274 Curves on Surfaces, Lecture 20

Dylan Thurston

Fall 2012

22 More about strong positivity

Today we will ignore tags.

Let D be a diagram. Recall that a crossing in D is positive if both of its resolutions are positive, where positivity means no singular 0-gons and 1-gons. Recall also that a diagram is taut if it has the minimum number of self-intersections.

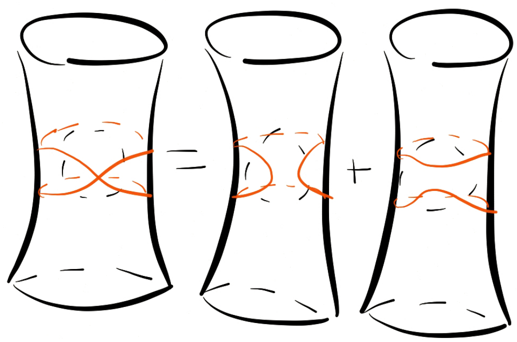

Figure 1: A crossing that is not positive.

Definition A *multi-bracelet* is a diagram where two components don't intersect and each component is a simple arc or a bracelet (and there are no 0-gons and 1-gons).

Lemma 22.1. *Any taut diagram that is not a multi-bracelet has a positive crossing.*

To show this we will use the following.

Lemma 22.2. *If D is taut but a crossing of D resolves into D_1, D_2 where D_1 is not positive, then D_1 has a singular 0-gon or 1-gon passing through the reducing disk (the disk surrounding the crossing in which we apply the skein relation) twice.*

Proof. Since D_1 is not positive, it has a singular 0-gon or 1-gon H. If H does not pass through the reducing disk, then we get a 0-gon or 1-gon for D, which contradicts tautness. If H passes through the reducing disk once, then we get a 1-gon or 2-gon for D, which also contradicts tautness. $\square$

Exercise 22.3. *Find more examples of resolutions $D = D_1 + D_2$ where D is taut and D_1 is not positive. Check the lemma above in your examples.*

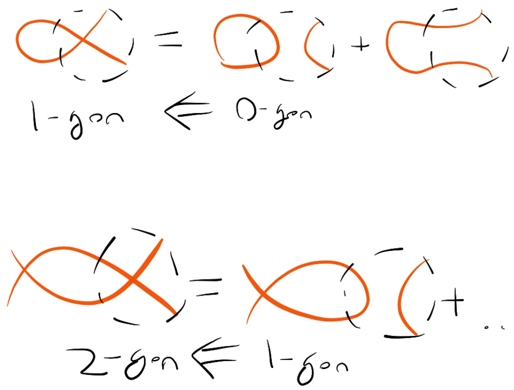

Figure 2: Singular polygons in D_1 and singular polygons in D.

Definition A *bracelet chain* in D is a 0-chain or 1-chain H such that the smoothing H^0 is homotopic to a bracelet. A *maximal bracelet chain* is a bracelet not contained in any larger bracelet chain for the same loop.

Lemma 22.4. *Every component of a taut diagram D with at least one self-crossing has a maximal bracelet chain.*

Proof. Let C be a component with a self-crossing. Then it has a 1-chain. Take a minimal 1-chain by inclusion. This is a 1-chain H such that H^0 is a simple loop L. Take H' to be a maximal bracelet chain containing H (which is also a bracelet for L). $\square$

Lemma 22.5. *The crossing at the end of a maximal bracelet 1-chain is positive.*

Proof. Let H be such a maximal bracelet. Tne resolution of the crossing is not connected to the rest of the diagram locally, so a 0-gon or 1-gon cannot pass through the reducing disk twice. There are two possible cases a), b) involving the other resolution which must be ruled out.

To rule out case a), write the maximal bracelet as $\gamma^k \in \pi_1$ for k maximal and γ a loop. Write the rest of the diagram as $\rho \in \pi_1$. If we get a singular 1-gon in the first case, then $\rho\gamma^k = \mathrm{id}$, or $\rho = \gamma^{-k}$, which contradicts the maximality of H.

To rule out case b), with notation as above, we have $\rho\gamma^\ell = \mathrm{id}$ for some $\ell \leq k$. It follows that the entire component is a bracelet, which contradicts the maximality of H. $\square$

We are getting close to the proof of the first lemma; it suffices now to consider crossings between components.

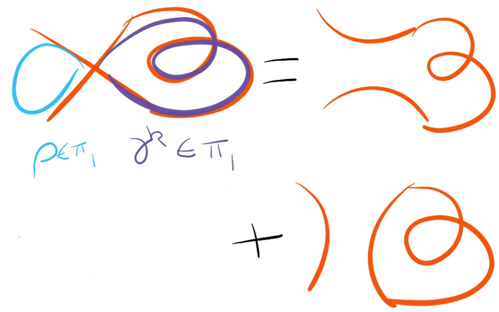

Figure 3: Part of a maximal bracelet.

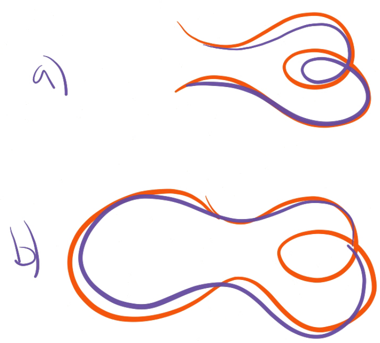

Figure 4: The two cases above.

