274 Curves on Surfaces, Lecture 21

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23 More about the geometry of skein relations

Recall that a twisted SL_2-representation of a surface Σ is an SL_2-representation of a $\mathbb{Z}$-extension $\tilde{\pi}_1(\Sigma) \cong \pi_1(\mathrm{UT}(\Sigma))$ of the fundamental group such that a 360° rotation acts by -1. Such a representation in particular descends to an ordinary PSL_2-representation of $\pi_1(\Sigma)$. Defining

$$\tilde{\rho}(D) = \prod_i \mathrm{tr}(\tilde{\rho}(\tilde{D}_i)) \tag{1}$$

where D is a curve diagram, $\tilde{D}_i$ are the lifts of the components of D to $\mathrm{UT}(\Sigma)$, and $\tilde{\rho}$ is a twisted representation, we claim that $\tilde{\rho}(D)$ satisfies the skein relations with $q = 1$.

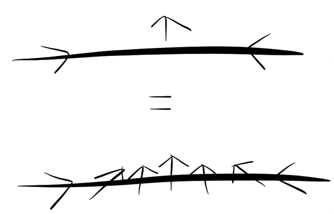

Figure 1: An immersed curve and its tangent vectors.

We should say more about paths in the unit tangent bundle. We can notate these by writing down an immersed curve in Σ together with arrows indicating how the unit tangent vectors should rotate. This notation satisfies some straightforward axioms.

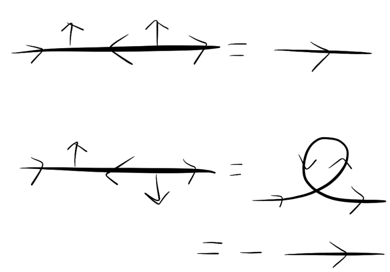

Figure 2: Some axioms.

To verify the skein relations, there are two cases depending on how the skeins close up into curves. We will apply the trace relation $\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$, but keeping track of tangent vectors.

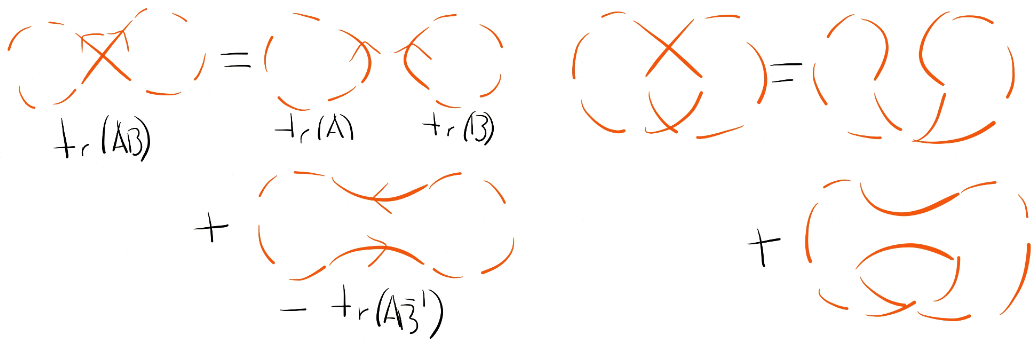

Figure 3: The two cases.

Exercise 23.1. *Verify the skein relation in the second case.*

The relationship to spin structures is the following. Concretely, a double cover of the frame bundle is a homomorphism η from π_1 of the frame bundle to $\mathbb{Z}/2\mathbb{Z}$. For a surface Σ the frame bundle is essentially the unit tangent bundle, so a spin structure on Σ assigns a sign to every path in $\mathrm{UT}(\Sigma)$, hence to immersed curves in Σ. This assignment satisfies various axioms.

Alternatively, we can think of η as an element of $H^1(\mathrm{UT}(\Sigma), \mathbb{Z}/2\mathbb{Z})$ which is nontrivial on the 360° rotation. There is an action of $H^1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ on the above cohomology group (by translation), hence $H^1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ acts on the set of spin structures of Σ. If spin structures exist, the set of all such spin structures is then a torsor over $H^1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. More concretely, if η_1, η_2 are two spin structures, their difference is an element of $H^1(\Sigma; \mathbb{Z}/2\mathbb{Z})$.

Proposition 23.2. *If $\rho : \pi_1(\Sigma) \to SL_2$ is an SL_2-representation of the fundamental group and $\eta : \pi_1(UT(\Sigma)) \to \mathbb{Z}/2\mathbb{Z}$ is a spin structure, then $\rho\eta : \pi_1(UT(\Sigma)) \to SL_2$ is a twisted SL_2-representation.*

Conversely, we have an identification

$$\mathrm{Rep}^{\mathrm{Twist}}_{\mathrm{SL}_2}(\Sigma) = (\mathrm{Rep}_{\mathrm{SL}_2}(\Sigma) \times \mathrm{Spin}(\Sigma))/H^1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \tag{2}$$

where $H^1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\pi_1, \mathbb{Z}/2\mathbb{Z})$ acts diagonally.

