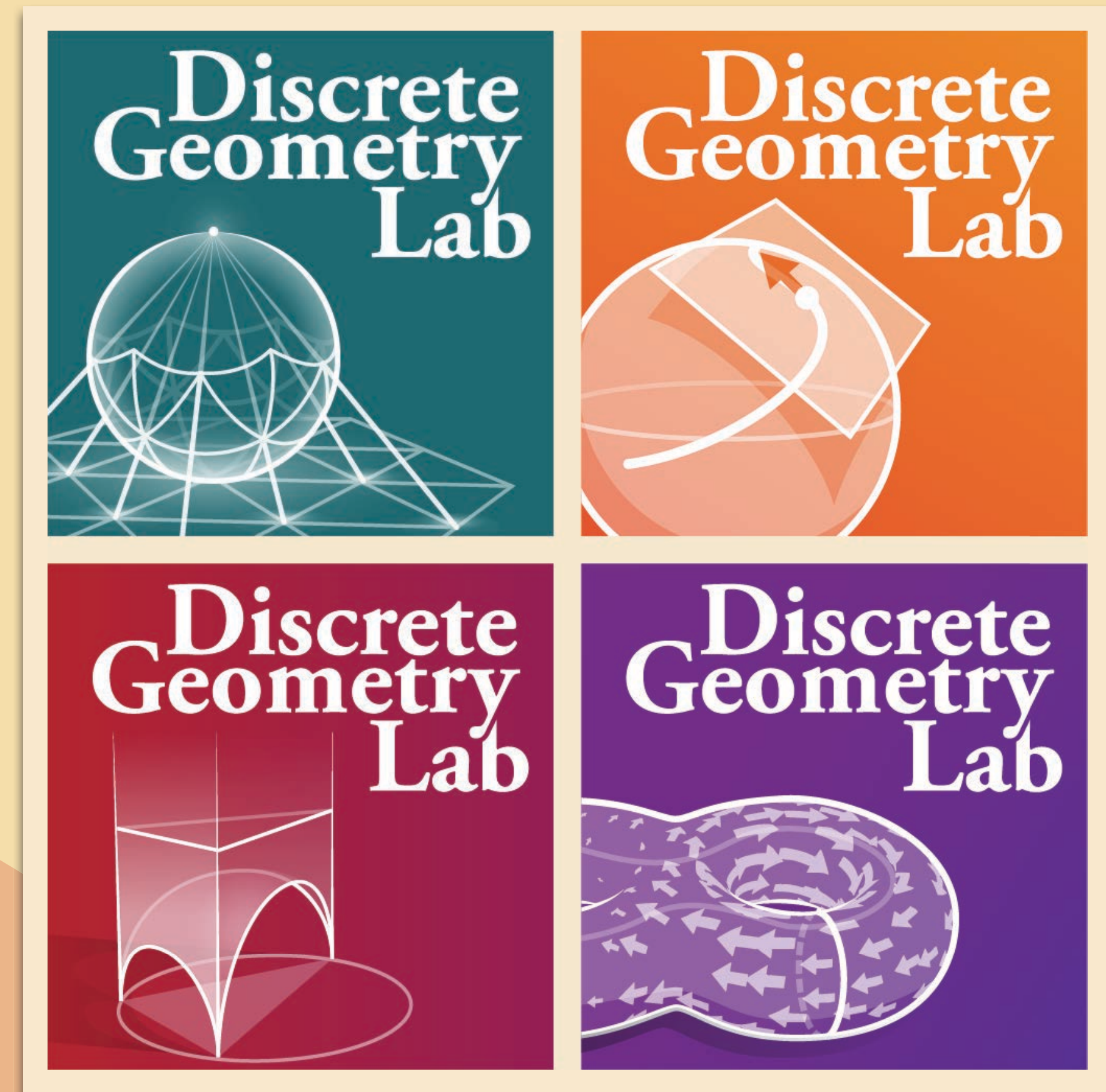


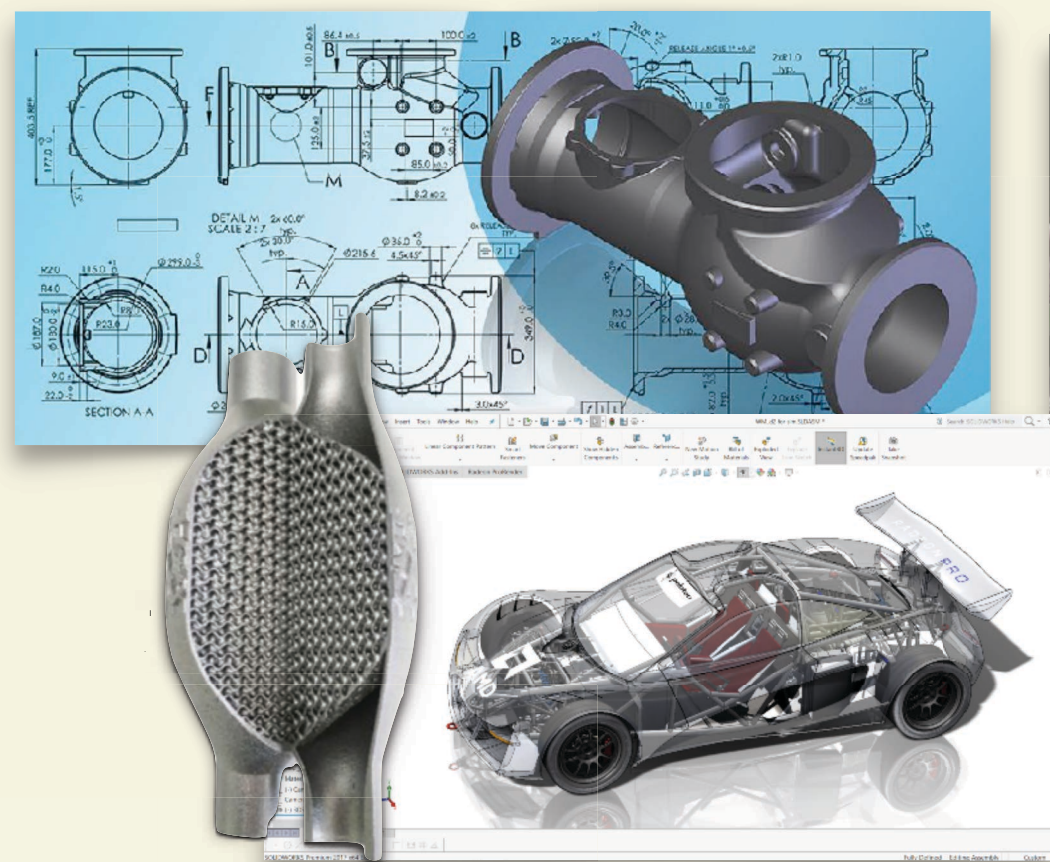
Open Problems in Geometry Processing

Mark Gillespie

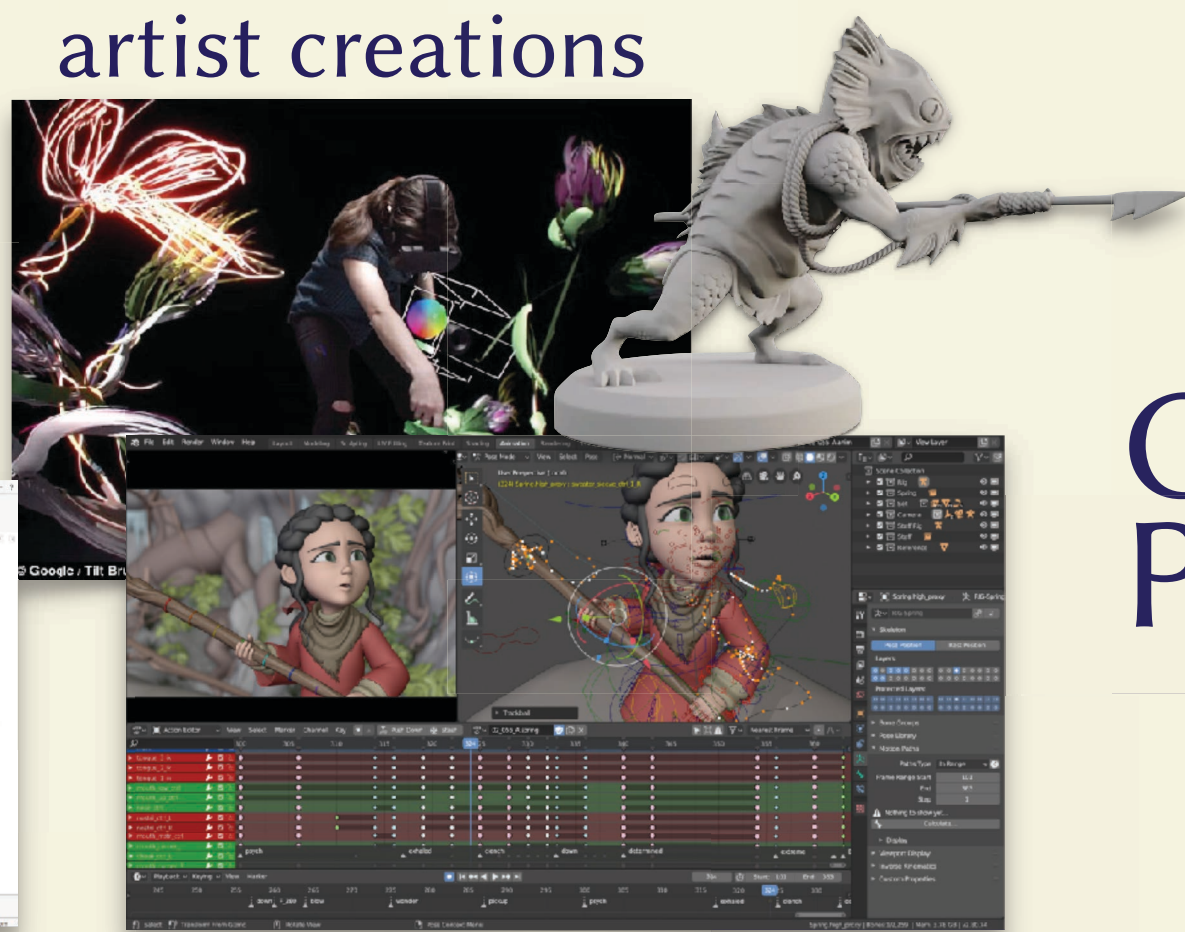


Geometry is all around us

technical assets



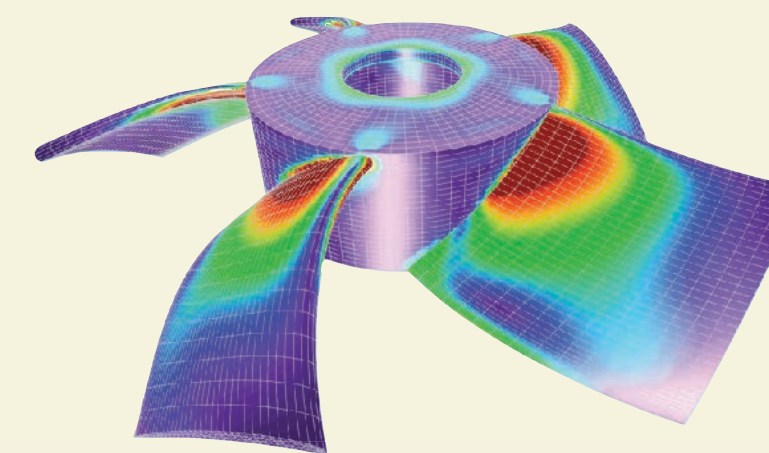
artist creations



Geometry Processing

design algorithms to be:

