274 Curves on Surfaces, Lecture 6

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6 Dehn-Thurston coordinates (Alex)

We want to study the action of the mapping class group on isotopy classes of curves on a surface. Ideally this action should be faithful. Dehn-Thurston coordinates are a way to parameterize isotopy classes of curves.

Let S_g be a compact orientable surface of genus g with negative Euler characteristic, possibly with boundary. (The Euler characteristic condition only excludes the sphere, the torus, the cylinder, and the disks.) We will consider *multicurves* on S_g, which are 1-dimensional submanifolds such that no component bounds a disk and such that no component is homotopic to an arc on the boundary. This gives a collection of non-intersecting, non-self-intersecting, non-null-homotopic curves.

We parameterize multicurves by first choosing a decomposition into pairs of pants.

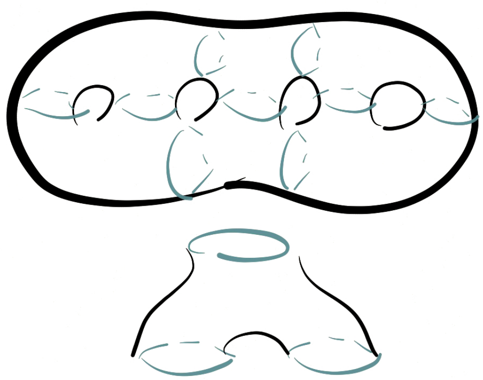

Figure 1: A pair of pants decomposition.

We want to know the intersections of a multicurve with the boundary of each pair of pants. This gives a collection of intersection numbers $m_1, ...m_n$. Additionally, we have N twisting numbers which tell us how to glue the pairs of pants together. If S_g has no boundary, then $N = 3g - 3$.

Definition The *geometric intersection number* of two curves is

$$(\gamma, \delta) = \min_{c,d} |c \cap d| \tag{1}$$

where c, d are curves isotopic to γ, δ respectively.

The claim we need for these intersections to determine a multicurve is that up to isotopy preserving the boundary componentwise, a multicurve on a pair of pants is determined by its intersection numbers with the boundary (except for components parallel to a boundary component).

Fix intervals on the boundary components of the pants; these are *windows*. We will require that our curves only intersect the boundaries in windows. This only gives a few possibilities for the components of a multicurve: it can either connect adjacent windows, loop around to connect a window with itself, or loop around a leg (parallel to a boundary component).

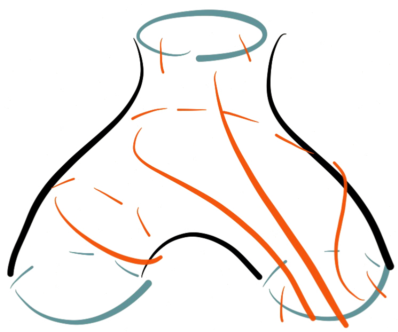

Figure 2: Curves on a pair of pants.

To define twisting numbers, we will now decompose S_g into pairs of pants and cylinders (and again fix windows).

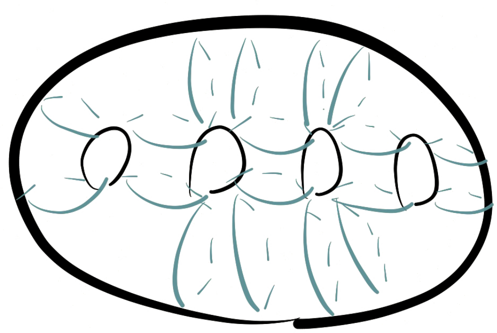

Figure 3: Pairs of pants and cylinders.

In a given cylinder, the twisting number is then the geometric intersection of a multicurve with either of two curves connecting the boundaries of the windows, with sign determined by handedness.

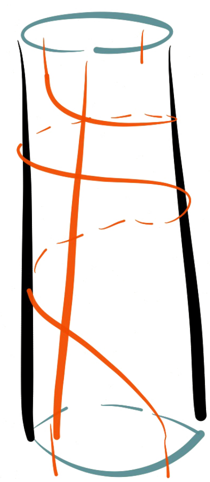

Figure 4: Twisting numbers in a cylinder.

