# 2D Plasma Simulation via Discrete Exterior Calculus

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Caltech

# Outline

1. Background
2. Goals
3. My Contributions

# Background

# Motivation

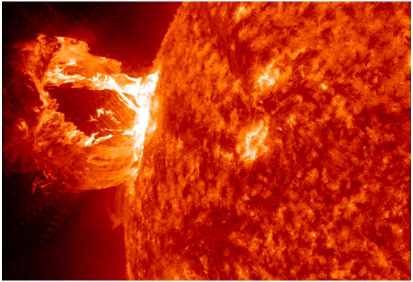

**Figure 1:** Coronal mass ejection photographed by NASA [1]

# Ordinary Fluids

Simplifying assumptions:

- Incompressible

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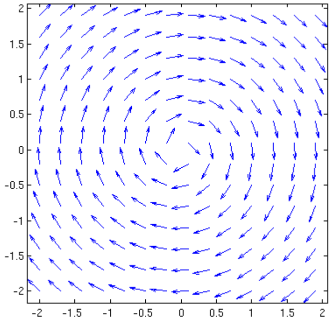
# Ordinary Fluids

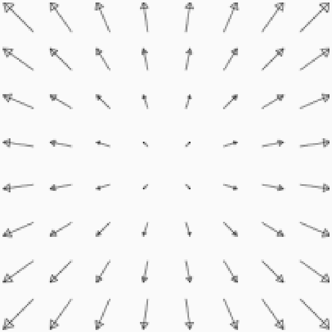
We represent the state of the fluid by a velocity vector field $v$.



# Incompressibility

Divergence-free vector field ($\delta v = 0$).

No regions like this:



# How does the fluid move over time?

Little particles in the fluid move with the flow. This drags our velocties along the velocity field.

$$\frac{\partial v}{\partial t} = -\mathcal{L}_{v^\sharp} v$$

But this dragging can violate incompressibility. So we introduce pressure.

$$\frac{\partial v}{\partial t} = -\mathcal{L}_{v^\sharp} v - dp$$

# Euler Equation for incompressible fluids

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Pressure to maintain incompressibility

# Euler Equation for incompressible fluids

$$\frac{\partial v}{\partial t} + \mathcal{L}_{v^\sharp} v + dp = 0$$
$$\delta v = 0$$

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- Fluid has no resistance

# Ideal MHD

Velocity equation: very similar to Euler equation

$$\frac{\partial v}{\partial t} + \mathcal{L}_{v^\sharp} v - \mathcal{L}_{(\star\beta)^\sharp}(\star\beta) + dp = 0$$

Force of magnetic field on charged particles (Lorentz force law)

# Ideal MHD

Magnetic field equation: just carried by velocity field

$$\frac{\partial \beta}{\partial t} + \mathcal{L}_{v^\sharp} \beta = 0$$

Change in magnetic field over time

Fluid pulling along magnetic field

# Ideal MHD Equations

Four equations total

$$\frac{\partial v}{\partial t} + \mathcal{L}_{v^\sharp} v - \mathcal{L}_{(\star\beta)^\sharp}(\star\beta) + dp = 0 \tag{1}$$

$$\frac{\partial \beta}{\partial t} + \mathcal{L}_{v^\sharp} \beta = 0 \tag{2}$$

$$\delta v = 0 \tag{3}$$

$$d\beta = 0 \tag{4}$$

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- Cross-helicity ($H$)
  - A measure of how linked $v$ and $\beta$ are

# Goals

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  - Treats $v$ and $\beta$ on equal footing
  - Good numerical behavior and solid justification
  - No proof of conservation laws
  - Only worked out in periodic domains

# My Goals

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3. Extend to 3D

# My Contributions

# Proof of Conservation Laws

I proved conservation of energy and cross-helicity, but not in way we expected.

Noether's theorem only predicts that energy changes at a constant rate in this case.

Further investigation is needed to understand why the standard technique didn't work, and what that means about this representation of the system.

# Implementation of Boundaries

The new perspective that I developed on the algorithm while proving the conservation laws made it simple to implement fixed boundary conditions.

# Implementation of Boundaries

# Experimental Test of Conservation (Energy)

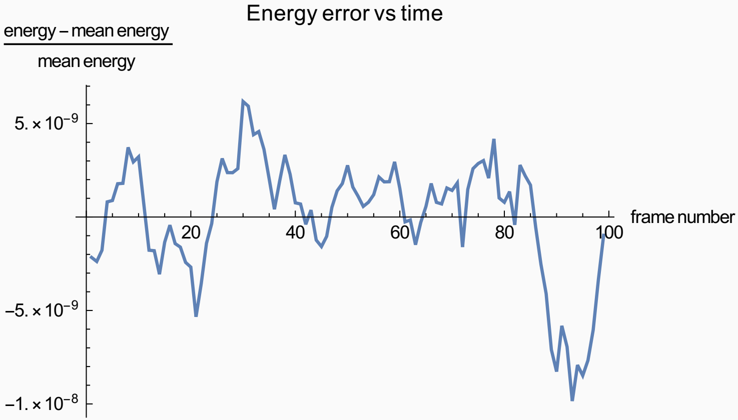


**Figure 2:** Energy drift of the Alfvén wave simulation

# Experimental Test of Conservation (Cross-helicity)

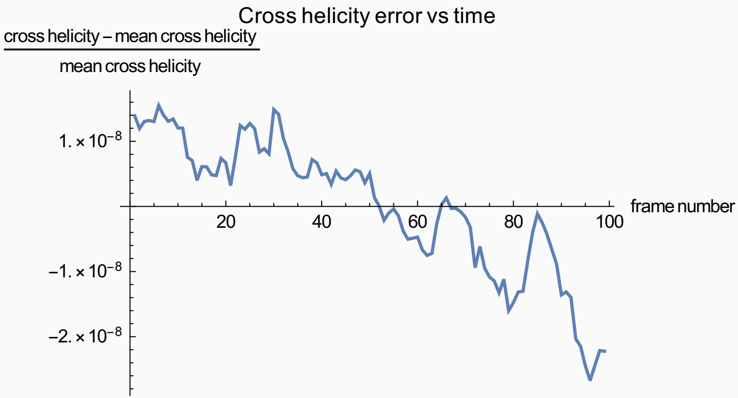


**Figure 3:** Cross-helicity drift of the Alfvén wave simulation

# Extension to 3D

I did not have time to work on the algorithm in 3D. Conceptually, it should not be very different from 2D. But there may be subtle differences that prove tricky to handle.

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# Acknowledgements

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