- simple
- efficient
- robust



simulation



robotics



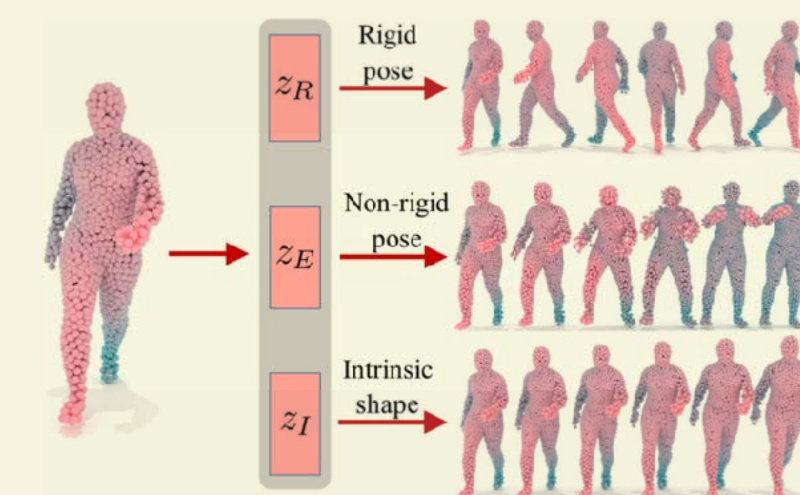
creative tools



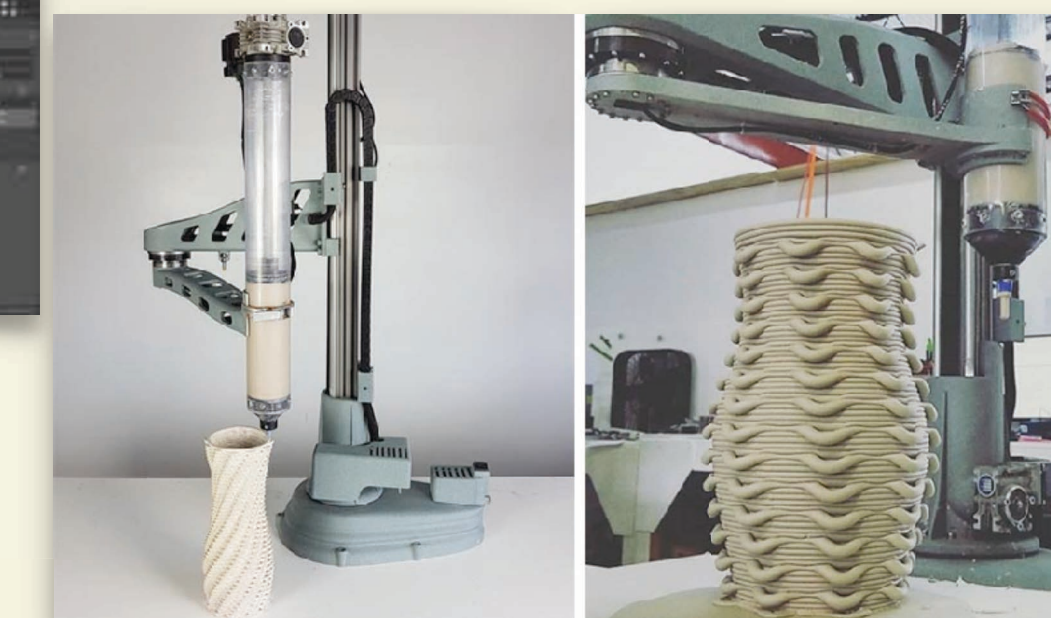
animation



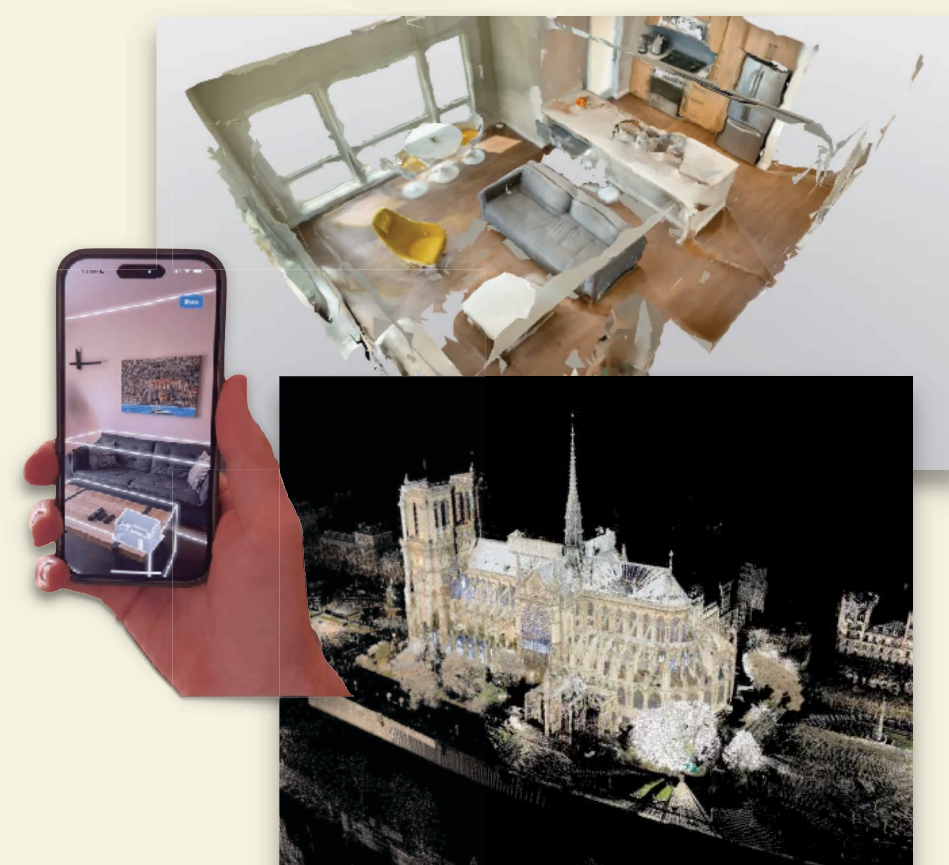
medical imaging



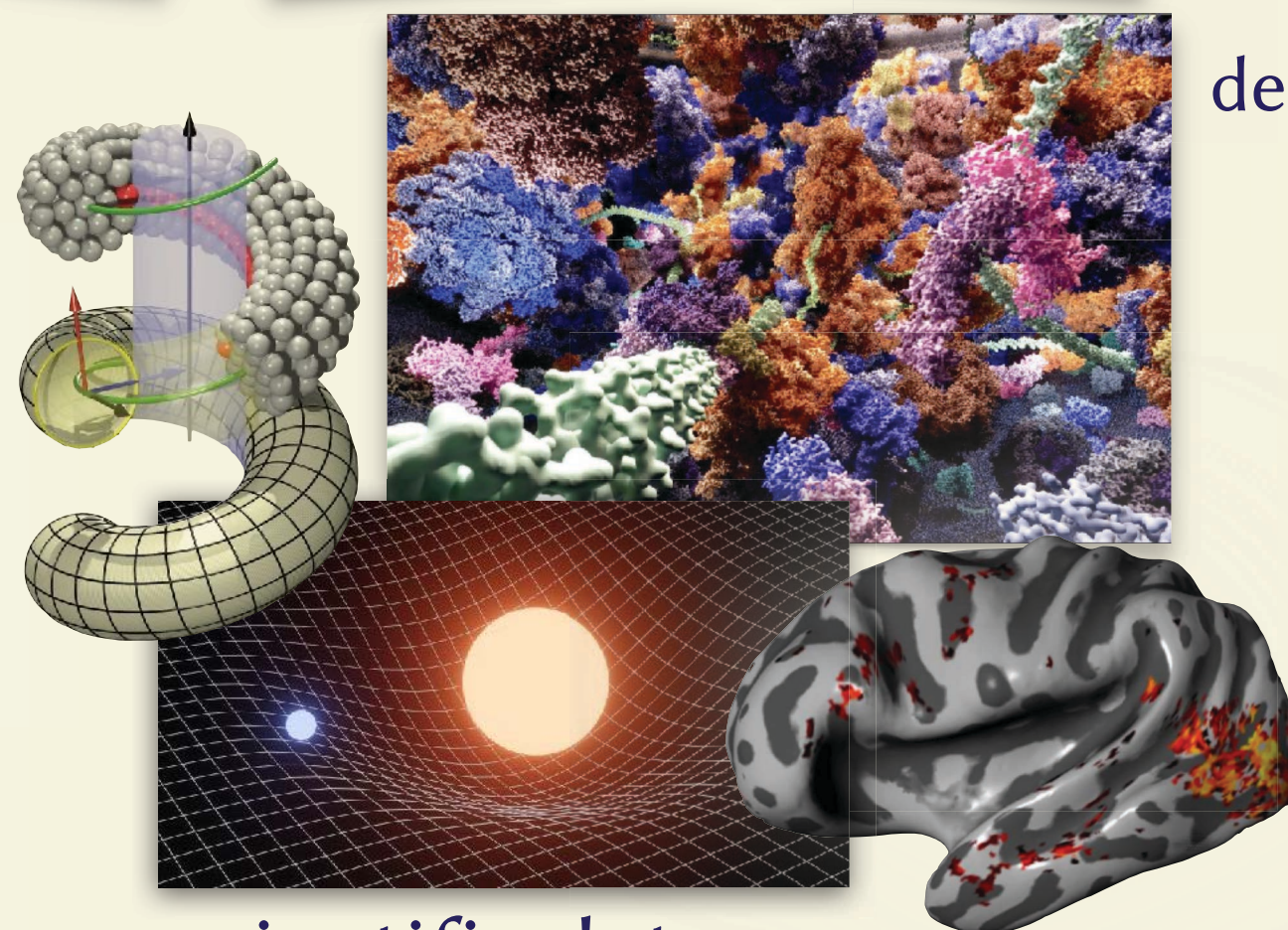
geometric ML



digital fabrication



3D scans



scientific data

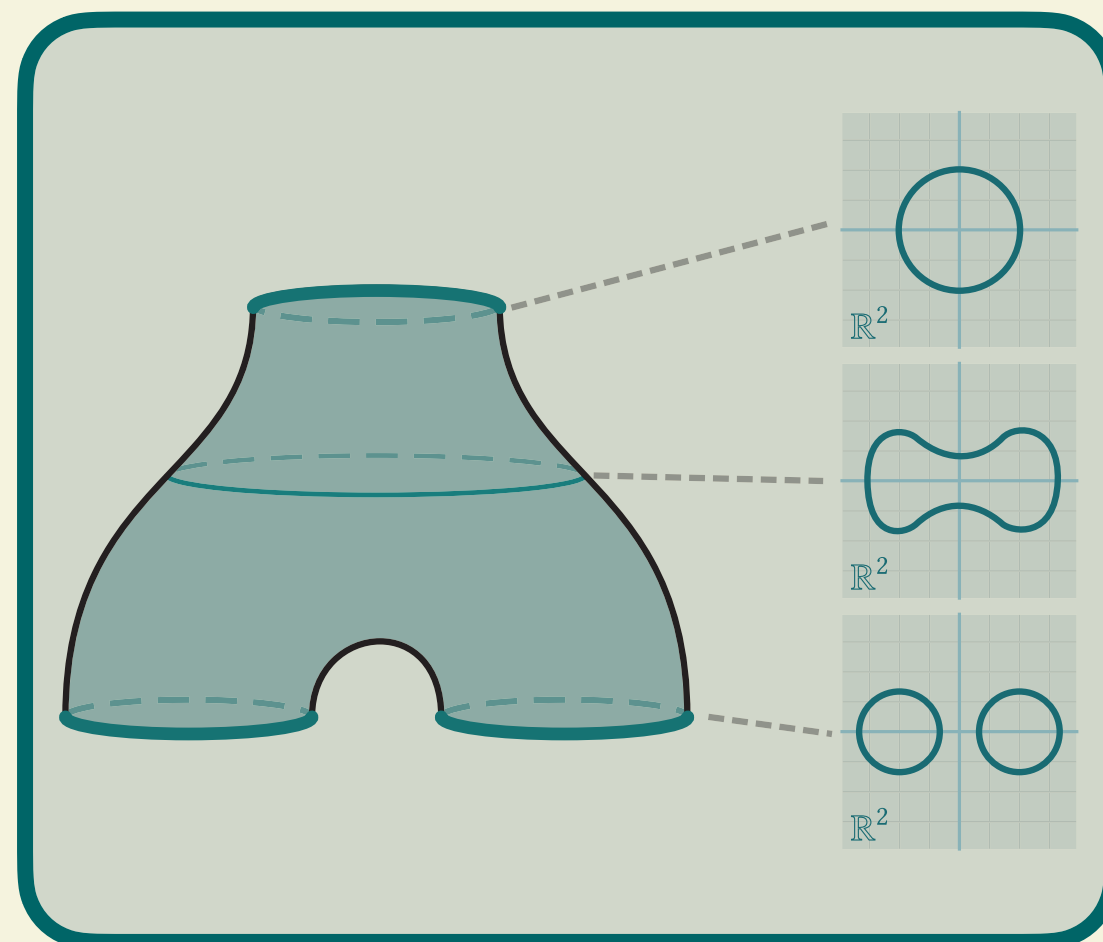
Talk Outline

I. SHAPE CORRESPONDENCE



use a new geometric perspective to map between surfaces

II. SURFACE MORPHING VIA 4D RAY TRACING



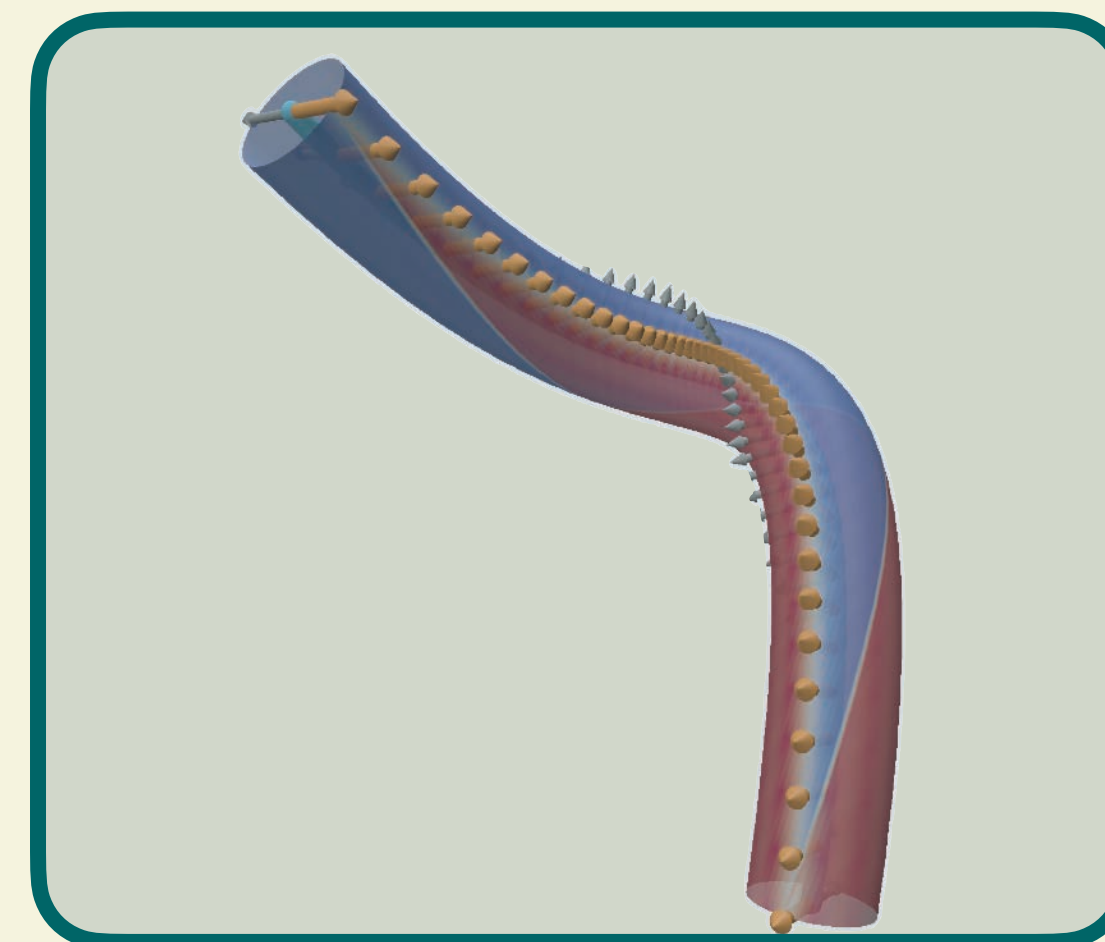
morph from one surface to another in 4D space