We also need twisting numbers counting components parallel to boundary components.

In summary, we parameterize multicurves by elements of the set $\mathbb{Z}_{\geq 0}^{3g-3} \times \mathbb{Z}^{3g-3}$ quotiented by the equivalence relation $(0, x) \sim (0, -x)$.

Theorem 6.1. *The mapping class group is generated by Dehn twists.*

If an element of the mapping class group does not act faithfully on multicurves, then it fixes all such curves (up to isotopy), hence commutes with all Dehn twists, hence lies in the center. To show that the action of the mapping class group on multicurves is faithful, it suffices to show that the center is trivial. This will be true whenever $g > 2$ and S_g does not have boundary.

To see this we will draw a suitable collection of circles on S_g. Any element of the center preserves (up to isotopy) every circle, so it preserves the graph describing how circles intersect.

But when $g > 2$ we can arrange these circles so that the corresponding graph has no automorphisms. It follows that up to isotopy an element of the center of the mapping class group fixes the graph pointwise.

The complement of the graph is a collection of disks, and a homeomorphism of the disk fixing the boundary is isotopic to the identity through homeomorphisms fixing the boundary, so we conclude.

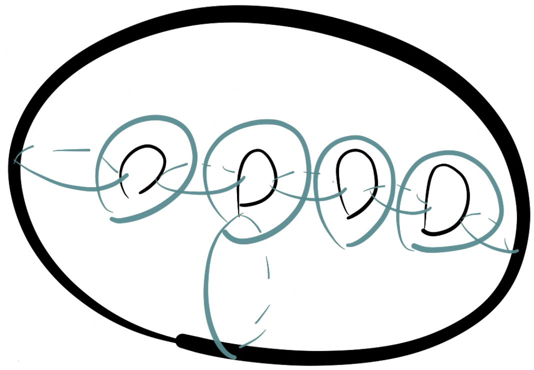

Figure 5: Circles on a 4-holed torus.

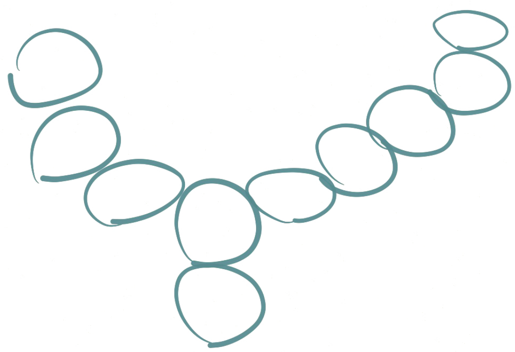

Figure 6: A graph describing the intersections of the circles which has no automorphisms.

This argument does not work when $g = 2$, when we can take a $180°$ rotation. This is a reflection of the fact that when $g = 2$ a smooth projective algebraic curve over $\mathbb{C}$ is hyperelliptic, so alwways has a hyperelliptic involution, but when $g > 2$ not all curves are hyperelliptic.

Dehn-Thurston coordinates were discovered by Dehn and rediscovered by Thurston.

7 More about ideal polygons

Last time we discussed two methods for understanding ideal polygons. One was to send three of their vertices to $0, 1, \infty$, and another was to choose horocycles around the vertices and count distances. We want a more natural version of this picture (which does not depend on a choice of triangulation).

Consider the hyperboloid model $x^2 + y^2 = z^2 - 1$. What do horocycles look like here? First, what do circles look like? Using the projection to a plane, they come from cones coming from the other hyperboloid.

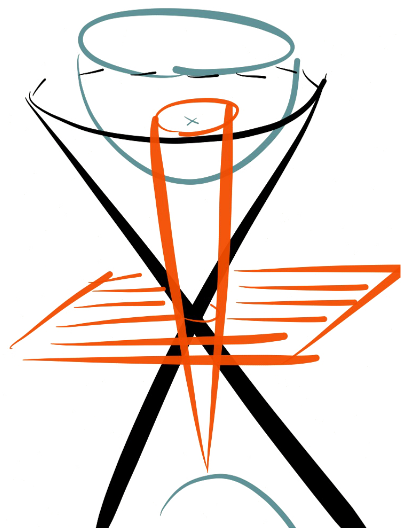

Figure 7: A cone and the corresponding circle.

