Ray Tracing Harmonic Functions
Supplemental Material

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1 EXPRESSING SPHERICAL HARMONICS AS POLYNOMIALS

Abstractly, spherical harmonics may be characterized as the Laplacian eigenfunctions of the sphere. However, the term "spherical harmonics" is often used to refer to a particular choice of eigenfunctions, expressed in spherical coordinates as

$$Y_\ell^m(\theta, \varphi) = N e^{im\varphi} P_\ell^m(\cos\theta). \tag{1}$$

Here N is a normalization constant and P_ℓ^m denotes the *associated Legendre function* of degree ℓ and order m. In this section, we show how to express these functions as the restrictions of harmonic polynomials of degree ℓ to the unit sphere. Converting to Cartesian coordinates, we find that

$$\begin{aligned} Y_\ell^m(x,y,z) &= N\left(\frac{x+iy}{\|x+iy\|}\right)^m P_\ell^m(z) \\ &= N(x+iy)^m \cdot \frac{P_\ell^m(z)}{(1-z^2)^{m/2}}, \end{aligned} \tag{2}$$

where we have used the fact that $x^2 + y^2 = 1 - z^2$ for points on the unit sphere. Using the definition of the associated Legendre functions, the rightmost term is a polynomial $q(z)$ of degree $\ell - m$, leaving us with

$$Y_\ell^m(x,y,z) = N(x+iy)^m \cdot q(z). \tag{3}$$

This polynomial is not homogenous, as q is not homogeneous. But we can homogenize it by considering instead the polynomial $\tilde{q}(x,y,z,\lambda) := \lambda^{\ell-m} q(z/\lambda)$. Setting $\lambda = \sqrt{x^2+y^2+z^2}$, we get a homogeneous polynomial defined on all of $\mathbb{R}^3$ which agrees with Y_ℓ^m on the surface of the unit sphere (where $\lambda = 1$).

To ensure that our spherical harmonics have unit L^2-norm over the sphere, we use the following normalization [Arfken 1985, p. 681]

$$N = \sqrt{\frac{2\ell+1}{2\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}. \tag{4}$$

The first several normalized spherical harmonics are given by:

$$\begin{aligned}
Y_1^{-1}(x,y,z) &= \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\, y \\
Y_1^{0}(x,y,z) &= \sqrt{\tfrac{3}{2\pi}}\, z \\
Y_1^{1}(x,y,z) &= \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\, x \\
Y_2^{-2}(x,y,z) &= \tfrac{1}{2}\sqrt{\tfrac{15}{\pi}}\, xy \\
Y_2^{-1}(x,y,z) &= \tfrac{1}{2}\sqrt{\tfrac{15}{\pi}}\, yz \\
Y_2^{0}(x,y,z) &= \tfrac{1}{2}\sqrt{\tfrac{5}{2\pi}}\left(-x^2 - y^2 + 2z^2\right) \\
Y_2^{1}(x,y,z) &= \tfrac{1}{2}\sqrt{\tfrac{15}{\pi}}\, xz \\
Y_2^{2}(x,y,z) &= \tfrac{1}{4}\sqrt{\tfrac{15}{\pi}}\left(x^2 - y^2\right) \\
Y_3^{-3}(x,y,z) &= \tfrac{1}{4}\sqrt{\tfrac{35}{2\pi}}\, y\left(y^2 - 3x^2\right) \\
Y_3^{-2}(x,y,z) &= \tfrac{1}{2}\sqrt{\tfrac{105}{\pi}}\, xyz \\
Y_3^{-1}(x,y,z) &= \tfrac{1}{4}\sqrt{\tfrac{21}{2\pi}}\, y\left(x^2 + y^2 - 4z^2\right) \\
Y_3^{0}(x,y,z) &= \tfrac{1}{2}\sqrt{\tfrac{7}{2\pi}}\left(-3x^2z - 3y^2z + 2z^3\right) \\
Y_3^{1}(x,y,z) &= \tfrac{1}{4}\sqrt{\tfrac{21}{2\pi}}\, x\left(x^2 + y^2 - 4z^2\right) \\
Y_3^{2}(x,y,z) &= \tfrac{1}{4}\sqrt{\tfrac{105}{\pi}}\, z\left(x^2 - y^2\right) \\
Y_3^{3}(x,y,z) &= \tfrac{1}{4}\sqrt{\tfrac{35}{2\pi}}\left(x^3 - 3xy^2\right) \\
Y_4^{-4}(x,y,z) &= \tfrac{3}{4}\sqrt{\tfrac{35}{\pi}}\, xy\left(x^2 - y^2\right) \\
Y_4^{-3}(x,y,z) &= \tfrac{3}{4}\sqrt{\tfrac{35}{2\pi}}\, z\left(3x^2y - y^3\right) \\
Y_4^{-2}(x,y,z) &= \tfrac{3}{4}\sqrt{\tfrac{5}{\pi}}\, xy\left(x^2 + y^2 - 6z^2\right) \\
Y_4^{-1}(x,y,z) &= \tfrac{3}{4}\sqrt{\tfrac{5}{2\pi}}\, yz\left(3x^2 + 3y^2 - 4z^2\right) \\
Y_4^{0}(x,y,z) &= \tfrac{3}{8\sqrt{2\pi}}\left(3x^4 + 6x^2\left(y^2 - 4z^2\right) + 3y^4 - 24y^2z^2 + 8z^4\right) \\
Y_4^{1}(x,y,z) &= \tfrac{3}{4}\sqrt{\tfrac{5}{2\pi}}\, xz\left(3x^2 + 3y^2 - 4z^2\right) \\
Y_4^{2}(x,y,z) &= \tfrac{3}{8}\sqrt{\tfrac{5}{\pi}}\left(x^2 - y^2\right)\left(x^2 + y^2 - 6z^2\right) \\
Y_4^{3}(x,y,z) &= \tfrac{3}{4}\sqrt{\tfrac{35}{2\pi}}\, z\left(x^3 - 3xy^2\right) \\
Y_4^{4}(x,y,z) &= \tfrac{3}{16}\sqrt{\tfrac{35}{\pi}}\left(x^4 - 6x^2y^2 + y^4\right)
\end{aligned}$$