III. VORONOI CROCHET PATTERN DESIGN



construct new crochet patterns via *geodesic Voronoi diagrams*

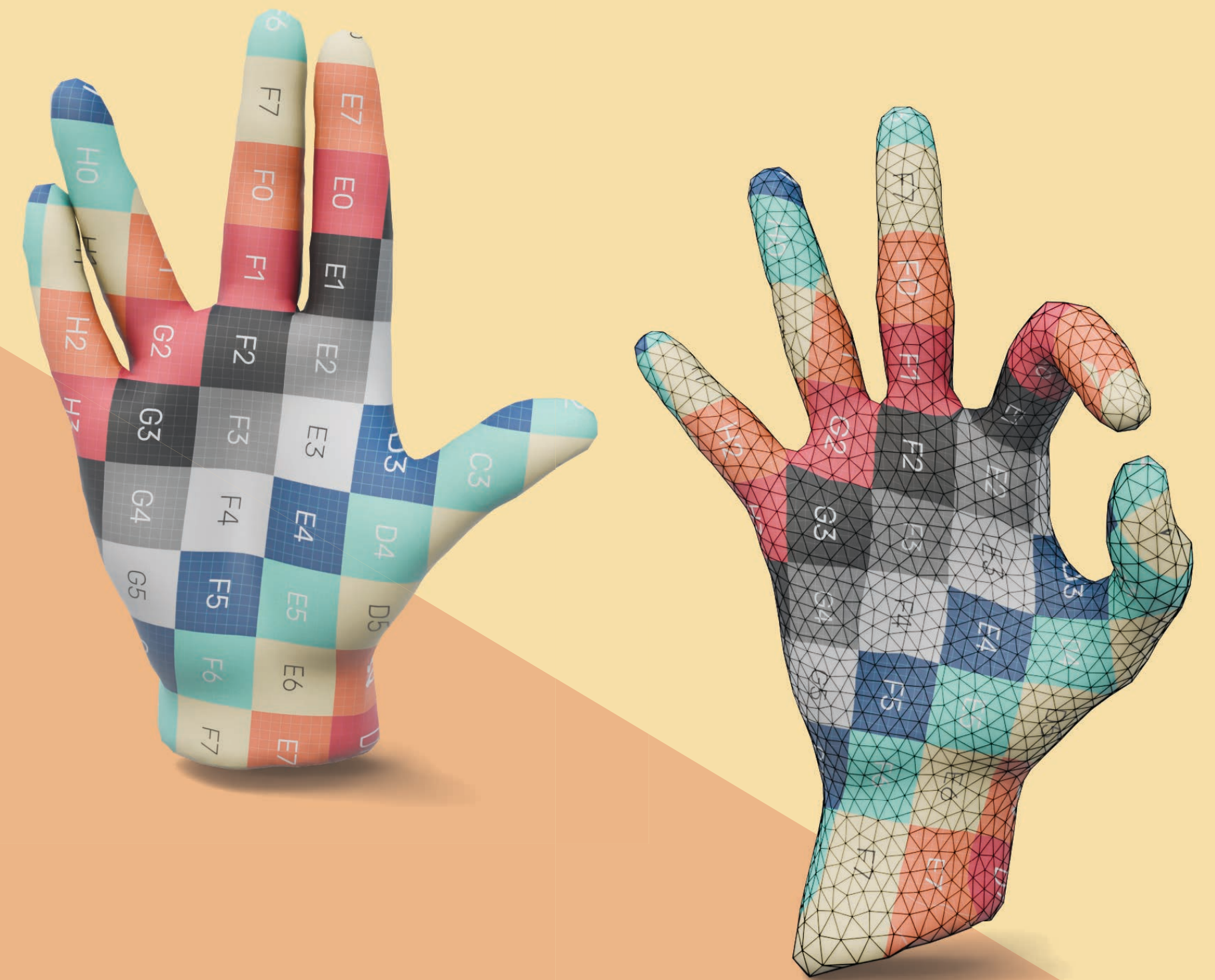
IV. IMPLICIT CURVE FRAMING



represent smooth normal vectors on curves via implicit functions

Shape Correspondence

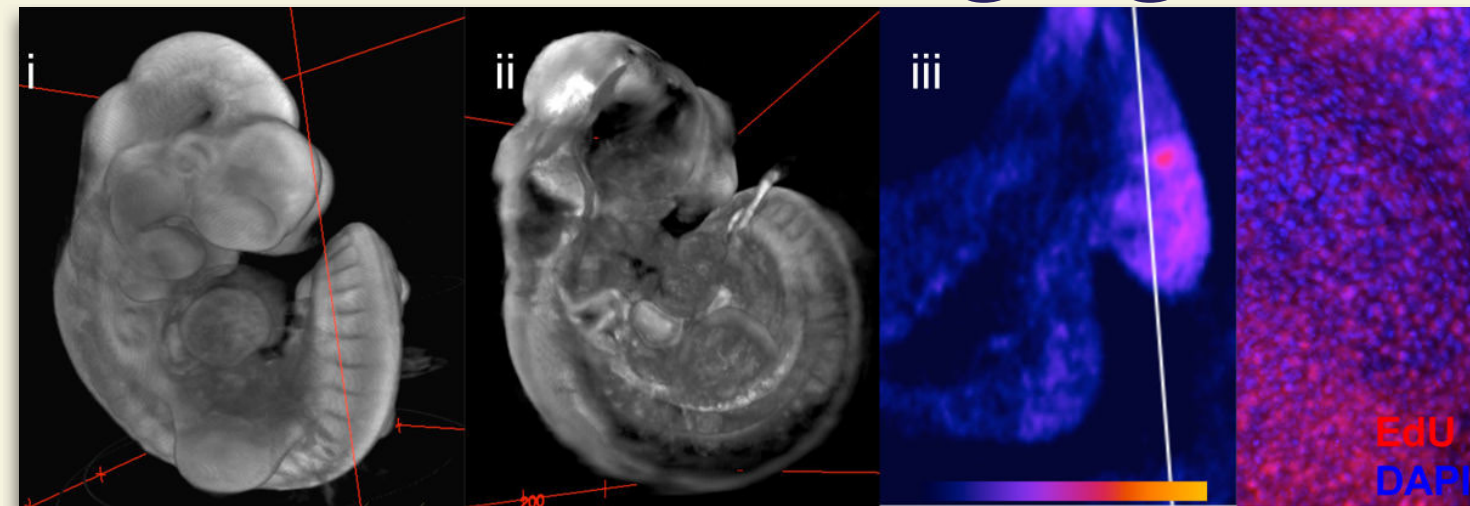
[Corman, Soliman, Magnet & **Gillespie**. 2026. *Implicit Minimal Surfaces for Bijective Correspondences*. ACM Trans. Graph.]



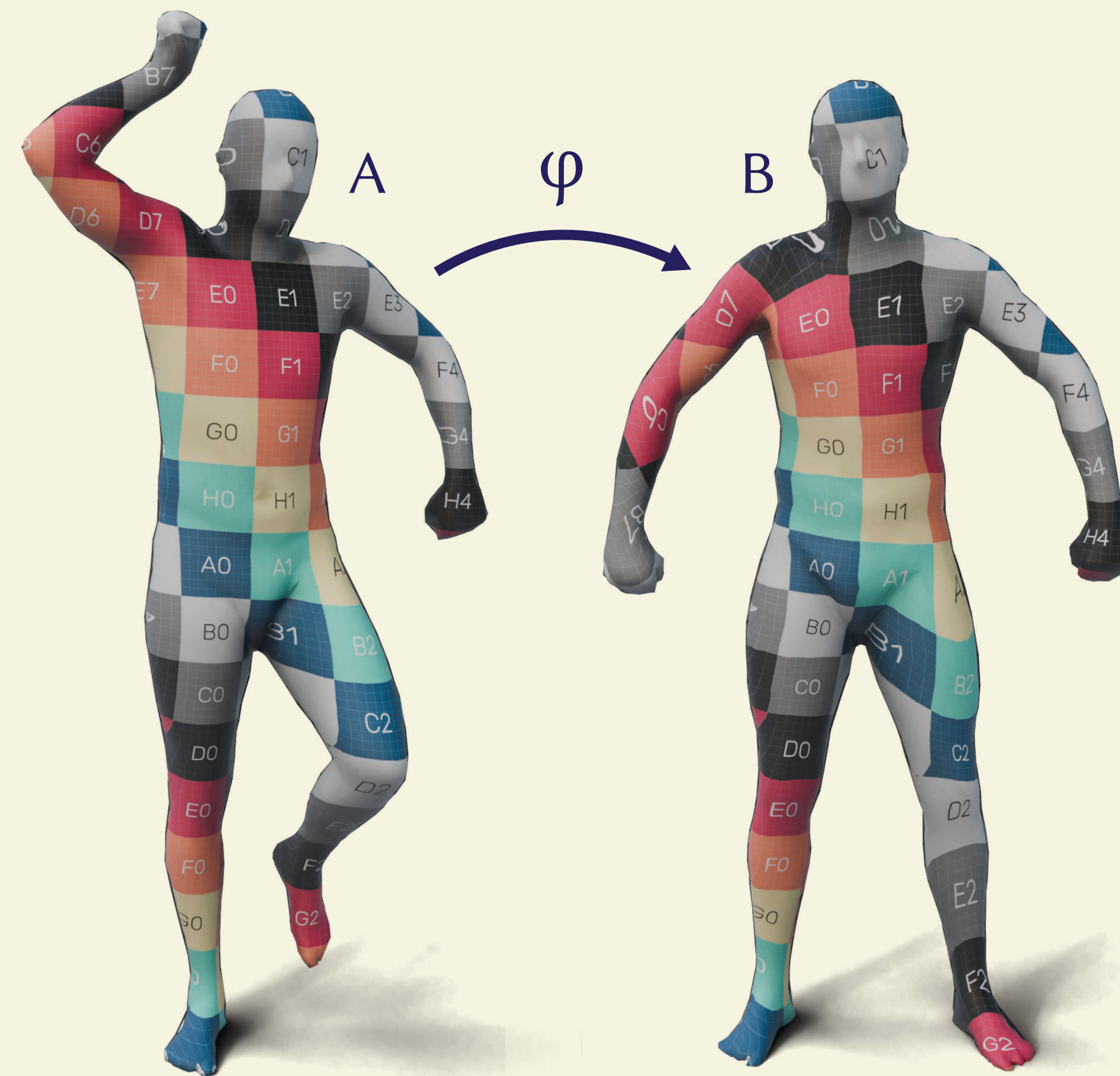
Problem: Mapping between Surfaces

required to study data over collections of shapes

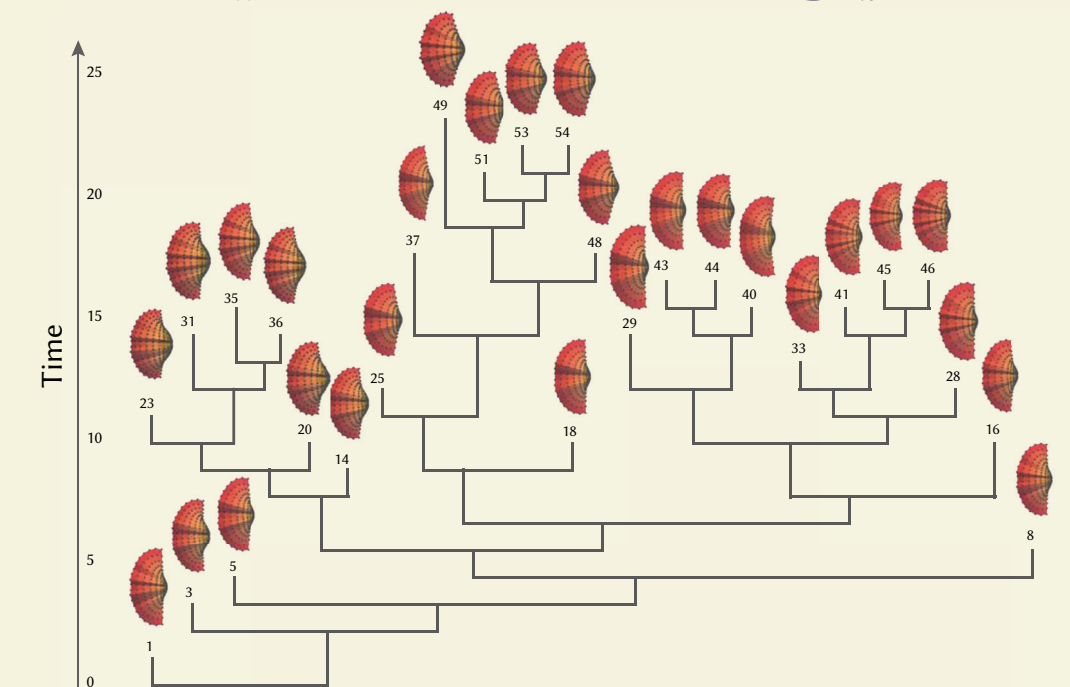
medical imaging



[Hallgrimsson et al. 2015]