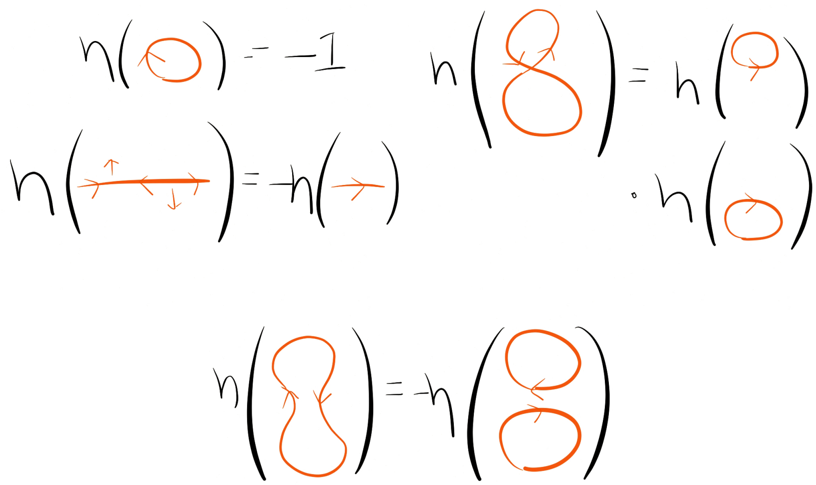

Figure 4: Some properties of spin structures.

Exercise 23.3. *What are the spin structures on the torus? What are their orbits under the action of the mapping class group?*

The skein algebra at $q = 1$ describes a class of functions on the set of twisted SL_2-representations. Moreover it has a strongly positive basis which is invariant under the action of the mapping class group. Passing to ordinary SL_2-representations by multiplication by a spin structure, we lose this invariance because not all spin structures are preserved by the action of the mapping class group.

Now suppose Σ is equipped with a hyperbolic structure. Then the universal cover of Σ is naturally identified with the hyperbolic plane $\mathbb{H}^2$, so we can write $\Sigma \cong \mathbb{H}^2/\Gamma$ where $\Gamma \cong \pi_1(\Sigma)$ is a discrete subgroup of $PSL_2(\mathbb{R})$. Thus a hyperbolic structure determines a (faithful) representation

$$\pi_1(\Sigma) \to PSL_2(\mathbb{R}). \tag{3}$$

This is part of the structure defining a twisted SL_2-representation. The different lifts of this representation to an SL_2-representation are classified by spin structures and people usually pick one.

However, there is a canonical such lift. To see this, write $\mathbb{H}^2$ as the quotient of $PSL_2(\mathbb{R})$ by $SO(2)$. This gives an identification of $UT(\mathbb{H}^2)$ with $PSL_2(\mathbb{R})$, hence an identification

$$\mathrm{UT}(\Sigma) \cong \mathrm{PSL}_2(\mathbb{R})/\Gamma \tag{4}$$

using the fact that the identification $\Sigma \cong \mathbb{H}^2/\Gamma$ respects tangent spaces. This in turn gives an identification

$$\mathrm{UT}(\Sigma) \cong \mathrm{SL}_2(\mathbb{R})/\tilde{\Gamma} \tag{5}$$

where $\tilde{\Gamma}$ is the preimage of Γ in $\mathrm{SL}_2(\mathbb{R})$ (some $\mathbb{Z}/2\mathbb{Z}$ central extension). This is not quite $\tilde{\pi}_1(\Sigma)$, but there is a diagram

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \tilde{\pi}_1(\Sigma) & \longrightarrow & \pi_1(\Sigma) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \tilde{\Gamma} & \longrightarrow & \Gamma \end{array} \tag{6}$$

relating them.

Alternatively, consider the boundary of $\mathbb{H}^2$, which is naturally identified with $\mathbb{RP}^1$. This is naturally acted on by $\mathrm{PSL}_2(\mathbb{R})$. On the other hand, given a tangent vector to a point in $\mathbb{H}^2$ we can follow a unique geodesic to the boundary, which gives a natural identification

$$\mathrm{UT}_p(\Sigma) \cong \mathrm{UT}_p(\mathbb{H}^2) \cong \mathbb{RP}^1. \tag{7}$$

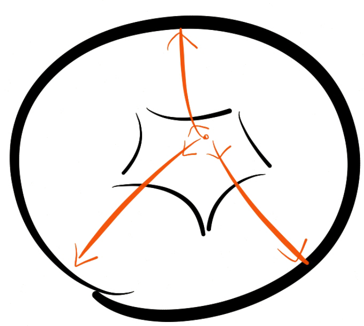

Figure 5: Identifying unit tangents with the boundary.

This identification can also be used to obtain the above result.

It follows that the Teichmuller space for Σ embeds into $\mathrm{Rep}^{\mathrm{Twist}}_{\mathrm{SL}_2}$.

Proposition 23.4. *The Teichmüller space for Σ is the totally positive part of $Rep_{SL_2}^{Twist}$: that is, it is the part where all elements in the positive basis for the skein algebra take positive values.*

To take decorations into account, we need a notion of twisted decorated SL_2-representation. We will first think of twisted SL_2-representations as twisted SL_2-local systems (2-dimensional real vector bundles V over $\mathrm{UT}(\Sigma)$ such that the projectivization of V is the pullback of the unit tangent bundle over Σ). To decorate them, we want the additional data of a choice of vector in V_p for each outward-pointing tangent vector at a boundary point.