274 Curves on Surfaces, Lecture 4

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4 Hyperbolic geometry

Last time there was an exercise asking for braids giving the torsion elements in $\mathrm{PSL}_2(\mathbb{Z})$. A 3-torsion element can be obtained by cyclically permuting punctures (a one-third-twist?), and a 2-torsion element can be obtained by swapping two punctures (a half-twist).

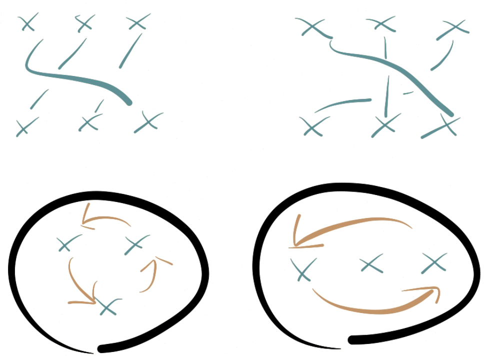

Figure 1: Braids which are torsion in $\mathrm{PSL}_2(\mathbb{Z})$.

Last time we also classified elements of $\mathrm{MCG}(T^2)$ as either periodic, parabolic, or hyperbolic. This classification generalizes to other surfaces; it is called the Nielsen-Thurston classification. We will get back to this later.

First, the hyperbolic plane. It is the unique complete simply-connected Riemannian surface with constant curvature -1, but this is not useful for computations.

The *Poincaré disk model* of the hyperbolic plane is the open disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with the metric

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - r^2)^2} \tag{1}$$

where $r^2 = x^2 + y^2$.

Since the metric is always a scalar multiple of the standard Euclidean metric, this angle is conformal with the Euclidean metric, so angles agree with Euclidean angles (even if lengths do not agree with Euclidean lengths). Geodesics are circles perpendicular to the boundary (including circles of infinite radius, or lines); in particular, there is a unique geodesic through any two points.

A geodesic triangle has the property that the sum of its angles is less than π. In fact,

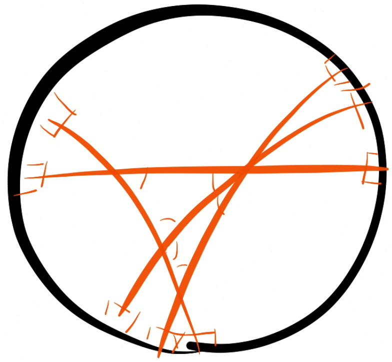

Figure 2: Geodesics in the disk model.

$$\text{Area(triangle)} = \pi - \text{sum of angles}. \tag{2}$$

Exercise 4.1. *The angle sum defect $\pi -$ sum of angles is additive on triangles.*

The *Poincaré half-plane model* is the upper half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{3}$$

Geodesics are circles perpendicular to the boundary (including circles of infinite radius). This metric is also conformal with the Euclidean metric.

Instead of thinking directly about these metrics, it is better to think about automorphisms (that is, about isometries). The automorphisms we want should be conformal; that is, they should preserve (Euclidean) angles. This is equivalent to complex analytic with derivative not equal to zero anywhere. We will look for such automorphisms within the group of Möbius transformations

$$z \mapsto \frac{az + b}{cz + d}, a, b, c, d \in \mathbb{C}, ad - bc \neq 0. \tag{4}$$

This is precisely the group of conformal automorphisms of the Riemann sphere. Abstractly, this is the group $\text{PGL}_2(\mathbb{C})$ (which naturally acts on $\mathbb{CP}^1$), and consequently it admits a morphism from $\text{GL}_2(\mathbb{C})$ (so Möbius transformations compose like matrices) sending $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to the above.

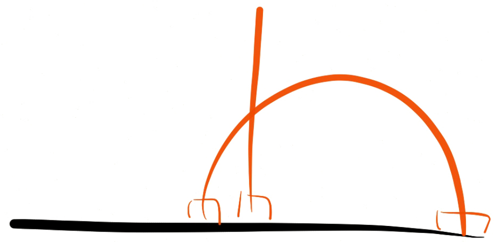

Figure 3: Geodesics in the half-plane model.