Since each polynomial $Y_\ell^m(x,y,z)$ is homogeneous of degree ℓ, its minimum value over a ball of radius h is simply h^ℓ times its minimum value over the unit ball. The minimum values over the unit ball are given by

$$\begin{aligned}
Y_1^{-1}(x,y,z) &\geq 0.488603 \\
Y_1^{0}(x,y,z) &\geq 0.690988 \\
Y_1^{1}(x,y,z) &\geq 0.488603 \\
Y_2^{-2}(x,y,z) &\geq 0.546274 \\
Y_2^{-1}(x,y,z) &\geq 0.546274 \\
Y_2^{0}(x,y,z) &\geq 0.446031 \\
Y_2^{1}(x,y,z) &\geq 0.546274 \\
Y_2^{2}(x,y,z) &\geq 0.546274 \\
Y_3^{-3}(x,y,z) &\geq 0.590044 \\
Y_3^{-2}(x,y,z) &\geq 0.556298 \\
Y_3^{-1}(x,y,z) &\geq 0.62938 \\
Y_3^{0}(x,y,z) &\geq 1.0555 \\
Y_3^{1}(x,y,z) &\geq 0.62938 \\
Y_3^{2}(x,y,z) &\geq 0.556298 \\
Y_3^{3}(x,y,z) &\geq 0.590044 \\
Y_4^{-4}(x,y,z) &\geq 0.625836 \\
Y_4^{-3}(x,y,z) &\geq 0.574867 \\
Y_4^{-2}(x,y,z) &\geq 0.608255 \\
Y_4^{-1}(x,y,z) &\geq 0.706531 \\
Y_4^{0}(x,y,z) &\geq 0.512926 \\
Y_4^{1}(x,y,z) &\geq 0.706531 \\
Y_4^{2}(x,y,z) &\geq 0.608255 \\
Y_4^{3}(x,y,z) &\geq 0.574867 \\
Y_4^{4}(x,y,z) &\geq 0.625836
\end{aligned}$$

2 ALTERNATIVE FORMULAS FOR SOLID ANGLE

In this section, we describe the two alternative methods for evaluating the solid angle function associated to a nonplanar polygon P which were mentioned in Section 4.2.2 of the main text. As before, we denote the vertices of our polygon P by $\mathbf{p}_1, \ldots, \mathbf{p}_k \in \mathbb{R}^3$. Given a unit sphere $S(\mathbf{x})$ centered around a point $\mathbf{x}$, we will denote the projection of $\mathbf{p}_i$ onto the sphere by $\mathbf{q}_i := (\mathbf{p}_i - \mathbf{x})/|\mathbf{p}_i - \mathbf{x}|$, and will denote the resulting spherical polygon by Q. Throughout we use $\operatorname{atan2}(y, x)$ to denote the two-argument arc tangent function, which yields values in the range $[-\pi, \pi)$; it is especially important when defining angle-valued functions to work in this full range (rather than the range $[-\pi/2, \pi/2]$ of the ordinary arc tangent function).

