274 Curves on Surfaces, Lecture 7

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8 Horocycles and lengths

Last time we saw that in the hyperboloid model there is a nice way to write down horocycles: they can be given by sets of the form

$$h_v = \{w : \langle v, w\rangle = \langle w, w\rangle = -1\} \tag{1}$$

where v is a null vector. On the other hand, it is often more convenient to work in the upper half-plane. Here the boundary is $\mathbb{RP}^1$ and we saw previously that there is a way to write down from a vector $(a, b) \in \mathbb{R}^2$ a null vector $(2ab, a^2 - b^2, a^2 + b^2)$, and this gives a map from $\mathbb{RP}^1$ to null vectors, which give horocycles. It would be nice to be able to avoid this and directly describe the horocycle associated to (a, b) in the upper half-plane.

Last time we also described the λ-length

$$\lambda(h_1, h_2) = \sqrt{-\frac{1}{2}\langle v_1, v_2\rangle} = \frac{1}{2}\left|\det(w_1, w_2)\right| \tag{2}$$

between two horocycles in terms of the inner product of the corresponding null vectors and then in terms of the determinant of the corresponding vectors in $\mathbb{R}^2$. This is some function $f(\ell(h_1, h_2))$ of the length which we have not yet worked out. To compute this function, we will work in the upper half-plane with a horocycle at 0 and a horocycle at ∞ (which is a horizontal line). We have one more degree of freedom, so we will choose the diameter of the horocycle at 0 to be 1.

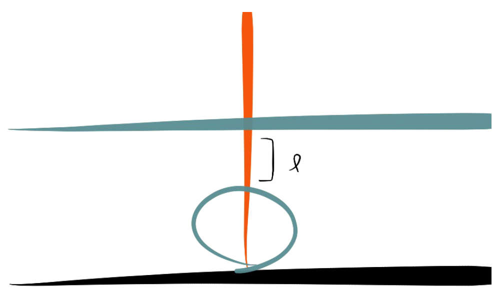

Figure 1: Two horocycles in the upper half-plane.

Then the horocycle at ∞ is the line $y = c$ for some c and the distance between them is

$$\int_1^c \frac{dy}{y} = \log c. \tag{3}$$

The nicest case is when we also have $c = 1$, in which case the distance is 0. We can do this by scaling the horocycle at ∞ by $\frac{1}{c}$ without scaling the horocycle at 0. This is not an isometry, so it will change the length and λ-length. In intrinsic hyperbolic terms, we move each point of the horocycle by $\log c$ away from ∞. To determine what this does to v_i or w_i we need to determine the corresponding element of $\mathrm{PSL}_2(\mathbb{R})$. As a fractional linear transformation, this is $x \mapsto \frac{x}{c}$, which is associated to the matrix

$$\begin{bmatrix} \frac{1}{\sqrt{c}} & 0 \\ 0 & \sqrt{c} \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}). \tag{4}$$

Now we should figure out what w_1 and w_2 are (the elements of $\mathbb{R}^2$ associated to our horocycles). We will have $w_1 = (0, y_1)$ for some y_1 and $w_2 = (x_2, 0)$ for some x_2. Scaling by $\frac{1}{c}$ multiplies w_2 by $\frac{1}{\sqrt{c}}$ and changes the λ-length to 1 by our normalization, so

$$\lambda(h_1, h_2) = \sqrt{c} = \exp\left(\frac{\ell(h_1, h_2)}{2}\right). \tag{5}$$

Exercise 8.1. *Find $\lambda(h_1, h_2)$ in terms of the Euclidean geometry of two horocycles in the upper half-plane.*

Given a decorated and triangulated ideal polygon, we can ask about how λ-lengths change when we change triangulations. Hyperbolically this is a messy computation, but algebraically it becomes nicer. For an ideal quadrilateral determined by four vectors w_1, w_2, w_3, w_4, we want to compute one of the corresponding determinants in terms of the others.