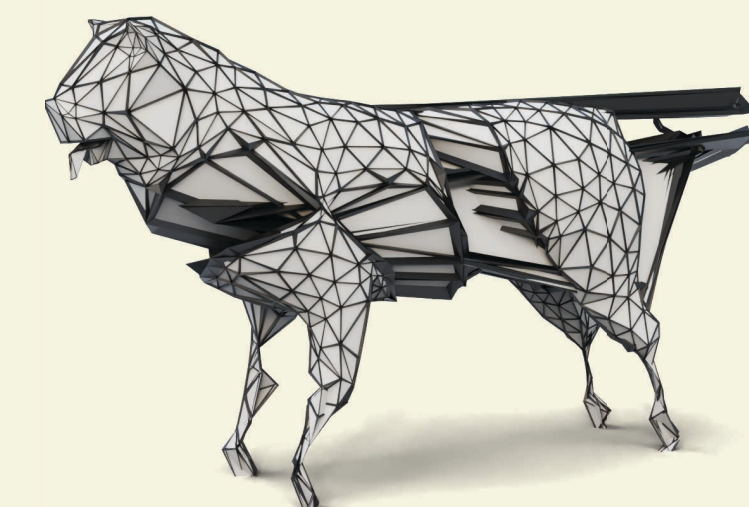
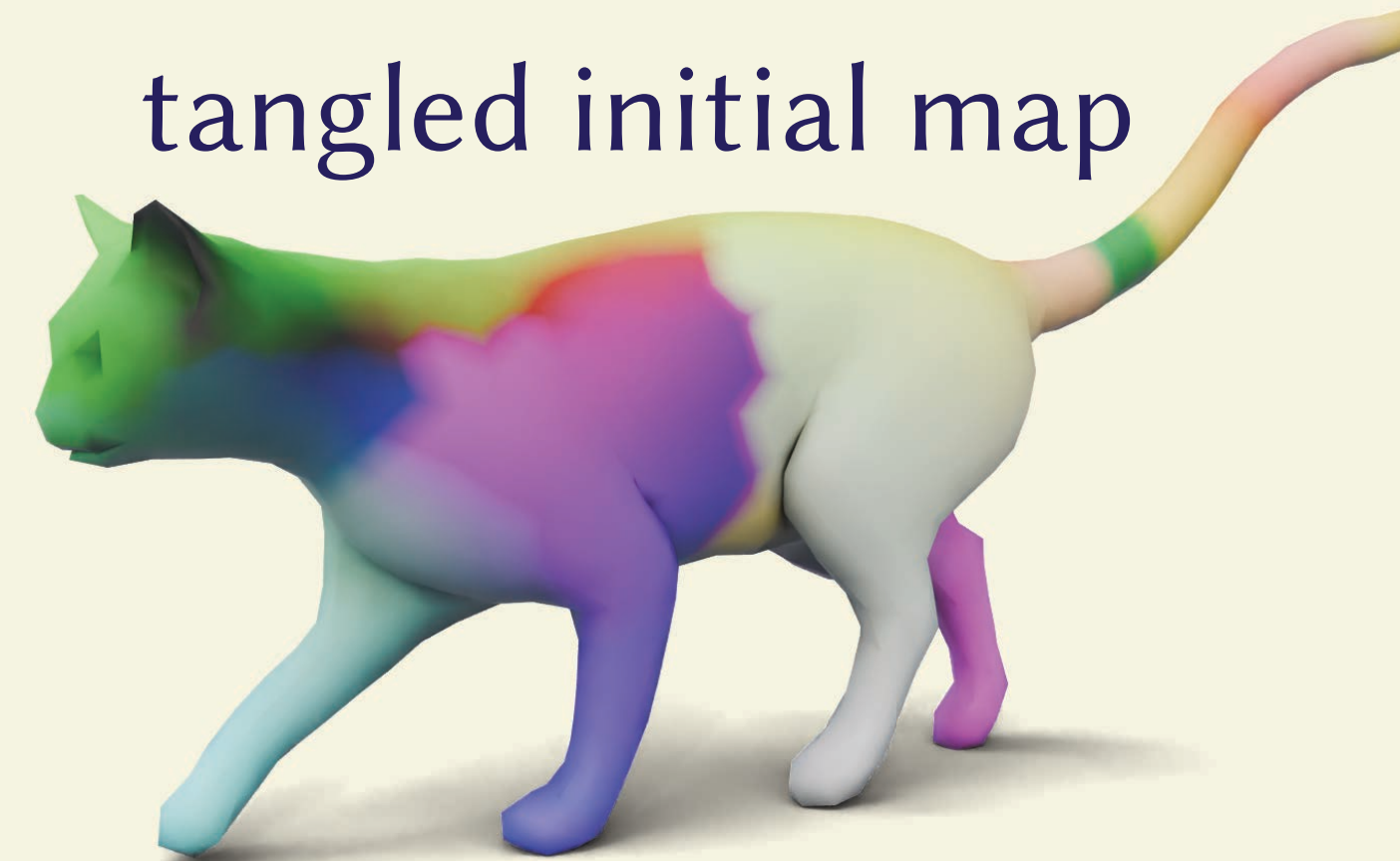
paleontology



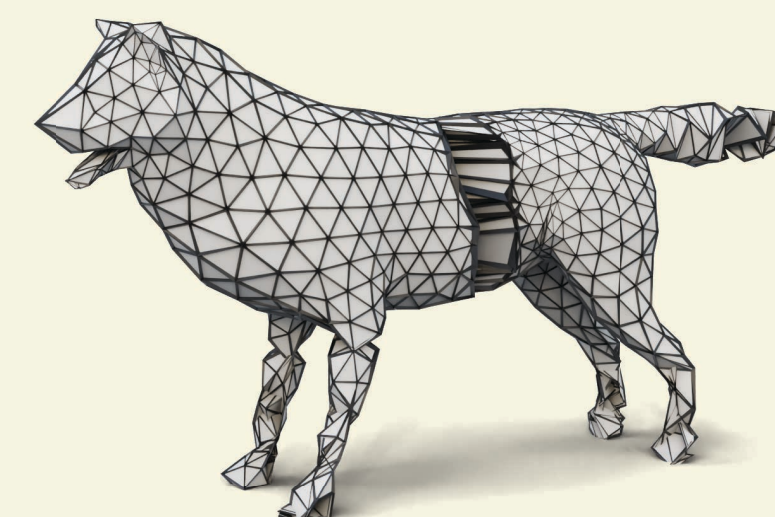
[Polly & Motz 2016]

Map Untangling

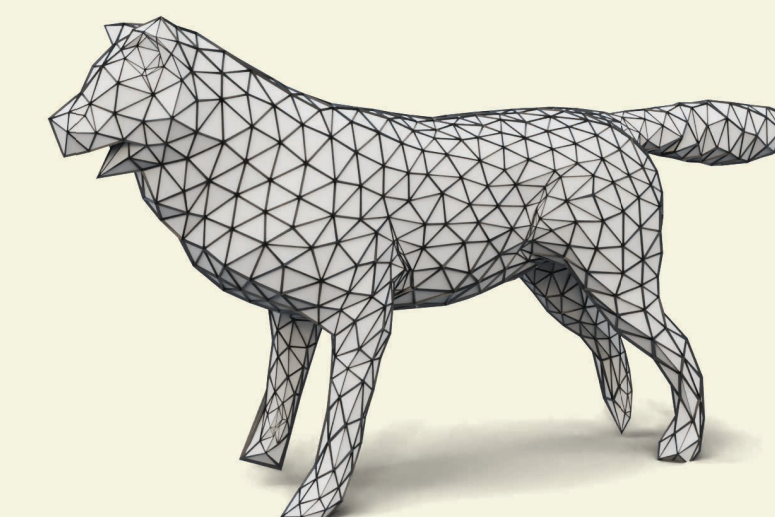
Fixing Flips



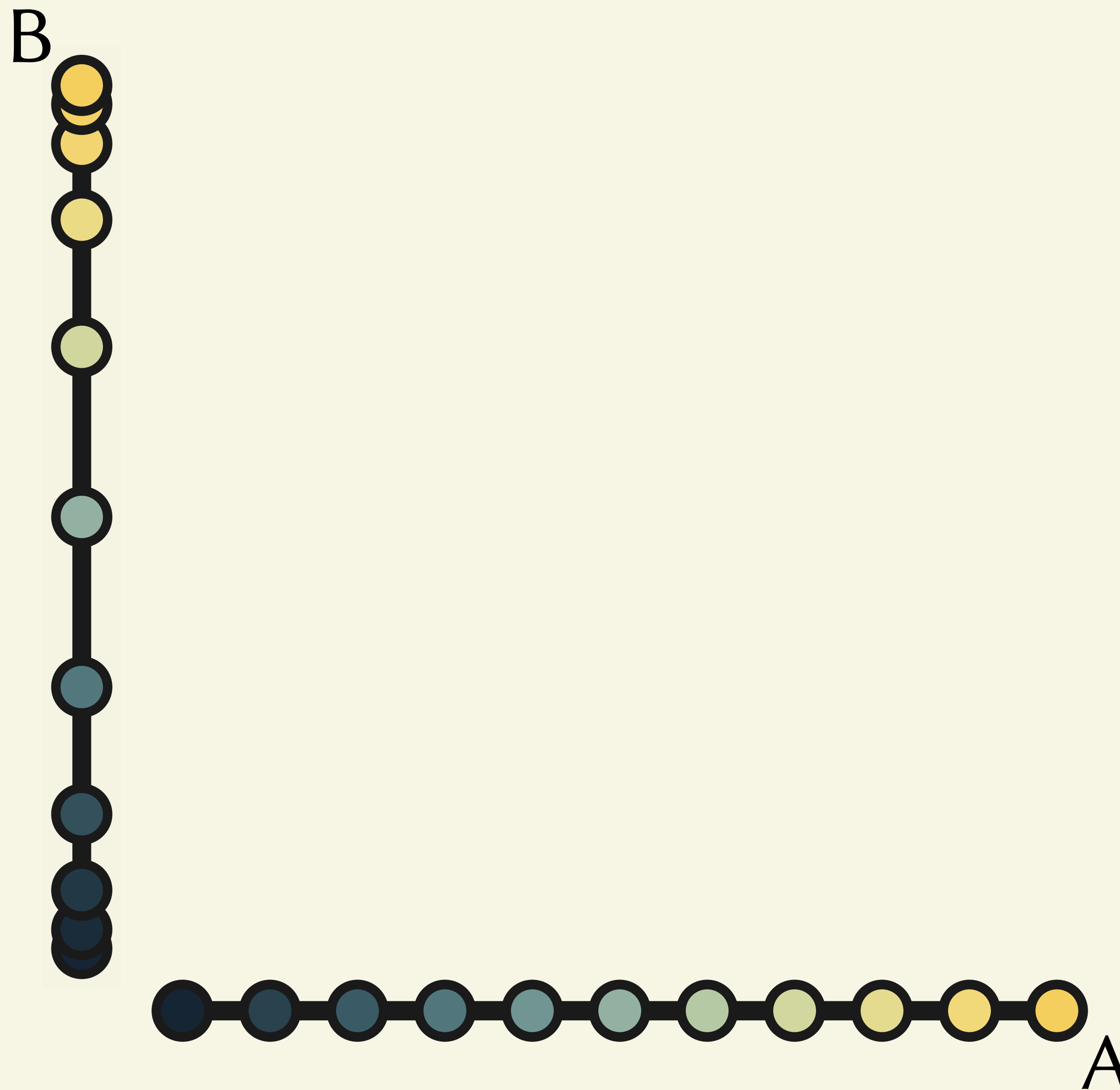
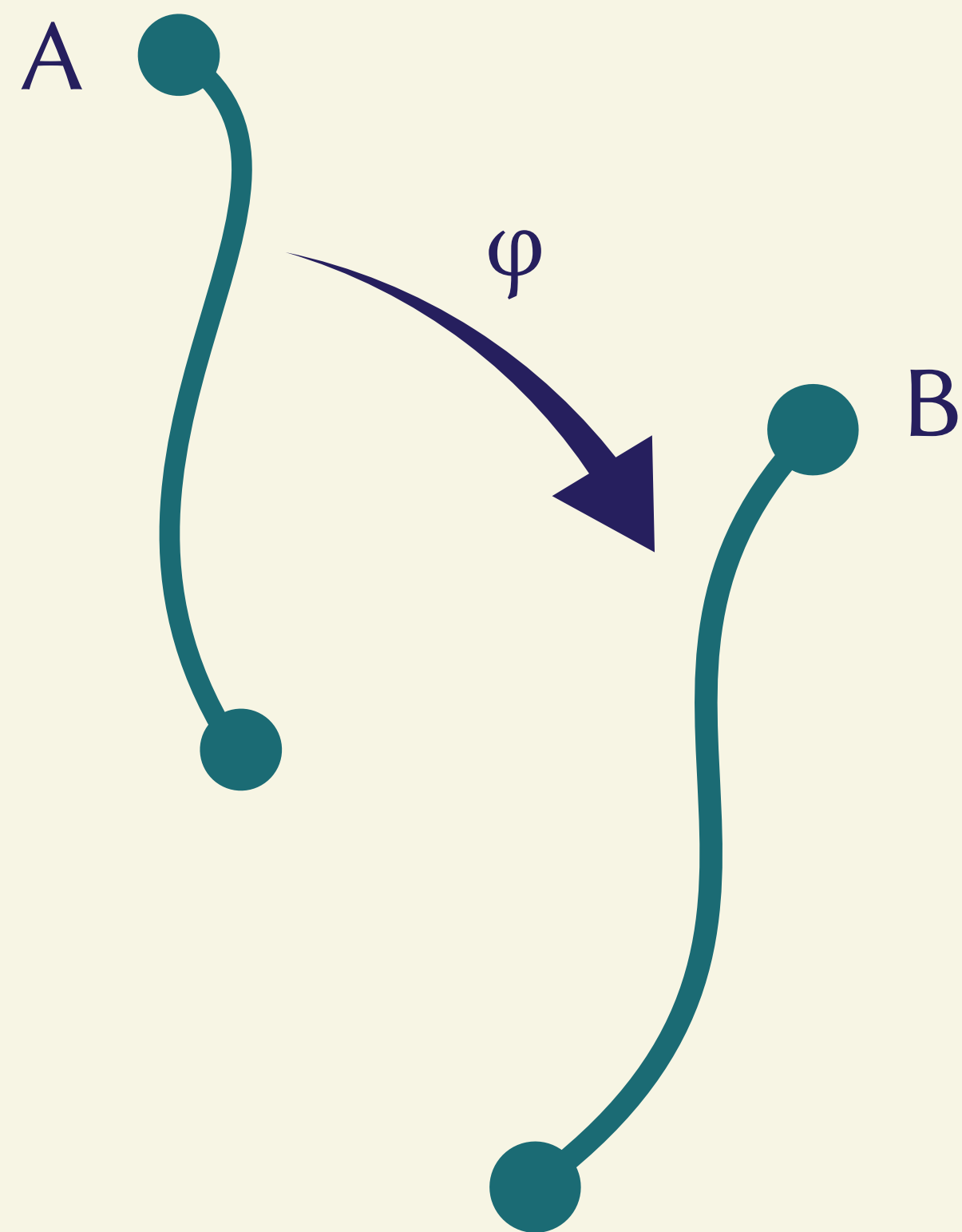
RHM
[Ezuz et al. 2019]



