Discrete Torsion of Connection Forms on Simplicial Meshes

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Connections on manifolds play a key role in the modern theory of differential geometry. It is thus hardly surprising that discrete connections are used throughout geometry processing from the design of vector fields and stripe patterns on surfaces [Crane et al. 2010; Knöppel et al. 2015; Liu et al. 2016], to vectorization of 2D sketches [Gutan et al. 2023], and design tools for several forms of fabrication [Montes Maestre et al. 2023; Mitra et al. 2023, 2024]. In the smooth setting, connections on surfaces can be studied using Cartan's method of moving frames, where they are characterized by two quantities: their scalar-valued curvature, and their vector-valued torsion. However, existing work on connections focuses almost exclusively on curvature, neglecting torsion entirely.

This work enriches the geometry processing toolbox by introducing a discretization of torsion on triangle meshes and demonstrating its relevance to vector and frame field design applications.

Background: Smooth Connections

While the directional derivative $D_X f$ of a scalar function f simply measures the rate of change of f as one moves along a tangential direction X, directional derivatives of vector fields on surfaces are more involved: vectors at different points on the surface belong to different tangent spaces, and a priori cannot be compared directly. A connection is a choice of operator $\nabla_X V$ acting as a directional derivative of a tangent vector field V in direction X; unlike the case of scalar functions, there are infinitely many meaningful connections.

If we express our vector field $V(x)$ using its coefficient functions $c^1(x)$ and $c^2(x)$, then any connection can be written in the form

$$\nabla_X \begin{pmatrix} c^1(x) \\ c^2(x) \end{pmatrix} = \begin{pmatrix} D_X c^1(x) \\ D_X c^2(x) \end{pmatrix} + \begin{pmatrix} \omega_1^1(X) & \omega_2^1(X) \\ \omega_1^2(X) & \omega_2^2(X) \end{pmatrix} \begin{pmatrix} c^1(x) \\ c^2(x) \end{pmatrix},$$

where the ω_i^j terms are *1-forms*. Different choices of 1-forms ω_i^j yield different connections. This equation is often written more concisely as

$$\nabla = d + \omega$$

where d is the componentwise exterior derivative, and ω is understood as a matrix-valued 1-form, known as the *connection 1-form*.

A connection is said to be *metric* if parallel transport preserves the lengths of vectors, which happens if and only if the connection 1-form is skew-symmetric, *i.e.,* of the form

$$\omega(X) = \begin{pmatrix} 0 & -\alpha(X) \\ \alpha(X) & 0 \end{pmatrix},$$

for some scalar-valued 1-form α.

Curvature and torsion. One can picture parallel transport as a description of the motion of a surface as it locally rolls around on the tangent plane of some point. Curvature and torsion capture two key features of this rolling behavior: curvature measures how much the surface rotates when rolled along an infinitesimal loop, while torsion measures how far it translates.
Both curvature and torsion have simple expressions in the language of differential forms. The curvature is the matrix-valued 2-form $\Omega^\nabla = d\omega + \omega \wedge \omega$. For metric connections on surfaces, $\omega \wedge \omega = 0$, so the curvature is just

$$\Omega^\nabla = d\omega$$

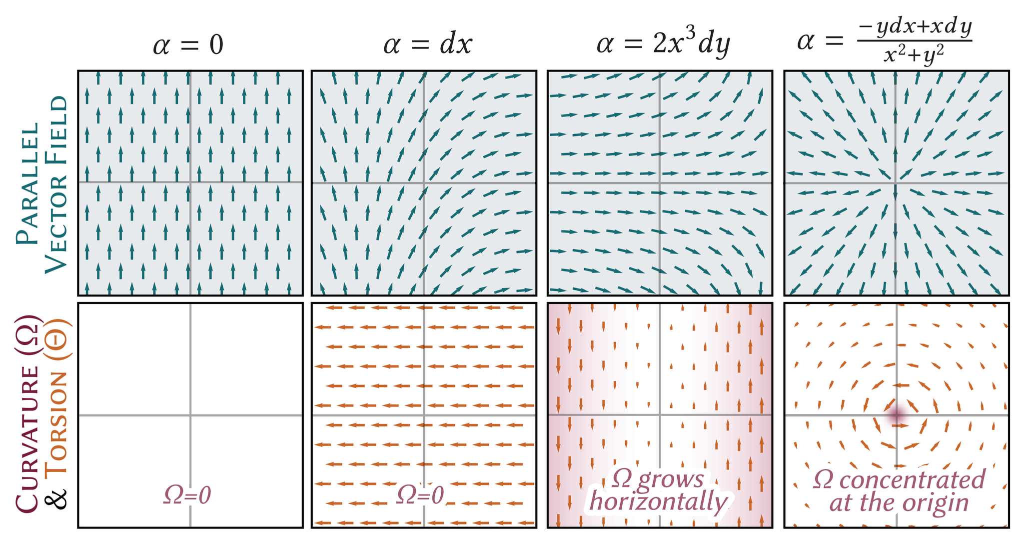
The torsion of our connection is the vector-valued 2-form Θ^∇, with