Writing the λ-lengths of the four sides as A, B, C, D and writing the λ-lengths of the diagonals as E, F, we get the following result.

Lemma 8.2. *(Ptolemy relation)* $\lambda(F)\lambda(E) = \lambda(A)\lambda(C) + \lambda(B)\lambda(D)$.

This relation is named after the corresponding relation for a quadrilateral inscribed in a circle in Euclidean geometry.

Proof. This is really a statement about a 2×4 matrix (assembled from the w_i)

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}, \tag{6}$$

namely a quadratic relation between the 2×2 minors. Both sides of this quadratic relation are invariant under scaling any of the columns, so assuming that the $y_i \neq 0$

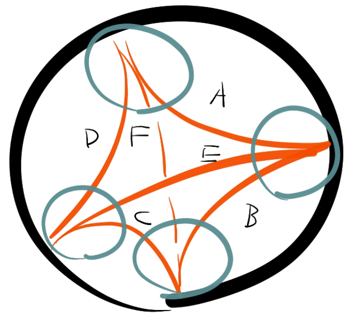

Figure 2: λ-lengths in a decorated ideal quadrilateral.

we may assume WLOG that the y_i are all equal to 1. Then the quadratic relation becomes

$$(x_2 - x_4)(x_1 - x_3) = (x_1 - x_2)(x_3 - x_4) + (x_1 - x_4)(x_2 - x_3). \tag{7}$$

Alternatively, by a suitable change of coordinates we can arrange $x_1 = 1, x_2 = 0, y_1 = 0, y_2 = 1$, which simplifies the relation considerably.

Alternatively, let $a = \sum x_i e_i$ and $b = \sum y_i e_i$ be vectors in $\mathbb{R}^4$ with e_i the standard basis. Then $a \wedge b$ has components which are the 2×2 minors above in the standard basis $e_i \wedge e_j, i < j$ of $\Lambda^2(\mathbb{R}^4)$. We have $(a \wedge b) \wedge (a \wedge b) = 0$ by standard properties of the exterior product, and expanding this out in the standard basis gives the relation above. $\square$

This is also known as the Plücker relation.

Exercise 8.3. *Relate $\lambda(h_1, h_2)$ to Euclidean geometry in the disk model. Prove the hyperbolic Ptolemy relation using the Euclidean one.*

Exercise 8.4. *Relate the cross-ratio of four ideal points to λ-lengths of a corresponding decorated ideal quadrilateral.*

Since a decorated ideal n-gon is determined by its n horocycles, specifying such an n-gon is equivalent to specifying a collection of n points in $\mathbb{R}^2/\{\pm 1\}$ modulo the action of $\mathrm{PSL}_2(\mathbb{R})$, which is very close to specifying a point in the Grassmannian $\mathrm{Gr}_{2,n}$ except that there are some cyclic order and positivity conditions. The positivity condition is equivalent to specifying a $2 \times n$ matrix all of whose minors have positive determinant.

Let's consider more general surfaces than the hyperbolic plane, such as the punctured torus. This surface admits a complete hyperbolic metric of constant curvature -1, and so we can talk about geodesics on it, which go to infinity (the puncture). We can write down three such geodesics giving an ideal triangulation of the torus.