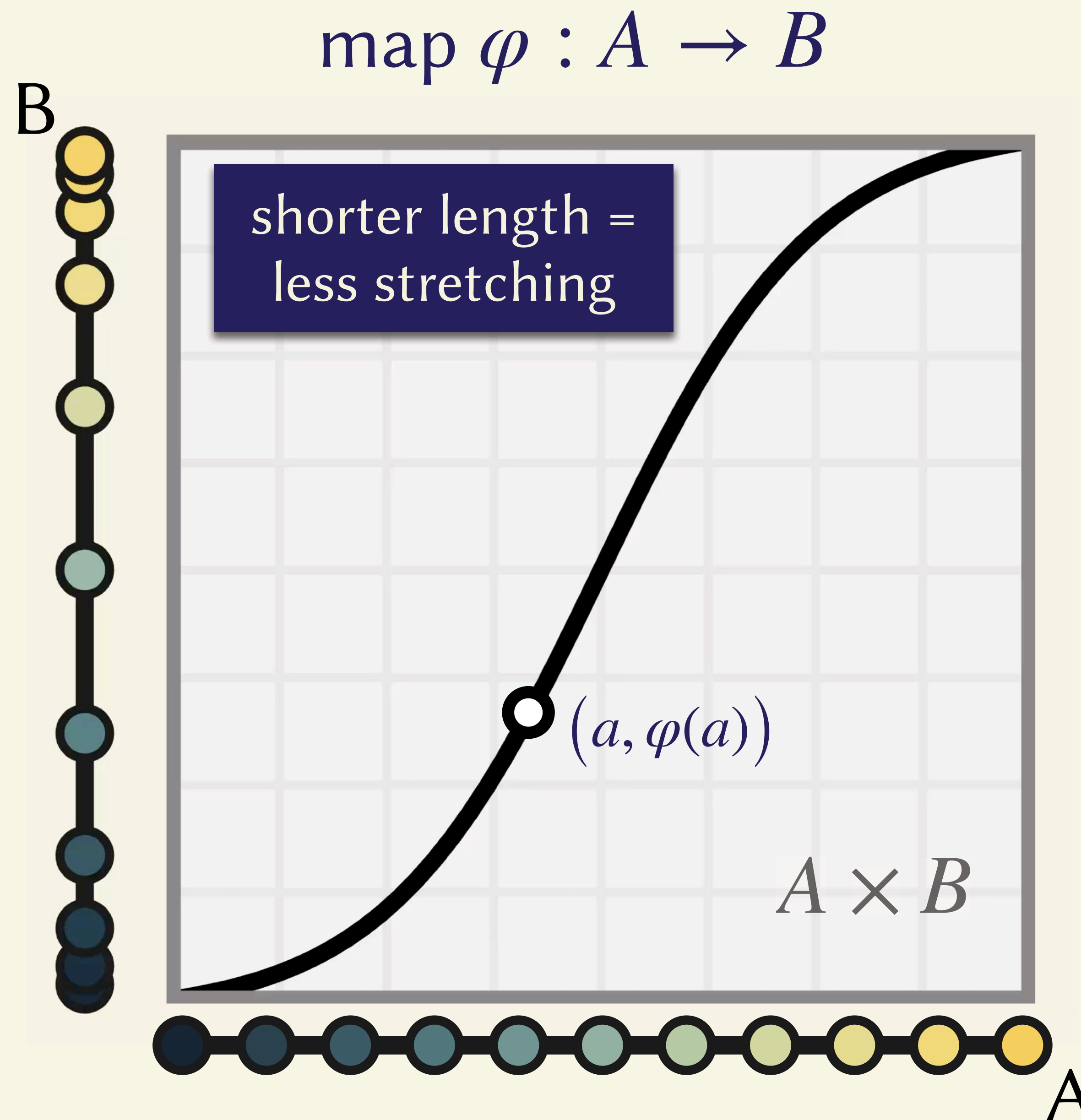
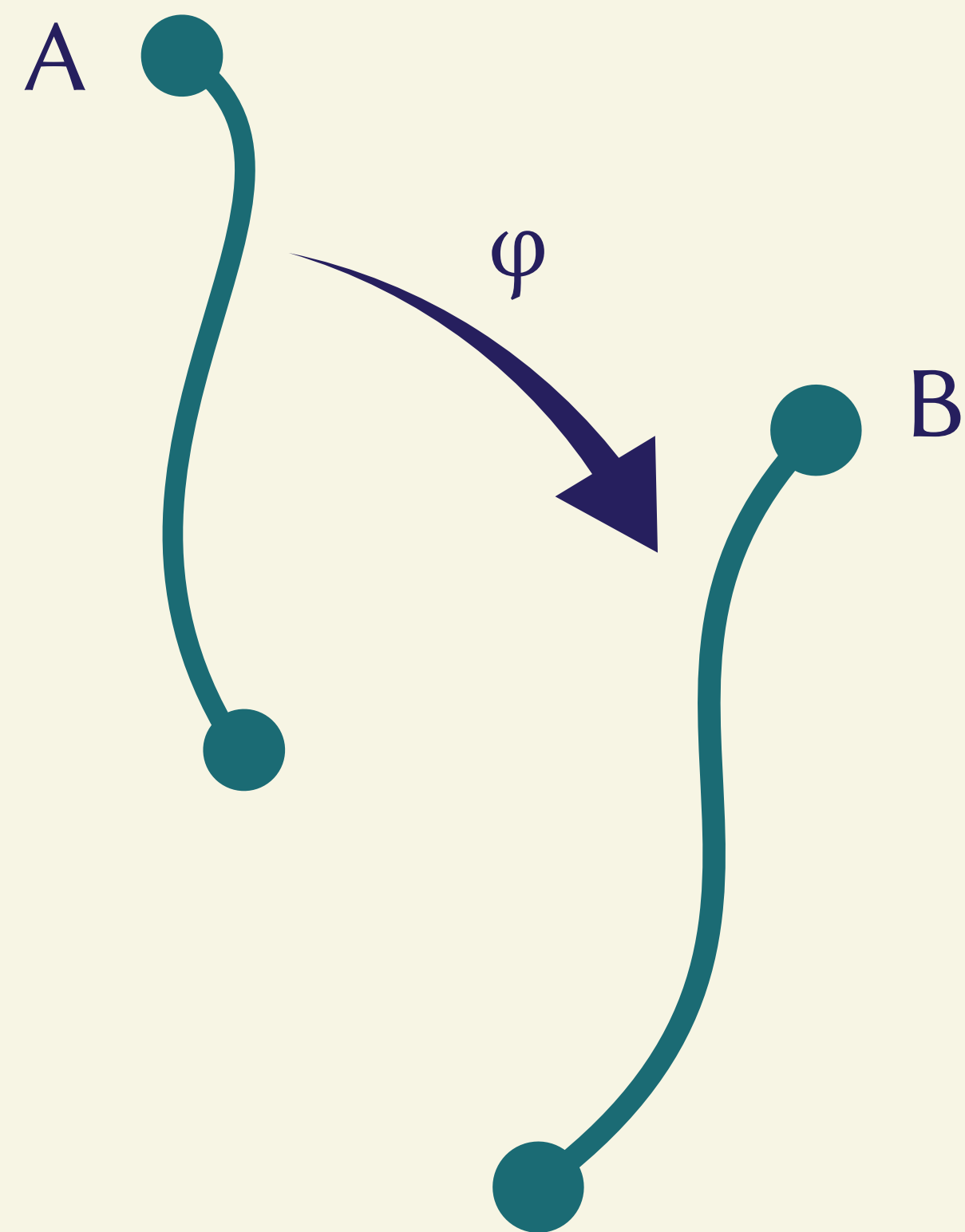
ours