Möbius transformations have many nice properties. For example, they preserve circles (including of infinite radius).

Some Möbius transformations are relatively easy to understand. Those of the form $z \mapsto az + b$ describe translations, scalings, and rotations. The only additional Möbius transformation needed to generate the entire group is $z \mapsto \frac{1}{z}$, which is closely related to circle inversion $z \mapsto \frac{1}{\bar{z}}$. Inversion has many nice properties: it sends circles inside the unit circle to circles outside the unit circle and sends circles intersecting the unit circle to circles intersecting the unit circle.

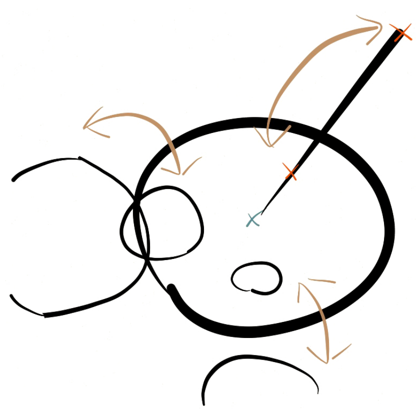

Figure 4: Circle inversion.

We want to consider Möbius transformations which in addition preserve the open disk or the upper half plane.

Theorem 4.2. *The orientation-preserving isometries of the upper half-plane are precisely $PSL_2(\mathbb{R}) \subset PGL_2(\mathbb{C})$.*

Proof. Since $\mathbb{H}^2$ has constant curvature, it has an isometry taking every pair of a point and a tangent vector to another point and a tangent vector. Every orientation-preserving isometry is conformal, hence complex-analytic, and furthermore extends to a conformal automorphism $\mathbb{CP}^1 \to \mathbb{CP}^1$ by the Schwarz reflection principle, hence must be a Möbius transformation. Moreover, it must preserve $\mathbb{RP}^1 \subset \mathbb{CP}^1$, so lies in $\mathrm{PGL}_2(\mathbb{R})$.

There is a commutative diagram of inclusions

$$\begin{array}{ccc} \mathrm{PSL}_2(\mathbb{R}) & \longrightarrow & \mathrm{PSL}_2(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{PGL}_2(\mathbb{R}) & \longrightarrow & \mathrm{PGL}_2(\mathbb{C}) \end{array} \tag{5}$$

and the inclusion $\mathrm{PSL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ is an isomorphism (we can always scale by the square root of the determinant) but the inclusion $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R})$ is not; the latter has two connected components, one consisting of matrices with positive determinant (which $\mathrm{PSL}_2(\mathbb{R})$ maps to isomorphically) and one consisting of matrices with negative determinant.

The elements of $\mathrm{PGL}_2(\mathbb{R})$ of negative determinant take the upper half-plane to the lower half-plane, so the elements of $\mathrm{PSL}_2(\mathbb{R})$ are the ones we want. □

We can now obtain the metric on the upper half-plane we wrote down earlier as follows: first assume that it is $ds^2 = dx^2 + dy^2$ at the point $(0, 1)$ and find an automorphism taking $(0, 1)$ to another point (x, y). It suffices to take $z \mapsto yz + x$, and we want this to be an isometry, which in fact forces $ds^2 = \frac{dx^2+dy^2}{y^2}$ by inspecting the Jacobian.

A similar idea works for the metric on the disk; alternately, there is a Möbius transformation taking the upper half-plane to the disk given by

$$z \mapsto \frac{z - i}{z + i}. \tag{6}$$

To see this, note that it takes the boundary of the upper half-plane to the boundary of the open disc and takes i to 0. (Multiplying by i gives a map which fixes ± 1 and which sends ∞ to i.) This gives us a description of the Möbius transformations fixing the disk by conjugating by the above map; explicitly, they have the form

$$z \mapsto e^{i\theta} \frac{z - a}{\bar{a}z - 1} \tag{7}$$

where a lies in the disk.