Exercise 22.6. *Show that any crossing between two components C_1, C_2 of a taut diagram D where C_1, C_2 are simple arcs or bracelets is positive.*

Use the fact that roots are unique in $\pi_1(\Sigma)$: that is, if $\gamma^k = \rho^\ell$, then there exists σ, s, t such that $\gamma = \sigma^s, \rho = \sigma^t$, and $sk = t\ell$.

Exercise 22.7. *Prove the multiplication rule $T_{(a,b)}T_{(c,d)} = T_{(a+c,b+d)} + T_{(a-c,b+d)}$ for the basis for the unpunctured torus (at $q = 1$) from last time.*

Now we will discuss a geometric interpretation of the skein relations. Here we will ignore marked points and arcs. The skein relation should have something to do with SL_2. More precisely, it should have something to do with the following fact: if $A, B \in \mathrm{SL}_2$, then

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}). \tag{1}$$

This follows from the Cayley-Hamilton theorem, which gives $B^2 - \mathrm{tr}(B)B + I = 0$, after dividing by B and multiplying by A, then taking traces.

A, B should be the monodromy of two loops, except that the signs don't match.

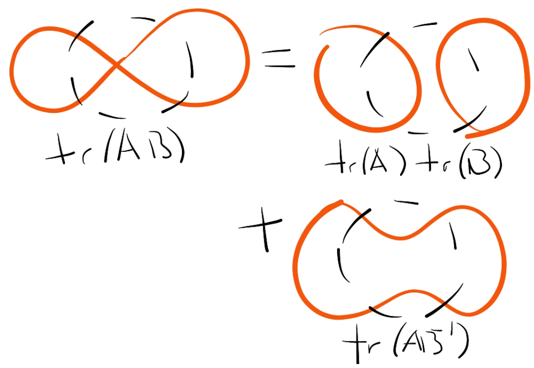

Figure 5: The signs are incorrect here.

This is because we were taking $q = 1$ and we actually need to take $q = -1$, which gives a skein relation in which three terms add up to 0. Geometrically we should take the negative of the trace. More precisely,

Proposition 22.8. *If Σ is a surface and $\rho : \pi_1(\Sigma) \to SL_2$ is a representation, then the assignment*

$$D \mapsto \prod_i -tr(\rho(D_i)) \tag{2}$$

where D is a diagram with components D_i satisfies the $q = -1$ skein relation.

However, taking $q = -1$ destroys positivity. To get back to $q = 1$ we need to twist. There is a fibration

$$S^1 \to \mathrm{UT}(\Sigma) \to \Sigma \tag{3}$$

where UT is the unit tangent bundle. This gives a long exact sequence in homotopy

$$\pi_2(\Sigma) \to \pi_1(S^1) \to \pi_1(\mathrm{UT}(\Sigma)) \to \pi_1(\Sigma) \to 0 \tag{4}$$

and if Σ is not S^2 then $\pi_2(\Sigma)$ vanishes, hence $\pi_1(\mathrm{UT}(\Sigma)) = \tilde{\pi}_1(\Sigma)$ is a canonical $\mathbb{Z}$-extension of $\pi_1(\Sigma)$. We define a twisted SL_2 representation to be a representation $\rho : \tilde{\pi}_1(\Sigma) \to \mathrm{SL}_2$ such that $\rho(360^\circ \text{ turn}) = -1$. The corresponding quotient map to PSL_2 gives an honest PSL_2-representation.

If γ is an immersed curve in Σ, by taking tangent vectors it lifts to a curve $\tilde{\gamma}$ in $\mathrm{UT}(\Sigma)$, and associated to this choice of lift is a trace. (If we had just used a PSL_2-representation, the trace would only be defined up to sign.)

Proposition 22.9. *For Σ a surface and $\rho : \tilde{\pi}_1(\Sigma) \to SL_2$ a twisted SL_2-representation, the assignment*

$$D \mapsto \prod_i tr(\rho(\tilde{D}_i)) \tag{5}$$

satisfies the $q = 1$ skein relation.

In the background here is the fact that a hyperbolic structure on Σ has a canonical twisted SL_2-representation lifting the PSL_2-representation given by considering the universal cover.

This is related to spin structures. Recall that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$; in particular it is not simply connected. When $n = 2$ we have $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$. In any case, for $n \geq 2$, it follows that $\mathrm{SO}(n)$ has a unique double cover called $\mathrm{Spin}(n)$, and for $n \geq 3$ this is the universal cover. There are exceptional isomorphisms $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ and $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$.

A *spin structure* on a smooth oriented (Riemannian for simplicity but this is not necessary) manifold M is a lift of the frame bundle to a $\mathrm{Spin}(n)$-bundle. Concretely, this gives us some information about which loops in the frame bundle lift and which do not. On an oriented surface Σ, rather than thinking about frames we can think about tangent vectors, and then the question is whether or not a path of tangent vectors lifts. We can generate paths of tangent vectors using an immersed curve. This gives us a rule assigning connected immersed curves signs ± 1 satisfying some rules.

A twisted SL_2-representation can then be described as the product of an ordinary SL_2-representation and a spin structure.