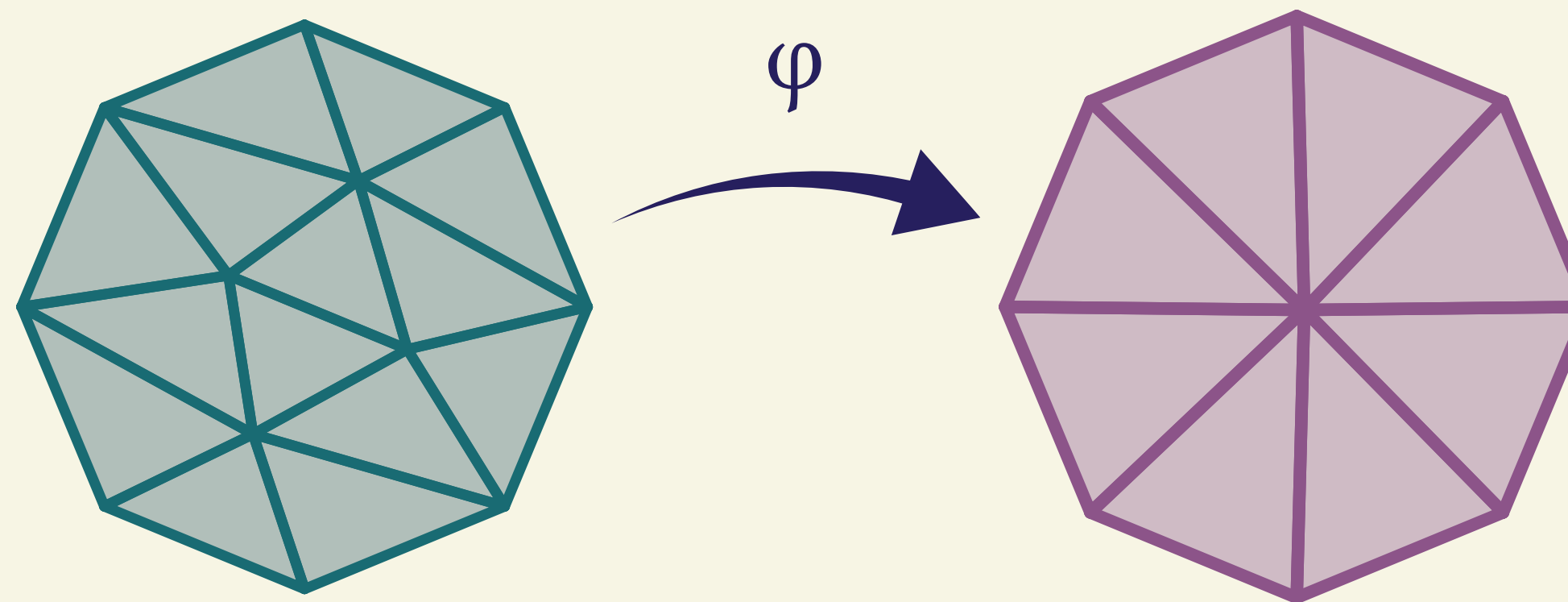
Example: 1D Curves



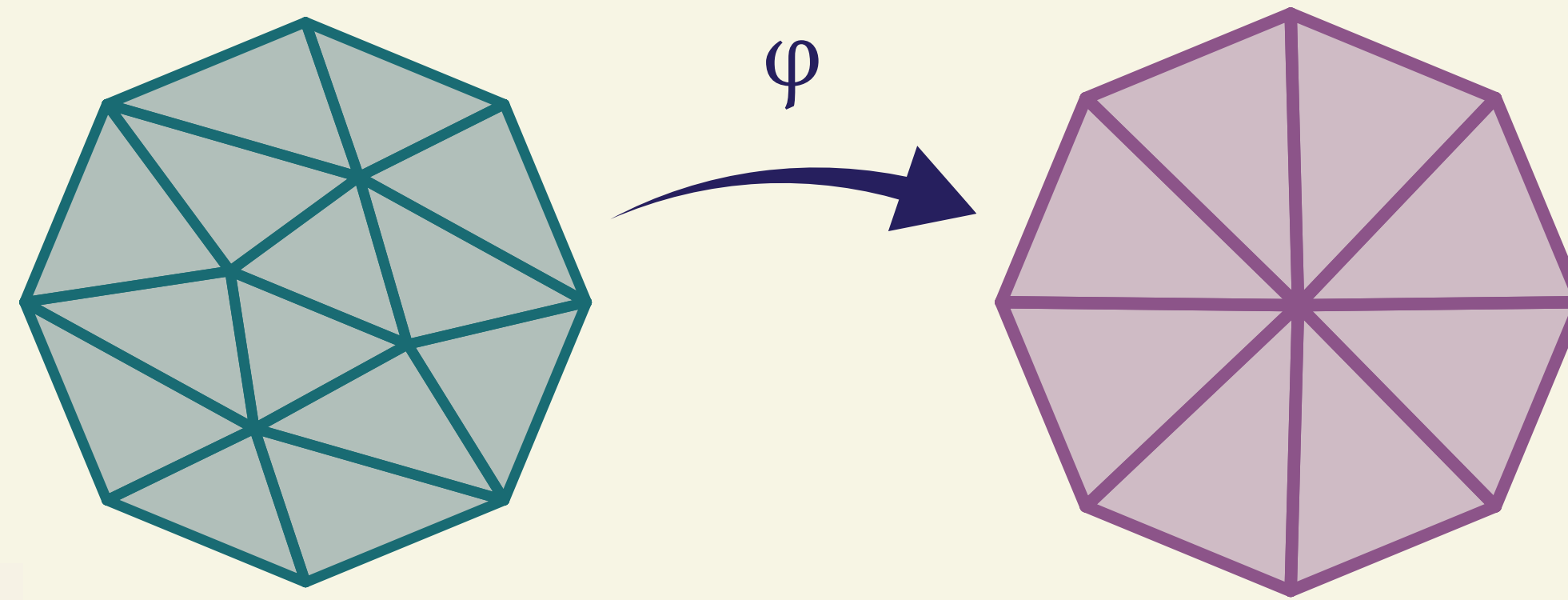
Example: 1D Curves



Example: 2D Disks



Example: 2D Disks



Find a good map by minimizing the area in the product space

Map Distortion

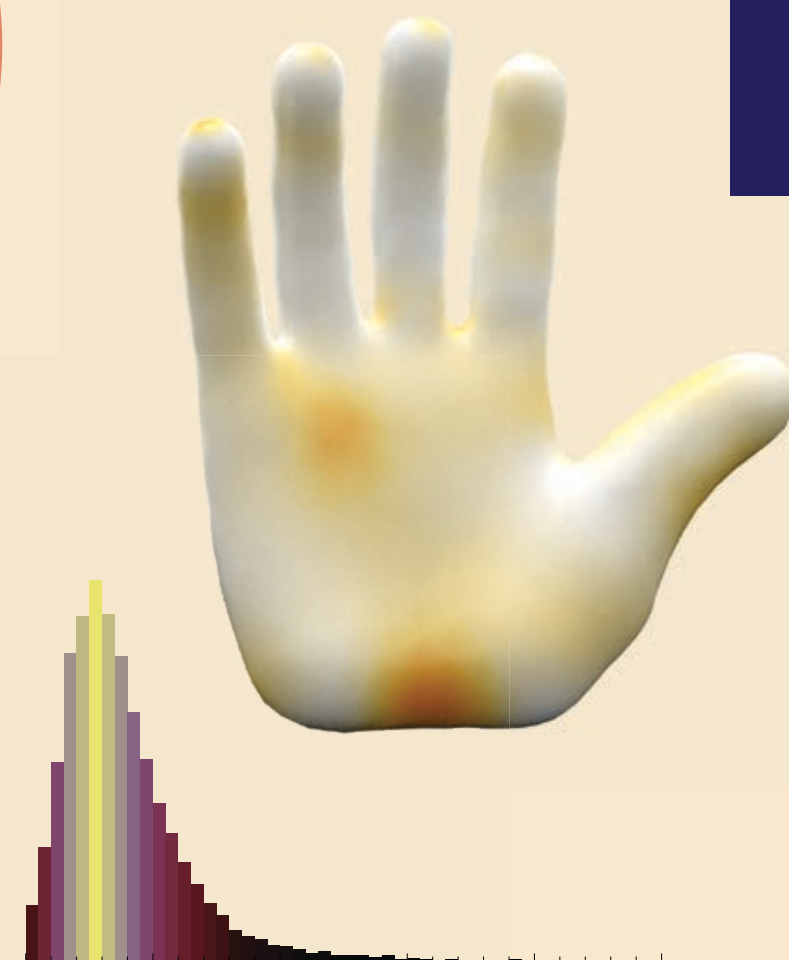


source hand

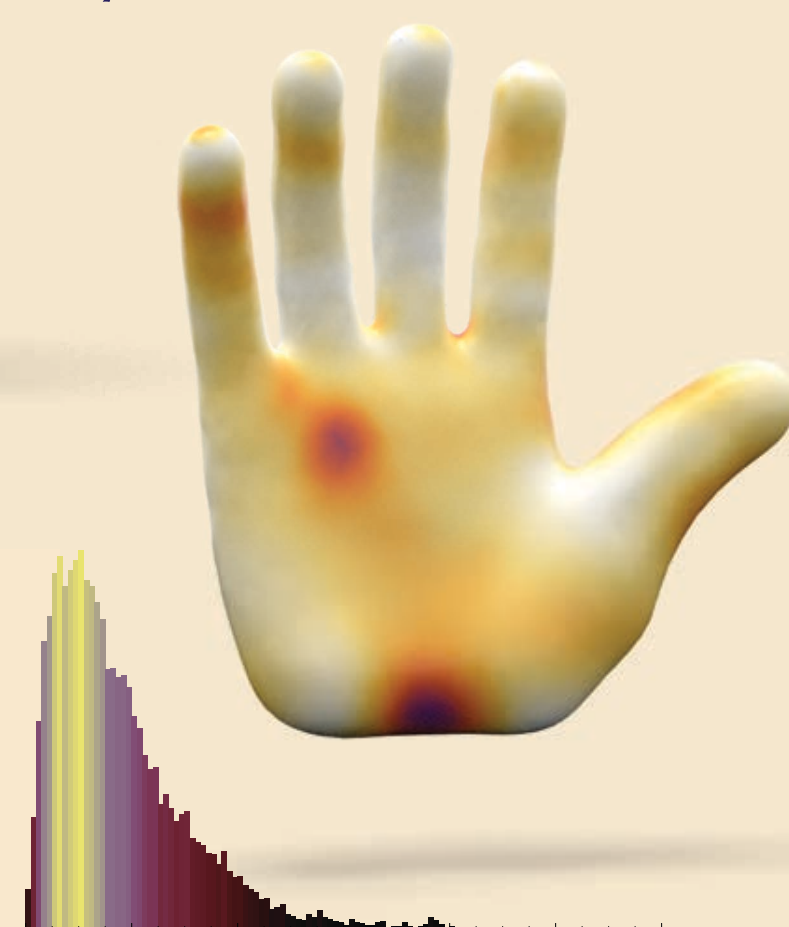


map
[ours]

distortion
(area)

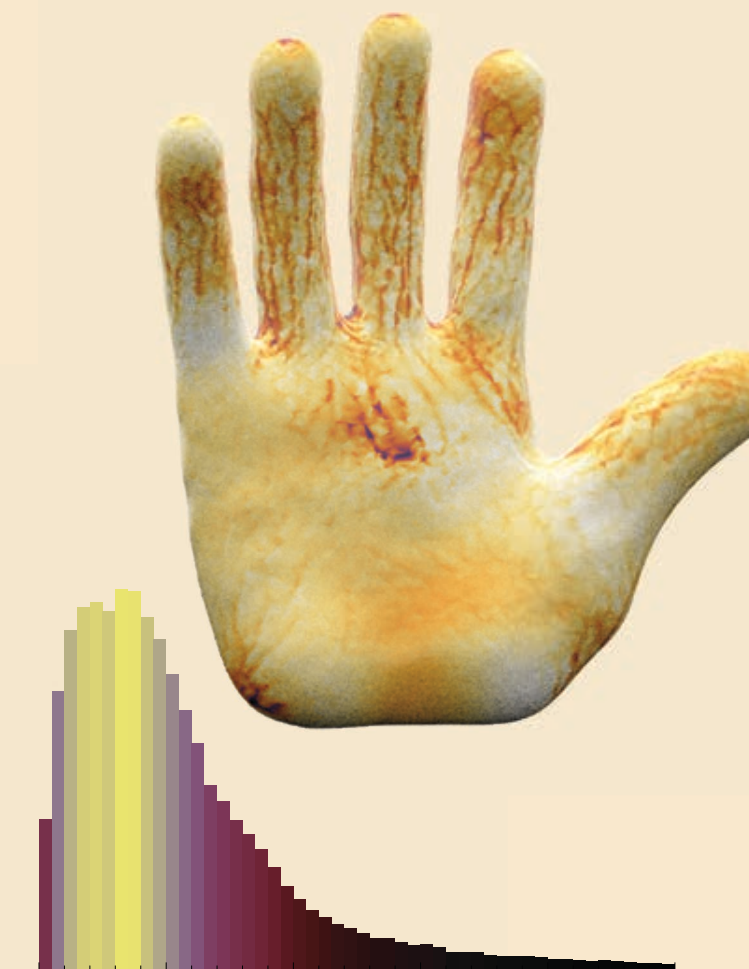


distortion
(symmetric Dirichlet)



our simpler optimization
yields lower distortion in
both senses

distortion
(area)