More generally, the Riemann mapping theorem asserts that for any open simply-connected subset U of $\mathbb{C}$ there exists a unique biholomorphic map from U to the open disc sending a particular point and tangent vector to a particular point and tangent vector in the open disc. This gives us a metric on U.

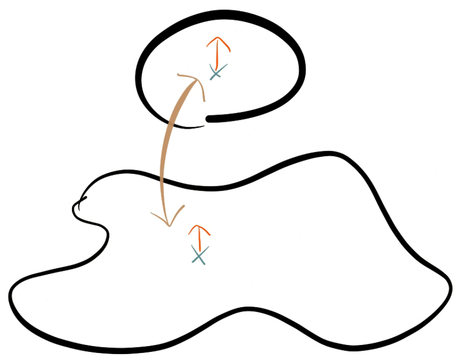

Figure 5: The Riemann mapping theorem.

Moving away from complex analysis, we can write down a different kind of model of the hyperbolic plane which generalizes better to higher dimensions as follows. There is an exceptional isomorphism

$$\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}^+(2,1) \tag{8}$$

where the RHS describes the group of all linear transformations of $\mathbb{R}^3$ preserving the quadratic form $x^2+y^2-z^2$ which have determinant 1 and which maps the upper half of the cone $x^2+y^2=z^2$ to itself.

This can be seen as follows. $\mathrm{SL}_2(\mathbb{R})$ has a natural representation on $\mathbb{R}^2$. This gives a representation of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathrm{Sym}^2(\mathbb{R})$, or equivalently on 2×2 real symmetric matrices as follows:

$$\begin{bmatrix} p & q \\ q & r \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ q & r \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \tag{9}$$

This action preserves the determinant $pr-q^2$. Writing $x=q, p=z+y, q=z-y$ this gives the quadratic form $z^2-y^2-x^2$. Alternately, $\mathrm{PSL}_2(\mathbb{R})$ has an adjoint action on its Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, which is 3-dimensional. This action preserves the Killing form on $\mathrm{sl}_2(\mathbb{R})$, which also has signature $(2,1)$ as above.

SO$(2,1)$ preserves the two-sheeted hyperboloid $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2-z^2 = -1\}$, and SO$^+(2,1)$ preserves the sheet $z > 0$. This has an induced metric

$$ds^2 = dx^2 + dy^2 - dz^2 \tag{10}$$

which gives another model of the hyperbolic plane, the *hyperboloid model.* This is a pleasant model for computations because of the lack of division.

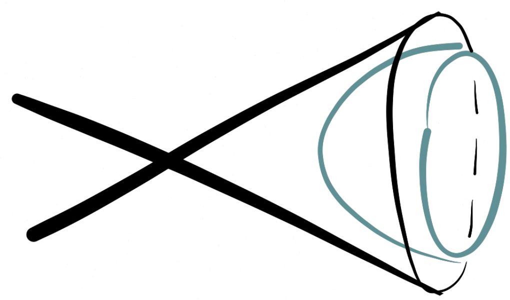

Figure 6: The hyperboloid model.

We need to check that the above metric is actually Riemannian; as a metric on $\mathbb{R}^3$ it is Lorentzian. SO$^+(2,1)$ acts transitively on the hyperboloid, so to check the signature of the metric it is enough to check the signature at a point. We will use the point $(0,0,1)$. The tangent plane at this point is the xy-plane, and the induced metric is Riemannian as desired.

The hyperboloid may be thought of as a sphere of radius i. This is not as silly as it sounds; it turns many trigonometric identities into hyperbolic trigonometric identities.

We now have three models for the hyperbolic plane. To do computations in the future, we will choose whichever model makes our computations easiest.