$$\Theta^\nabla = d\theta + \omega \wedge \theta.$$

Here θ is the dual frame associated to our basis vector fields.

However, we found that an alternative formulation of torsion on surfaces is more amenable to discretization. One can show that the torsion of a connection α is

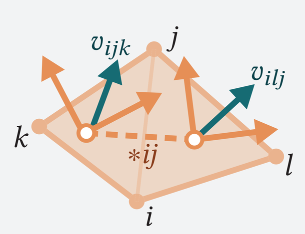
$$\Theta^\nabla = (\alpha - \alpha^{LC})^\sharp \, dA$$

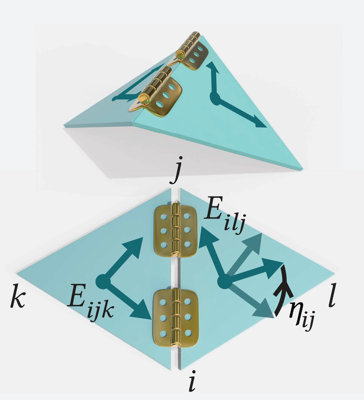


Metric Connections. Even in the plane equipped with the standard metric and standard basis, parallel transport of vectors by a metric connection may look very different from a simple translation. Here we consider four connection 1-forms α, and plot parallel vector fields for each connection *(top row)*, as well as the curvature *(bottom row, scalar density)* and torsion *(bottom row, vector field)* of each connection. All four connections are compatible with the standard metric—as their parallel vector fields are unit—but the rotations and singularities in the parallel-transported vector fields can become quite complicated.

Background: Discrete Connections

A discrete metric connection provides a rotation angle $\alpha_{\vec{ij}}$ on each halfedge describing how the coordinates of a vector v_{ilj} written in the frame of the right- hand face should be transformed to obtain a parallel-transported vector v_{ijk} in the frame of the left-hand face.





A canonical connection on triangle meshes, that was leveraged by Crane et al. [2010] and which we call the hinge connection η, is found by unfolding the hinge between two faces and measuring the angle η_{ij} between the chosen bases E_{ijk} and E_{ijl} on these two faces. The hinge connection is the Levi-Civita connection of the mesh, viewed as a polyhedral surface, but we introduce a new discrete Levi-Civita connection which offers lower torsion (seen as the deviation from the Levi-Civita connection of an underlying smooth surface)

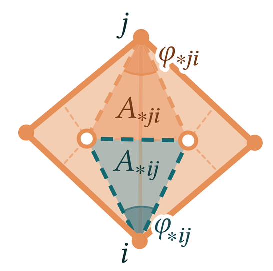
A New Discrete Levi-Civita Connection

The hinge map is the exact Levi-Civita connection of a triangle mesh, viewed as a polyhedral surface. However, this singular geometry differs in many ways from the geometry of the smooth surface which the mesh is supposed to approximate.

We instead introduce a new discrete connectionderived by spreading the curvature to be constant on each 1-ring instead. We find that this constant-curvature local approximation of the surface yields a connection with consistently lower error than the hinge map compared to the continuous Levi- Civita connection without much added complexity. Our new connection can be expressed in closed form as an offset from the hinge map:

$$\alpha_{ij}^{LC} = \eta_{ij} + \frac{1}{\lozenge_{ij}} \left(K_i A_{*ij} \left(\frac{A_{*ij}}{A_i} - \frac{\varphi_{*ij}}{\Phi_i} \right) - K_j A_{*ji} \left(\frac{A_{*ji}}{A_j} - \frac{\varphi_{*ji}}{\Phi_j} \right) \right),$$

using the local geometric quantities illustrated to the side.