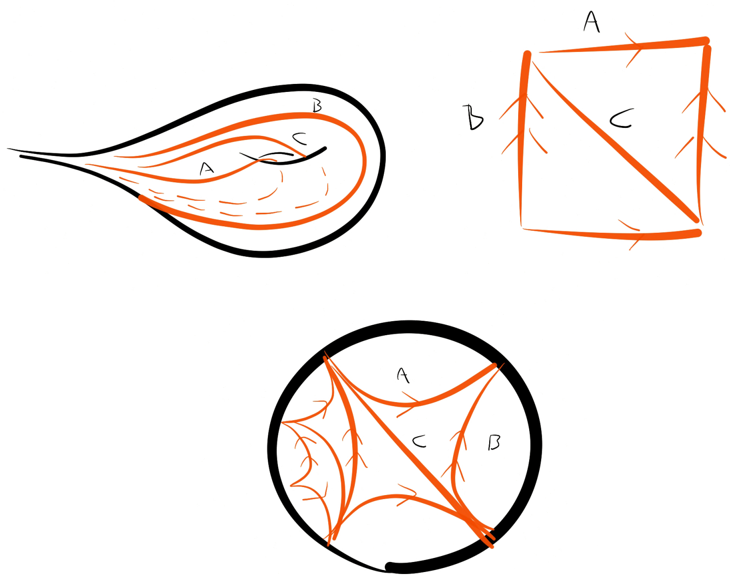

Figure 3: Three views of a punctured torus and three geodesics on it.

We want to measure lengths by decorating using horocycles as before. Here by horocycle we mean a curve which lifts to a horocycle in the universal cover (which is the hyperbolic plane).

Theorem 8.5. *Decorated hyperbolic structures on the punctured torus are parameterized by the three λ-lengths in a triangulation.*

Alternatively, let $T^{g,n}$ be the space of complete hyperbolic structures on a (compact, orientable) surface of genus g with n punctures up to isotopy (*Teichmüller space*), and let $\tilde{T}^{g,n}$ be the space of correspondingly decorated hyperbolic structures, where we also choose horocycles (*decorated Teichmüller space*). Then $\tilde{T}^{1,1}$ is homeomorphic to $\mathbb{R}^3$ with the homeomorphism given by λ-lengths.

There is an interesting relationship between these ideas and number theory. A *Markov triple* is a solution to $x^2 + y^2 + z^2 = 3xyz$. There is an obvious solution $(1, 1, 1)$, and new solutions can be generated from old solutions by permutation or by applying

$$(x, y, z) \mapsto (x, y, \frac{x^2 + y^2}{z}). \quad (8)$$

Consider the Ptolemy relation for a decorated ideal triangulation of a punctured torus. This gives $\lambda(C)\lambda(C') = \lambda(A)^2 + \lambda(B)^2$ where C' is the fourth diagonal, or

$$\lambda(C') = \frac{\lambda(A)^2 + \lambda(B)^2}{\lambda(C)}. \quad (9)$$

What is the relationship? In the special case that $\lambda(A) = \lambda(B) = \lambda(C) = 1$ (equilateral; all of the horocycles touch), the Ptolemy relation allows us to compute other λ-lengths, such as the lengths of various diagonals, and these are precisely the Markov triples we get starting from $(1, 1, 1)$.

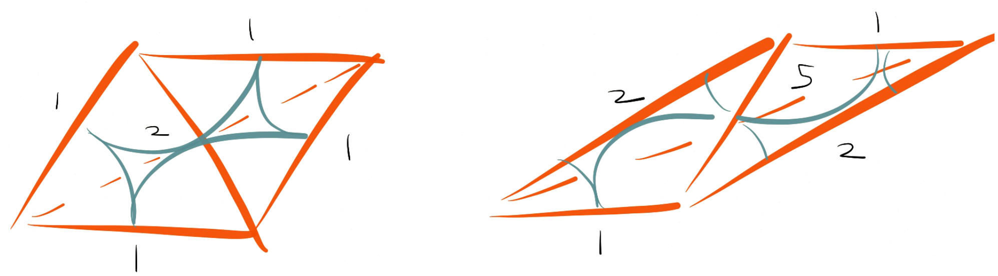

Figure 4: Changing triangulations on an equilateral torus.

A *Markov number* is a number occurring in a Markov triple.

Conjecture 8.6. *For every Markov number n, the Markov triples containing n can be connected by the transformation above without removing n.*

Equivalently, the *simple length spectrum* of the equilateral punctured torus is simple up to symmetries. The length spectrum is the multiset of lengths of closed geodesics. The simple length spectrum is the multiset of lengths of closed simple (non-intersecting) geodesics. A multiset is simple if every element occurs with multiplicity 1.