2.1 Quaternionic Formula

Chern and Ishida [2023, Cor. 3.4.1] introduced a formula to calculate the solid angle of a polygon P using quaternions:

$$\Omega_P(\mathbf{x}) := -2 \arg\left(\operatorname{Rot}\big(\mathbf{e}_1, \mathbf{p}_1 - \mathbf{x}\big), \prod_{i=1}^{k} \operatorname{Rot}\big(\mathbf{p}_i - \mathbf{x}, \mathbf{p}_{i+1} - \mathbf{x}\big)\right). \quad (5)$$

Here $\mathbf{e}_1$ is the basis vector $(1, 0, 0)$, $\operatorname{Rot}(\mathbf{v}, \mathbf{w})$ is the (non-normalized) quaternion encoding the shortest rotation from vector $\mathbf{v}$ to vector $\mathbf{w}$

$$\operatorname{Rot}(\mathbf{v}, \mathbf{w}) := \big(\mathbf{v} \cdot \mathbf{w} + \|\mathbf{v}\|\|\mathbf{w}\|, \mathbf{v} \times \mathbf{w}\big), \quad (6)$$

(see Chern and Ishida [2023, Def. 3.1] or Thomson [2015]), and $\arg(a, b)$ is defined for two quaternions a, b to be the angle

$$\arg(a, b) := \operatorname{atan2}\Big(\operatorname{Im}\left[\bar{b}a\right]_0, \operatorname{Re}\left[\bar{b}a\right]\Big), \quad (7)$$

which gives the angle from the origin to the quaternion $\bar{b}a$ in the plane spanned by the real axis and the first imaginary axis (see Chern and Ishida [2023, Sec. 3.2]). In Equation 7, we use $\operatorname{Im}[q]_0$ to denote the first imaginary component of a quaternion q—explicitly, $\operatorname{Im}[a + bi + cj + dk]_0 = b$. In our experiments, we did not observe a significant improvement in accuracy compared to the triangulation scheme.

2.2 Angle Sum

There is also a classic formula for the signed area of Q which uses the corner angle sum [Legendre 1817, §505; Lee 2018, Proof 9.3]. If we let κ_i denote the exterior turning angle of Q at vertex i, and let τ denote the turning number of Q on S^2, then the area of Q is given by

$$\operatorname{area}(Q) = 2\pi\tau - \sum_{i=1}^{k} \kappa_i. \quad (8)$$

One can compute τ as the planar turning number of Q in any chart [Lee 2018, Proof 9.2], and one can compute the angles κ_i as

$$\kappa_i = \operatorname{atan2}(\mathbf{q}_i \cdot (\mathbf{n}_{i-1/2} \times \mathbf{n}_{i+1/2}), \mathbf{n}_{i-1/2} \cdot \mathbf{n}_{i+1/2}),$$

where $\mathbf{n}_{i+1/2} := (\mathbf{p}_i - \mathbf{x}) \times (\mathbf{p}_{i+1} - \mathbf{x})$ are the (unnormalized) vectors orthogonal to the edge of Q between vertices i and $i+1$, and $\mathbf{q}_i$ is the projection of $\mathbf{p}_i$ onto the unit sphere centered at $\mathbf{x}$. Note, however, that when the evaluation point $\mathbf{x}$ is anywhere on the line through $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$, the normal vector $\mathbf{n}_{i+1/2}$ is equal to zero, hence the angle $\kappa_i = \operatorname{atan2}(0, 0)$ is not well-defined (and likewise for $\mathbf{p}_{i-1}, \mathbf{p}_i$). Although the singularities arising from each corner should cancel out in exact arithmetic, this function is numerically ill-behaved in floating point. Moreover, as discussed by Chern and Ishida [2023], the normal vectors $\mathbf{n}$ are not well-defined for zero-length edges, and hence exhibit poor numerical behavior for very short edges.

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