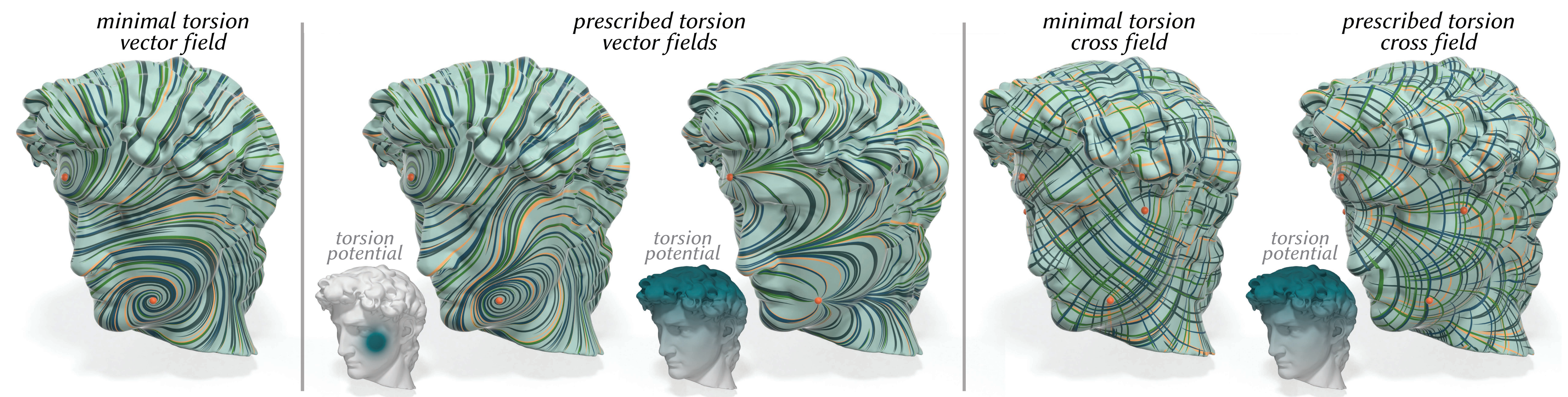
Discrete Torsion

In the continuous case, the torsion of a connection measures its deviation from the Levi-Civita connection. We thus define the discrete torsion of a discrete connection as the dual 1-form

$$\mathrm{Torsion}(\alpha) := \alpha - \alpha^{LC}.$$

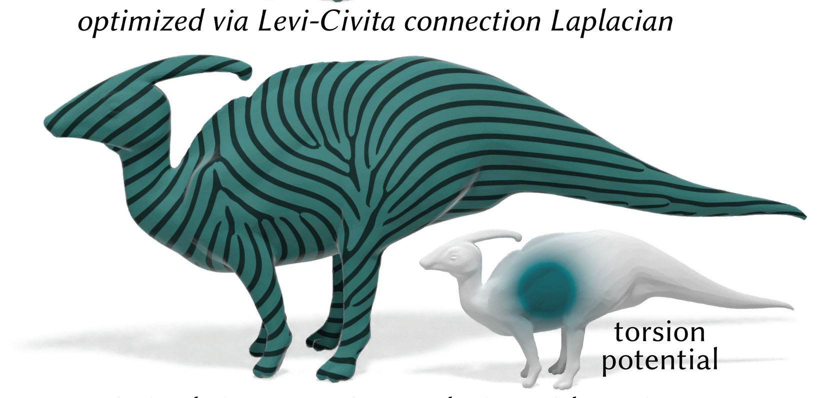
This definition exactly parallels the continuous definition and provides a simple and straightforward way not only to define torsion, but also to offer torsion processing on connections through Hodge decomposition [Crane et al. 2013] of this dual 1-form by controlling its curl, divergence, and harmonic components

