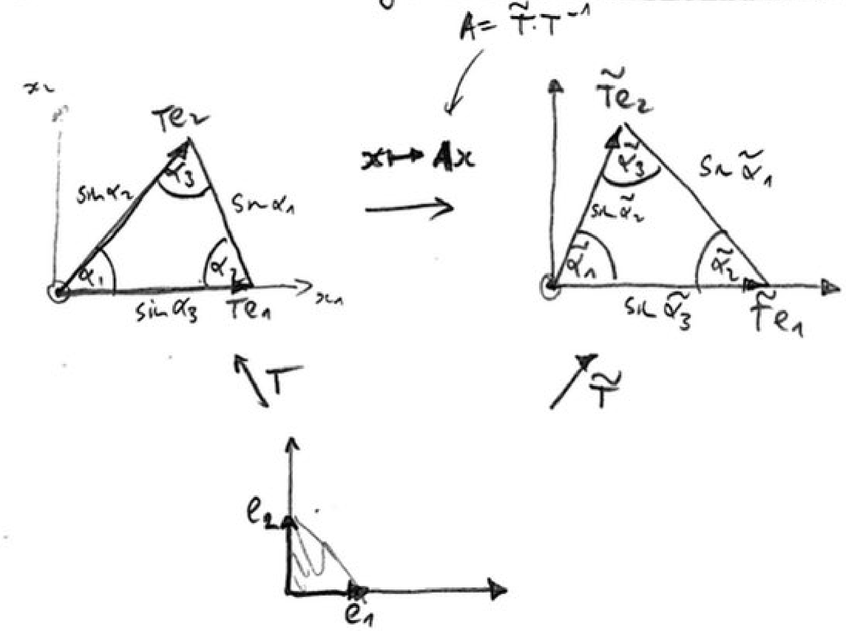
[Distortion of a linear map between triangles]

Quasikonforme Verzerrung einer lin. Abb. zwischen Dreiecken



$$T = \begin{pmatrix} \sin\alpha_3 & \cos\alpha_1 \sin\alpha_2 \\ 0 & \sin\alpha_1 \sin\alpha_2 \end{pmatrix}, \quad \det T = \sin\alpha_1 \sin\alpha_2 \sin\alpha_3$$

$$T^{-1} = \frac{1}{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3} \begin{pmatrix} \sin\alpha_1 \sin\alpha_2 & -\cos\alpha_1 \sin\alpha_2 \\ 0 & \sin\alpha_3 \end{pmatrix}$$

$$A = \tilde{T} T^{-1} = \frac{1}{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3} \begin{pmatrix} \sin\tilde{\alpha}_3 & \cos\tilde{\alpha}_1 \sin\tilde{\alpha}_2 \\ 0 & \sin\tilde{\alpha}_1 \sin\tilde{\alpha}_2 \end{pmatrix} \begin{pmatrix} \sin\alpha_1 \sin\alpha_2 & -\cos\alpha_1 \sin\alpha_2 \\ 0 & \sin\alpha_3 \end{pmatrix}$$

$$= \frac{1}{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3} \begin{pmatrix} \sin\alpha_1 \sin\alpha_2 \sin\tilde{\alpha}_3 & -\cos\alpha_1 \sin\alpha_2 \sin\tilde{\alpha}_3 + \cos\tilde{\alpha}_1 \sin\tilde{\alpha}_2 \sin\alpha_3 \\ 0 & \sin\tilde{\alpha}_1 \sin\tilde{\alpha}_2 \sin\alpha_3 \end{pmatrix}$$

$$\operatorname{Det} A = \frac{\sin\tilde{\alpha}_1 \sin\tilde{\alpha}_2 \sin\tilde{\alpha}_3}{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3}$$

[Let λ_1, λ_2 be the singular values of A, so the eigenvalues of A^tA are λ_1^2, λ_2^2.]