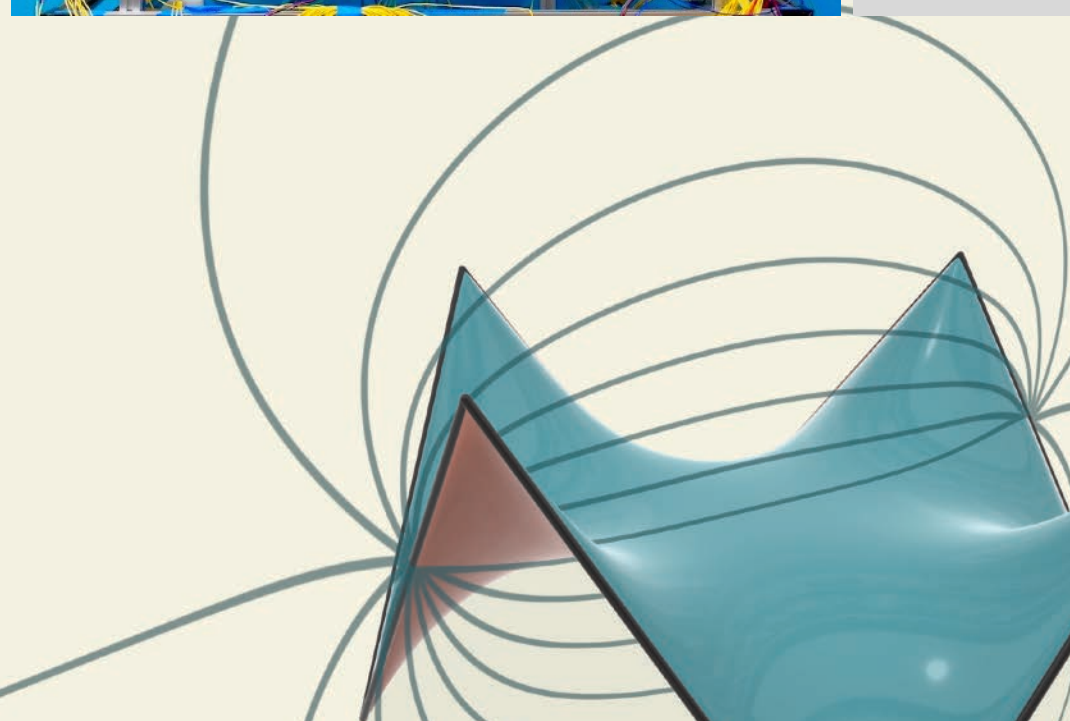
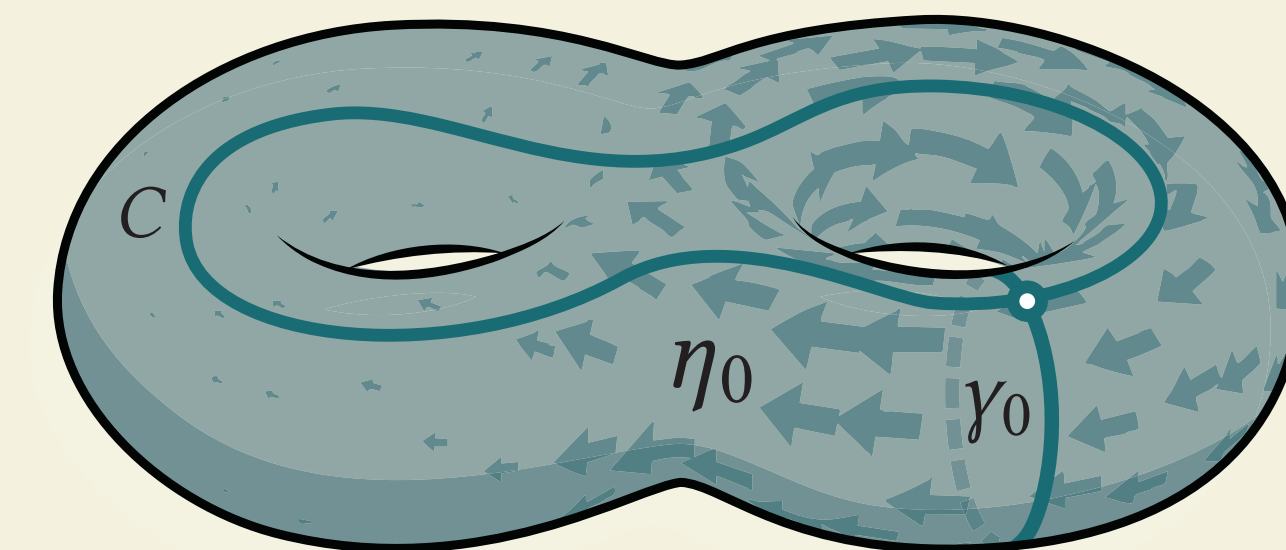
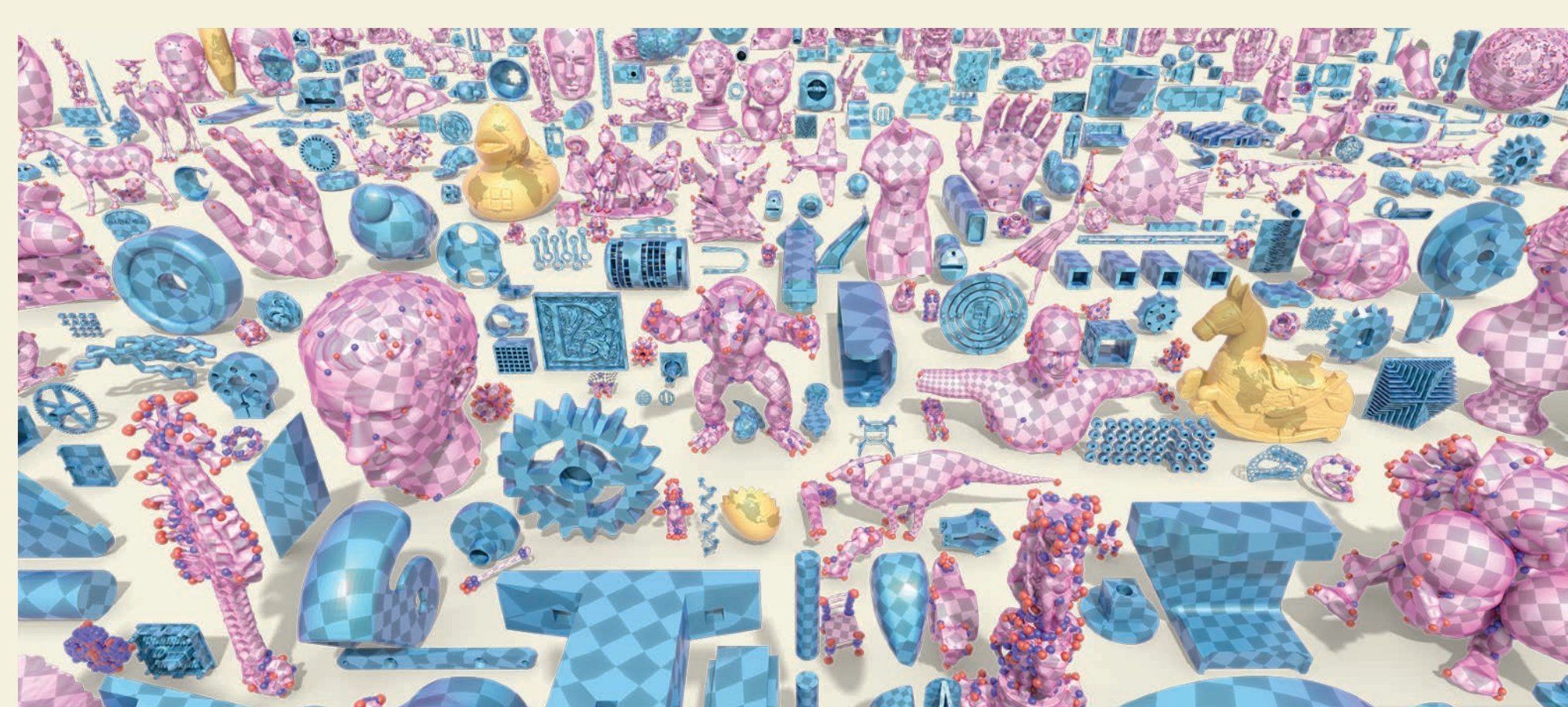
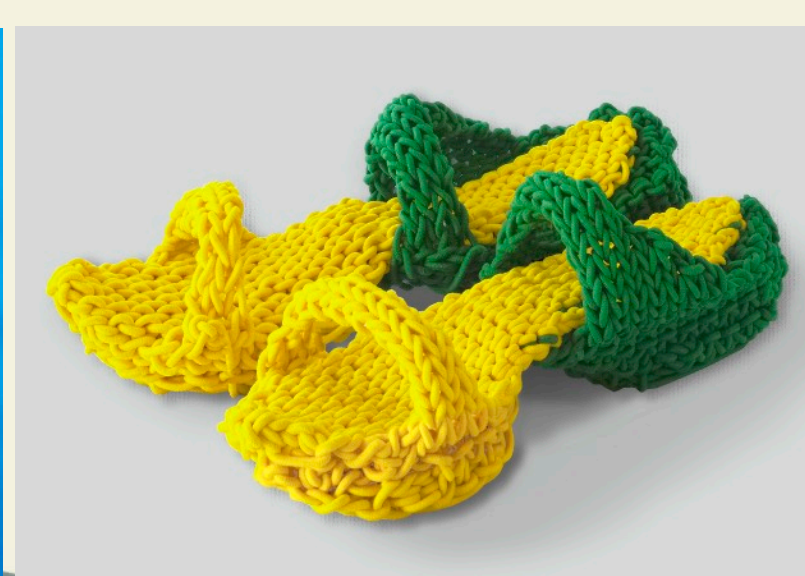
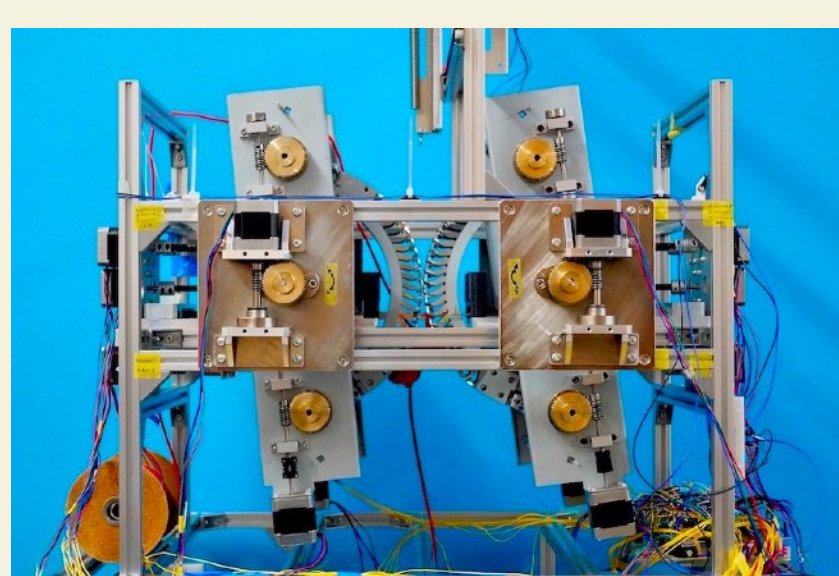


distortion
(symmetric Dirichlet)

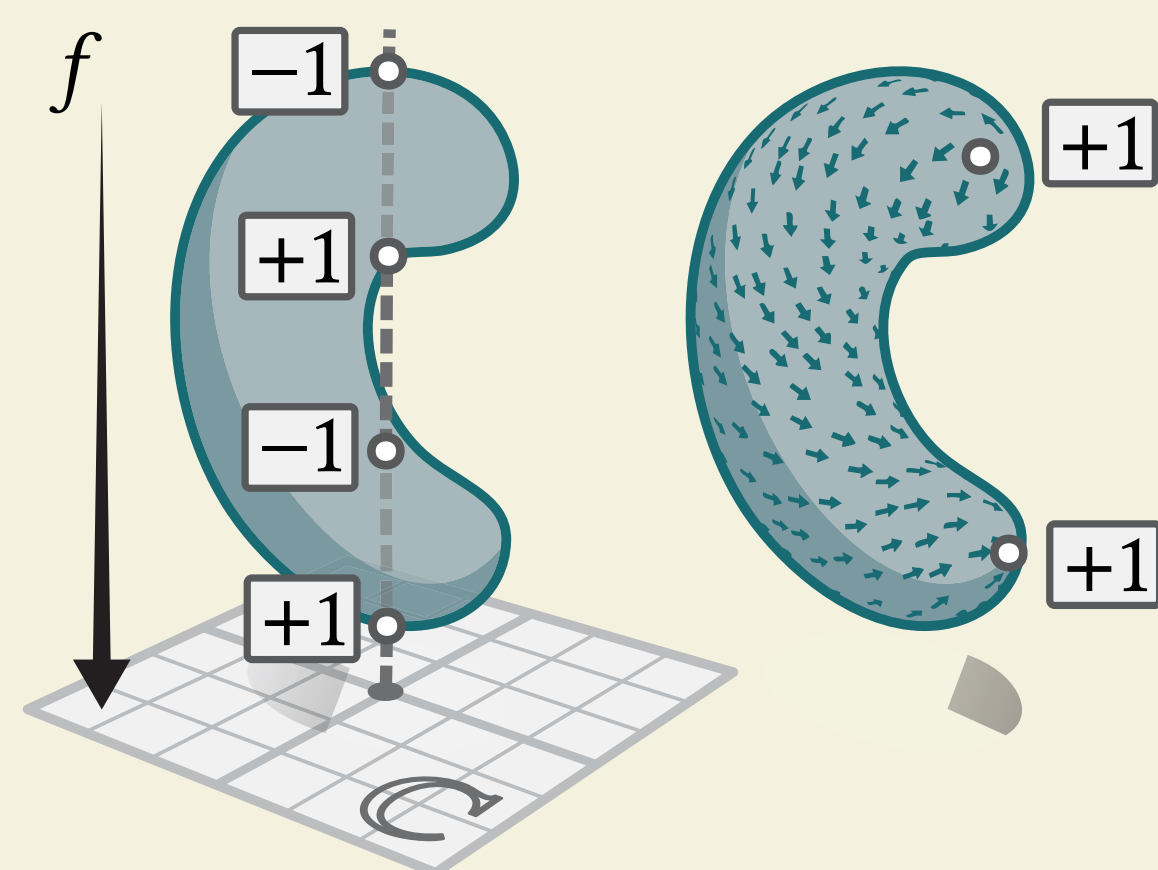
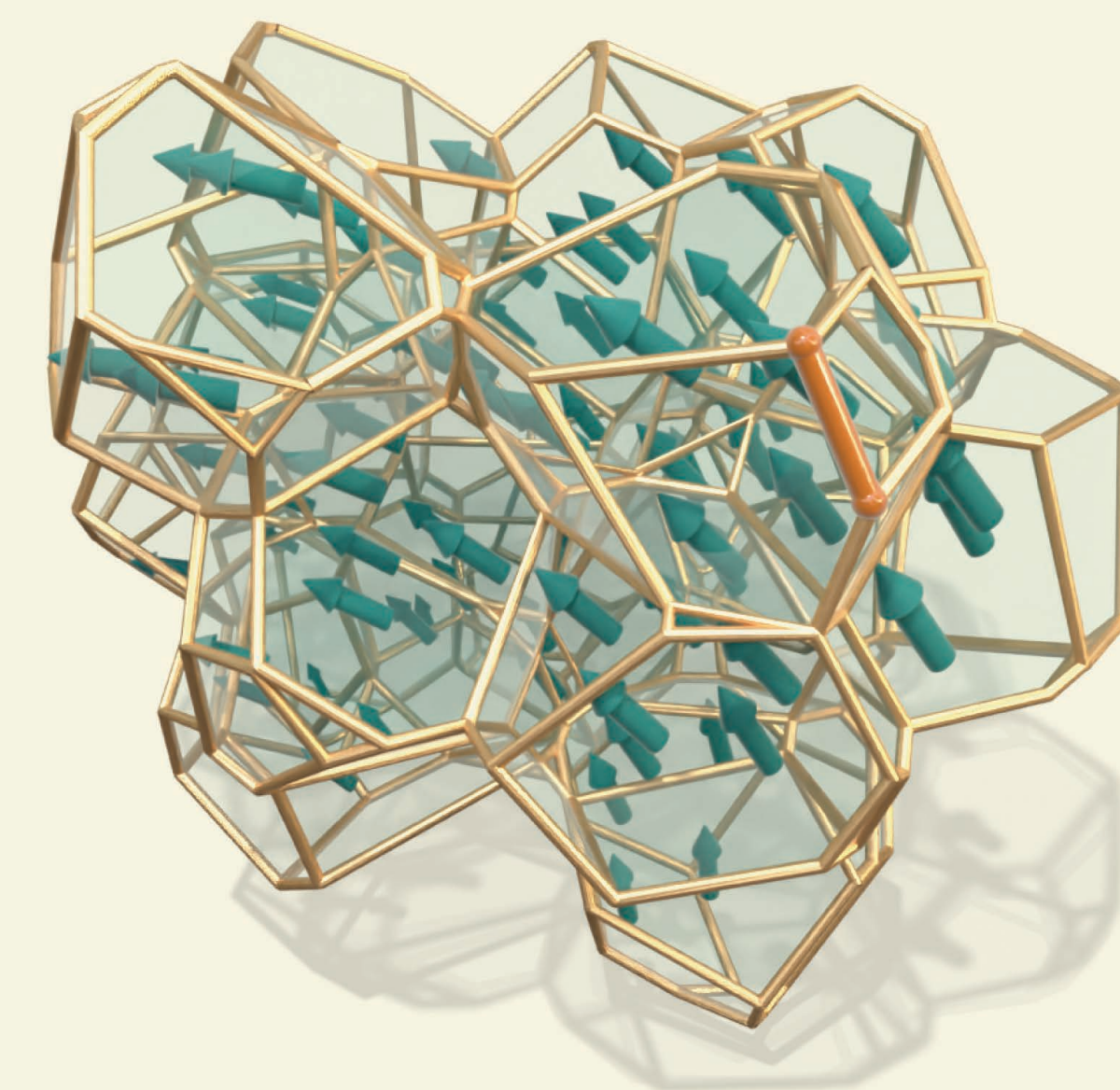