Examples



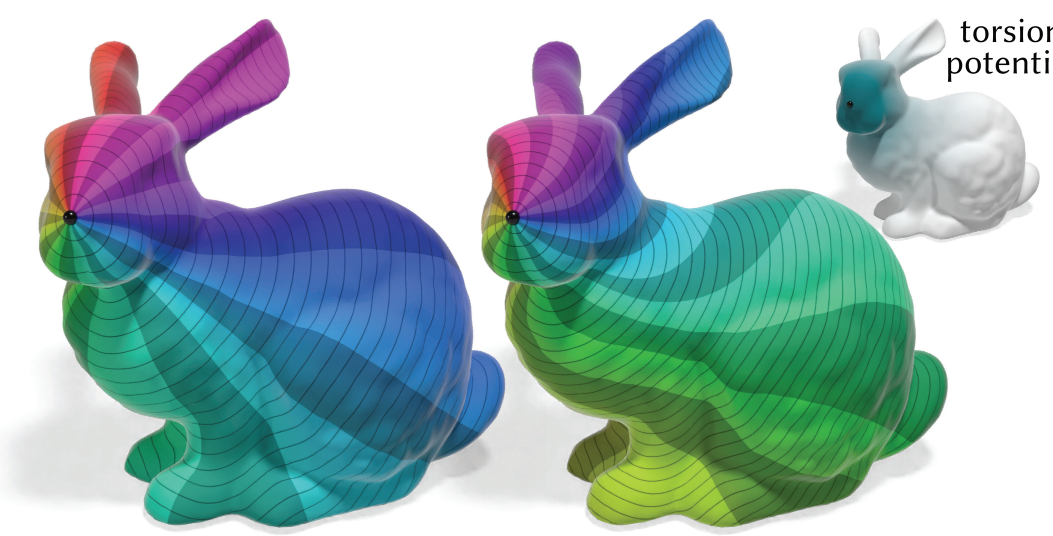
Torsion-controlled trivial connections. Here we explore the impact of torsion on vector field design: a parallel vector field for a minimal-torsion trivial connection with prescribed singularities is shown on the left, while parallel fields for trivial connections with additional gradient components in their torsion are shown in the center, along with the prescribed scalar potentials. A compact potential changes the field locally (center left), while a globally-supported potential causes larger-scale changes (center right). Torsion can also be incorporated into the design of n-vector fields such as cross fields (right).

optimized via Levi-Civita connection Laplacian



optimized via connection Laplacian with torsion

Optimal vector fields for different connections. Just as the ordinary Laplacian measures the smoothness of scalar functions, the connection Laplacian defines a notion of smoothness for vector fields. Computing an optimal vector field in the sense of Knöppel et al. [2013] using the discrete Levi-Civita connection to build the connection Laplacian yields a field whose streamlines are as straight as possible in 3D space (top), whereas using a connection with a nontrivial torsion (here, the gradient of a potential) introduces twists into the streamlines (bottom).



torsion-free connection Laplacian *connection Laplacian with torsion*

Torsion-controlled vector heat method. The connection Laplacian used in the vector heat method of Sharp et al. [2019] can be formed for any connection. In this example, we compute a logarithmic map—a sort of global polar parameterization—using the connection Laplacian of an ordinary torsion-free connection (left), and using a connection with prescribed non-zero torsion equal to the gradient of a torsion potential (right), which introduces a distinct twist into the radial lines of the parameterization.

