces the structure constants of the skein algebra in the bracelets basis in the sense that the Grothendieck ring of $C(\Sigma)$ should be $\mathrm{Sk}(\Sigma)$.