Alternatively, we can intersect the hyperboloid with a plane (analogous to what happens with a sphere). On a sphere, the center of the corresponding circle is the unique point whose tangent plane is parallel to the intersecting plane, and the same is true on the hyperboloid.

This is clearest to see for the lowest point on the hyperboloid, and everything is invariant under $\mathrm{SO}^+(2,1)$, so it follows everywhere.

To get horocycles, we take tangent planes to the cone (circles centered at infinity), then translate them so that they intersect with the hyperboloid.

We want to describe this situation algebraically in terms of the inner product on $\mathbb{R}^3$ with corresponding quadratic form $x^2 + y^2 - z^2$. The hyperboloid is the set of vectors v such that $\langle v, v \rangle = -1$. The (null) cone is the set of vectors v such that $\langle v, v \rangle = 0$ (*null vectors*). Planes can be described in the form

Figure 8: A plane intersecting a sphere and the corresponding circle.

Figure 9: A plane intersecting a hyperboloid and the corresponding circle.

$$P_v = \{w : \langle v, w\rangle = k\}. \tag{2}$$

When v is a null vector, $v \in P_{v,0}$. To get horocycles, we take hyperplanes of the form $P_{v,-1}$ and intersect them with the hyperboloid. In other words, horocycles have the form

$$h_v = \{w : \langle v, w\rangle = \langle w, w\rangle = -1\}. \tag{3}$$

We therefore have a natural correspondence between horocycles and nonzero null vectors in the upper cone ($z \geq 0$). Now, given two vectors v_1, v_2 at which we have centered two horocycles h_1, h_2, we want to describe algebraically the corresponding length. We begin by defining the *λ-length*

$$\lambda(h_1, h_2) = \sqrt{-\frac{1}{2}\langle v_1, v_2\rangle}. \tag{4}$$

This is some function $f(\ell(h_1, h_2))$ of the length. To explain the factor of $\frac{1}{2}$, first take $v_1 = (1, 0, 1)$. The corresponding horocycle contains $x = (0, 0, 1)$, and so does the horocycle corresponding to $v_2 = (-1, 0, 1)$. In fact, the horocycles are tangent at x, so $\ell(h_1, h_2) = 0$ in this case. On the other hand, $\langle v_1, v_2\rangle = -2$, so the above normalization gives $\lambda(h_1, h_2) = 1$.

Now we should talk more about the relationship between the upper half-plane model and the hyperboloid model. In the former the isometry group is $\mathrm{PSL}_2(\mathbb{R})$ while in the latter the isometry group is $\mathrm{SO}^+(2, 1)$. In the former the ideal points are $\mathbb{RP}^1$ while in the latter the ideal points are rays in the null cone. We would like a correspondence between them, hence a way to take vectors in $\mathbb{R}^2$ to null vectors in $\mathbb{R}^{2,1}$.

We can do this by thinking of $\mathbb{R}^{2,1}$ as symmetric 2×2 matrices with the negative of the determinant as the quadratic form. Given a vector in $\mathbb{R}^2$, we can now tensor it with itself to get such a symmetric matrix, giving

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \tag{5}$$

Exercise 7.1. *Check that diagonalizing this quadratic form gives a map $(a, b) \mapsto (a^2 - b^2, 2ab, a^2 + b^2)$ from $\mathbb{R}^2$ to null vectors in $\mathbb{R}^{2,1}$.*

This gives a map from $\mathbb{R}^2$ to horocycles. What is the λ-length in these terms? Given (a, b) and (c, d), the dot product of the corresponding null vectors is

$$abcd - \frac{1}{2}(a^2d^2 + b^2c^2) = -\frac{1}{2}(ad - bc)^2 = -\frac{1}{2}\det\begin{bmatrix} a & c \\ b & d \end{bmatrix}^2. \tag{6}$$

So the corresponding λ-length is

$$\frac{1}{2}\left|\det\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right|. \tag{7}$$

Exercise 7.2. *Was this detour necessary? Is there a natural way to write down horocycles as subsets of $\mathbb{CP}^1$ (which contains both the disk and the half-plane models) and do these computations there instead?*