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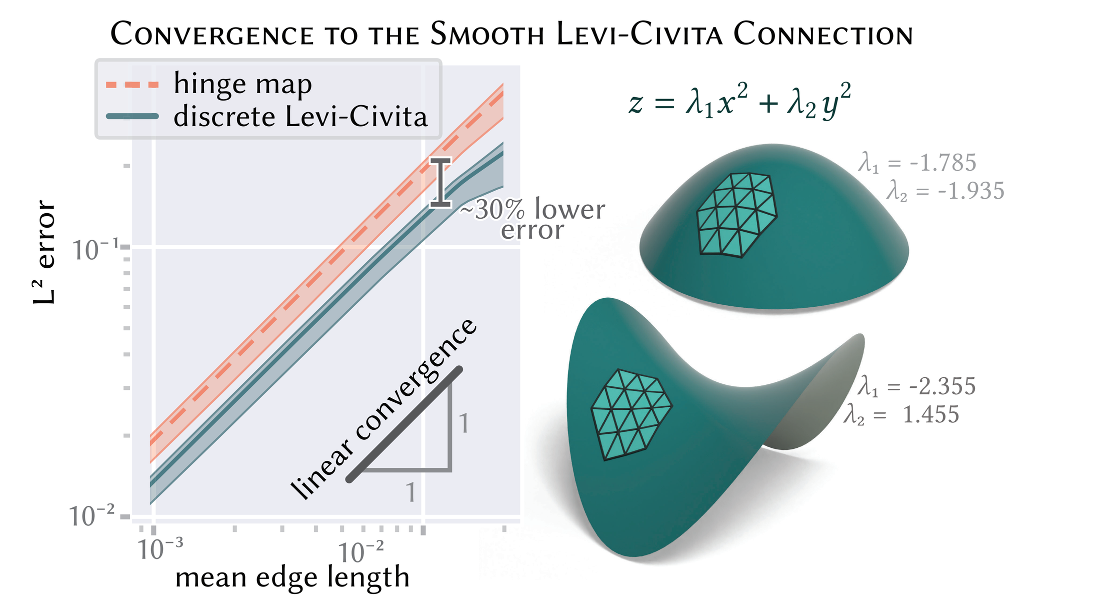
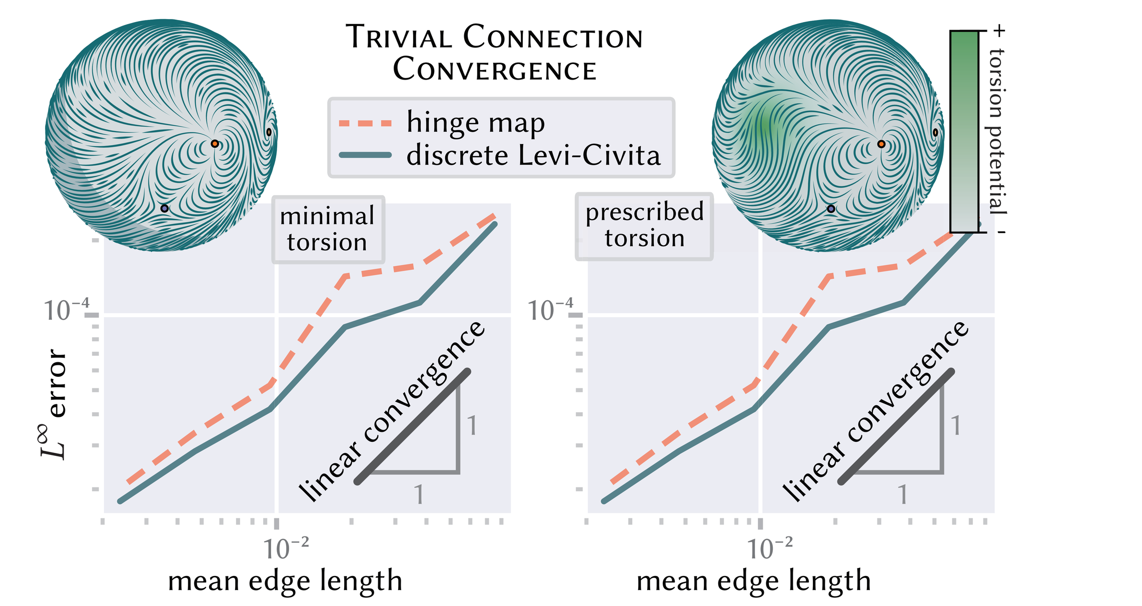
Convergence Tests



Here we plot the L2 errors of the hinge map vs. our discrete Levi-Civita connection compared to the smooth Levi-Civita connection on several hundred randomly generated quadratic surfaces. The plotted lines show the average error across all experiments, while shaded en- velopes show the maximum and minimum errors. Both discrete connections converge at the same linear rate, but our discrete Levi-Civita connection offers lower error in every case. Two sample surfaces are shown on the right



Here we plot the L∞ error of a discrete trivial connections compared to an analytic reference solution with minimal torsion (left) and with prescribed torsion (right). Since the solutions blow up around singularities, we measure the error on a patch of the sphere away from the singularities.

Spherical Trivial Connections

The smooth formulation of torsion which we use allows us to derive closed form expressions for trivial connections with prescribed singularities on smooth surfaces. In the appendix we derive an analytic expression for minimal-torsion trivial connections on the sphere. Our closed-form expressions can even be used to render these vector fields directly in a shader.

https://www.shadertoy.com/view/M3VBRz