Seien λ_1, λ_2 die Singulärwerte von A. Eigenwerte von A^tA sind λ_1^2, λ_2^2.

$$\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2} = \frac{\operatorname{Tr} A^tA}{\operatorname{Det} A}$$

$$= \frac{\sin^2\alpha_1 \sin^2\alpha_2 \sin^2\tilde\alpha_3 + \sin^2\tilde\alpha_1 \sin^2\tilde\alpha_2 \sin^2\alpha_3 + \begin{pmatrix} -\cos\alpha_1 \sin\alpha_2 \sin\tilde\alpha_3 \\ +\cos\tilde\alpha_1 \sin\tilde\alpha_2 \sin\alpha_3 \end{pmatrix}^2}{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 \sin\tilde\alpha_1 \sin\tilde\alpha_2 \sin\tilde\alpha_3}$$

$$= \frac{\sin^2\alpha_2 \sin^2\tilde\alpha_3 + \sin^2\tilde\alpha_2 \sin^2\alpha_3 - 2\cos\alpha_1 \cos\tilde\alpha_1 \sin\alpha_2 \sin\alpha_3 \sin\tilde\alpha_2 \sin\tilde\alpha_3}{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 \sin\tilde\alpha_1 \sin\tilde\alpha_2 \sin\tilde\alpha_3}$$

$$= \frac{\sin\alpha_2}{\sin\alpha_3 \sin\alpha_1} \cdot \frac{\sin\tilde\alpha_3}{\sin\tilde\alpha_1 \sin\tilde\alpha_2} + \frac{\sin\alpha_3}{\sin\alpha_1 \sin\alpha_2} \cdot \frac{\sin\tilde\alpha_2}{\sin\tilde\alpha_3 \sin\tilde\alpha_1}$$

$$-2 \cot\alpha_1 \cot\tilde\alpha_1$$

$$\boxed{\frac{\sin\alpha_i}{\sin\alpha_j \sin\alpha_k} = \frac{\sin(\alpha_j + \alpha_k)}{\sin\alpha_j \sin\alpha_k} = \cot\alpha_j + \cot\alpha_k}$$

$$= (\cot\alpha_3 + \cot\alpha_1)(\cot\tilde\alpha_1 + \cot\tilde\alpha_2)$$
$$+ (\cot\alpha_1 + \cot\alpha_2)(\cot\tilde\alpha_3 + \cot\tilde\alpha_1)$$
$$-2\cot\alpha_1 \cot\tilde\alpha_1$$

$$= \cot\alpha_1 \cot\tilde\alpha_2 + \cot\alpha_2 \cot\tilde\alpha_3 + \cot\alpha_3 \cot\tilde\alpha_1$$
$$+ \cot\alpha_1 \cot\tilde\alpha_3 + \cot\alpha_2 \cot\tilde\alpha_1 + \cot\alpha_3 \cot\tilde\alpha_2$$

$$\boxed{\frac{1}{2}\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right) = \frac{1}{2} \begin{pmatrix} \cot\alpha_1 \\ \cot\alpha_2 \\ \cot\alpha_3 \end{pmatrix}^t \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cot\tilde\alpha_1 \\ \cot\tilde\alpha_2 \\ \cot\tilde\alpha_3 \end{pmatrix}}$$

wobei λ_1, λ_2 Singulärwerte der lin. Abb. A, die Dreieck mit Winkeln $\alpha_1, \alpha_2, \alpha_3$ auf Dreieck mit Winkeln $\tilde\alpha_1, \tilde\alpha_2, \tilde\alpha_3$ abbildet.

Singulärwerte und Beltrami-Koeffizient

$f(z) = az + b\bar z$

OBdA $a>0, b>0$. [betrachte $e^{i\varphi} f(e^{i\psi} z)$]

$f(x+iy) = (a+b)x + i(a-b)y = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Singulärwerte: $\lambda_1 = a+b > a-b = \lambda_2$, also $a = \frac{1}{2}(\lambda_1 + \lambda_2)$, $b = \frac{1}{2}(\lambda_1 - \lambda_2)$

Beltrami-Koeff hat Betrag $|\mu| = \frac{b}{a} =: d_f$

$|\mu| = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}$, $\quad \frac{\lambda_1}{\lambda_2} = \frac{1+|\mu|}{1-|\mu|} =: D_f$

Ahlfors, Lectures on Quasiconf. Maps

f heißt K-quasikonform, wenn $\frac{\lambda_1}{\lambda_2} \le K$.

$\iff \quad |\mu| \le k = \frac{K-1}{K+1}$

$$\frac{1}{2}\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right) = \frac{1+|\mu|^2}{1-|\mu|^2}, \quad |\mu|^2 = \frac{\frac{1}{2}\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right) - 1}{\frac{1}{2}\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right) + 1}$$