map
[Schmidt et al. 2020]

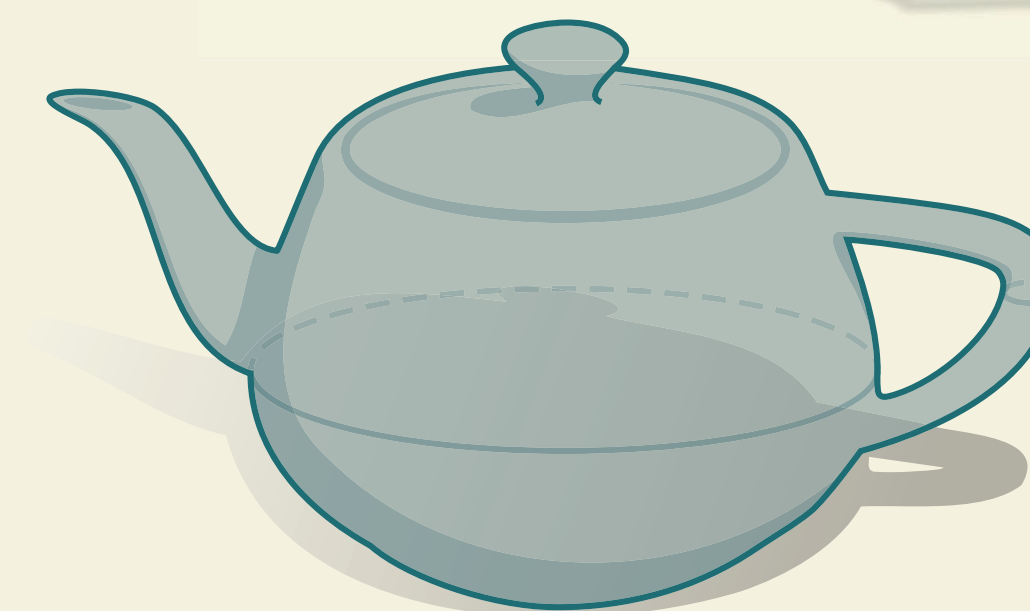
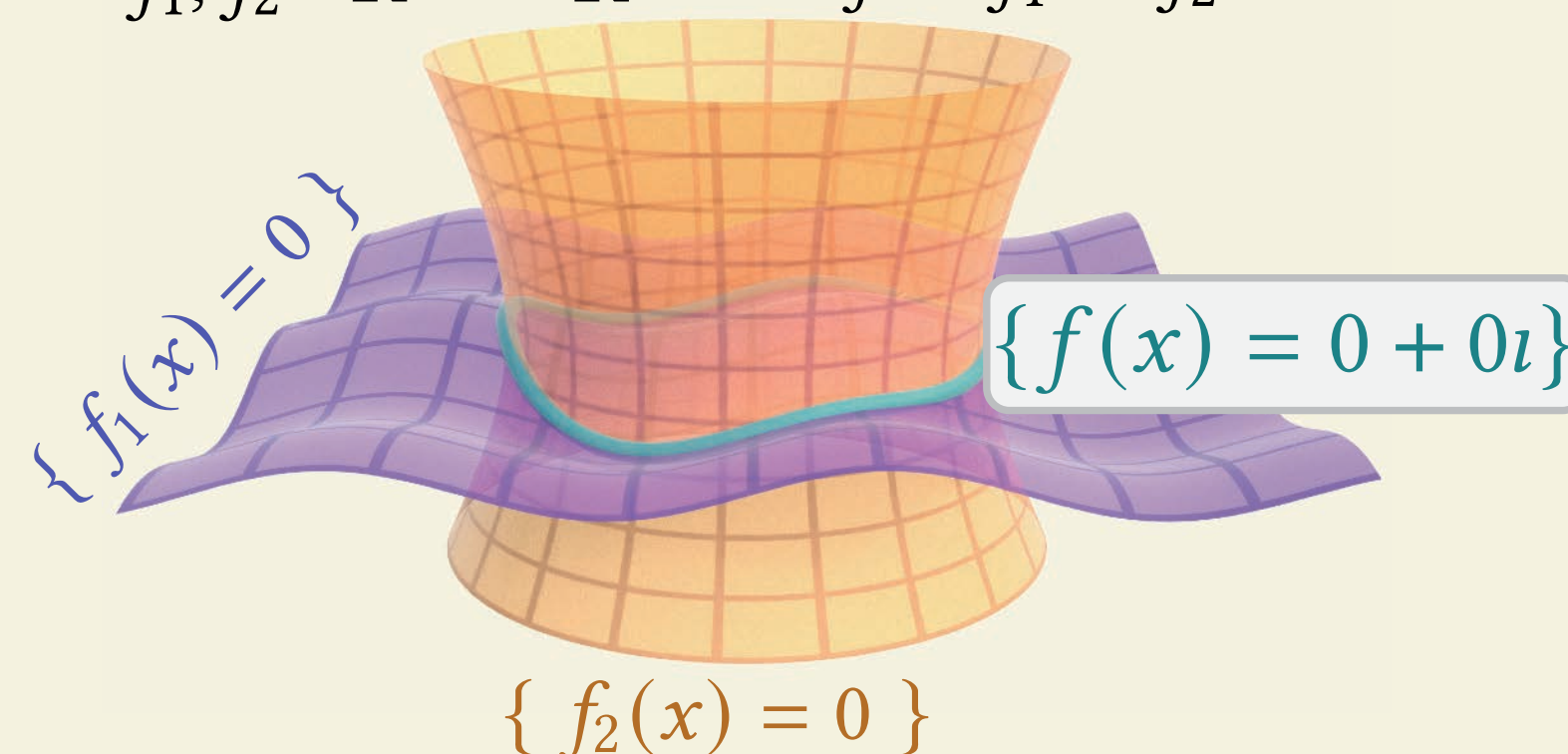




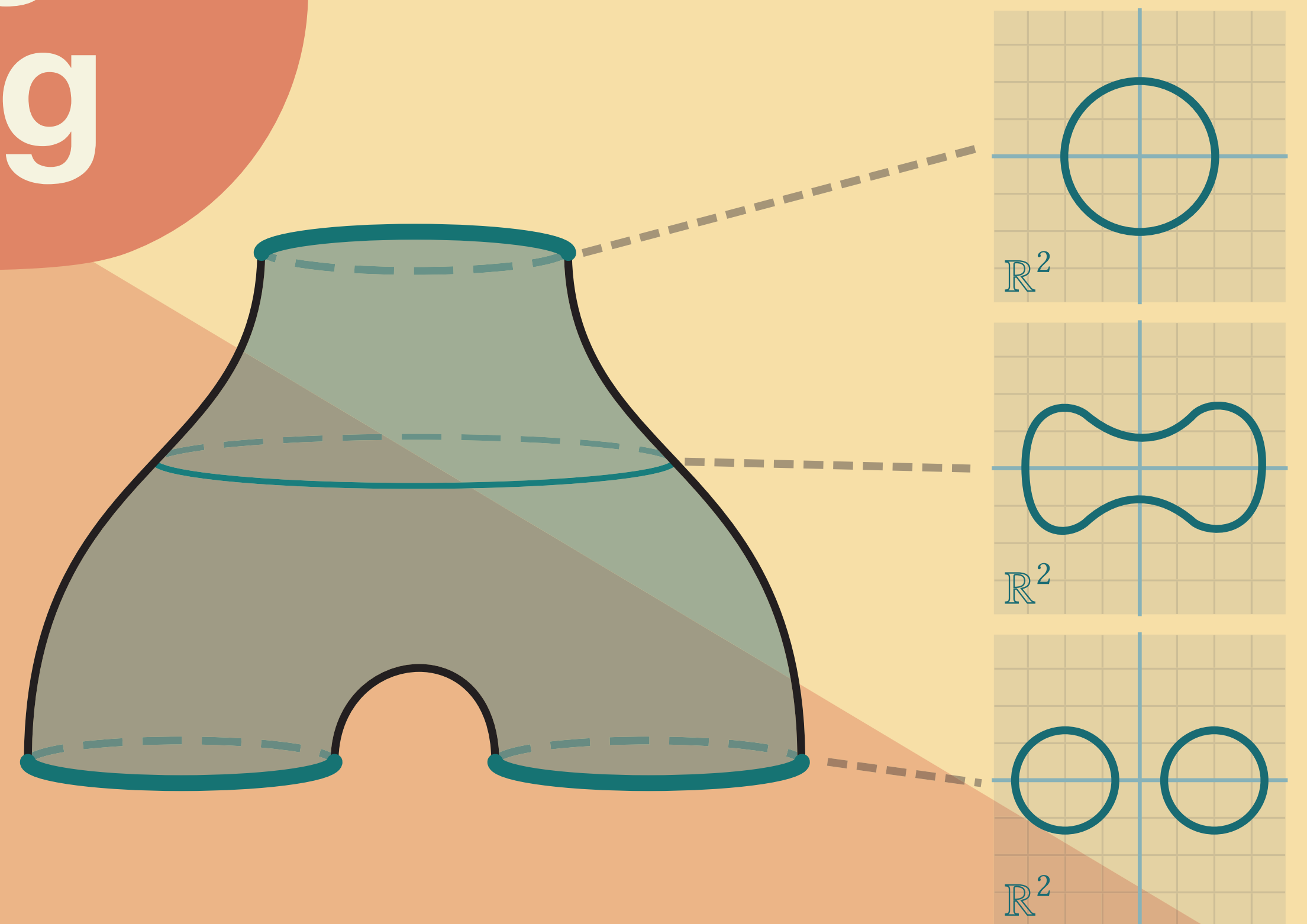
Thinking deeply about the **geometry** yields
more efficient algorithms
and **higher-quality** outputs



$$f_1, f_2 : \mathbb{R}^3 \rightarrow \mathbb{R} \iff f = f_1 + \iota f_2 : \mathbb{R}^3 \rightarrow \mathbb{C}$$



Shape Morphing via 4D Raytracing



Harmonic functions

special kind of function

heat transfer

gravitation

$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$$

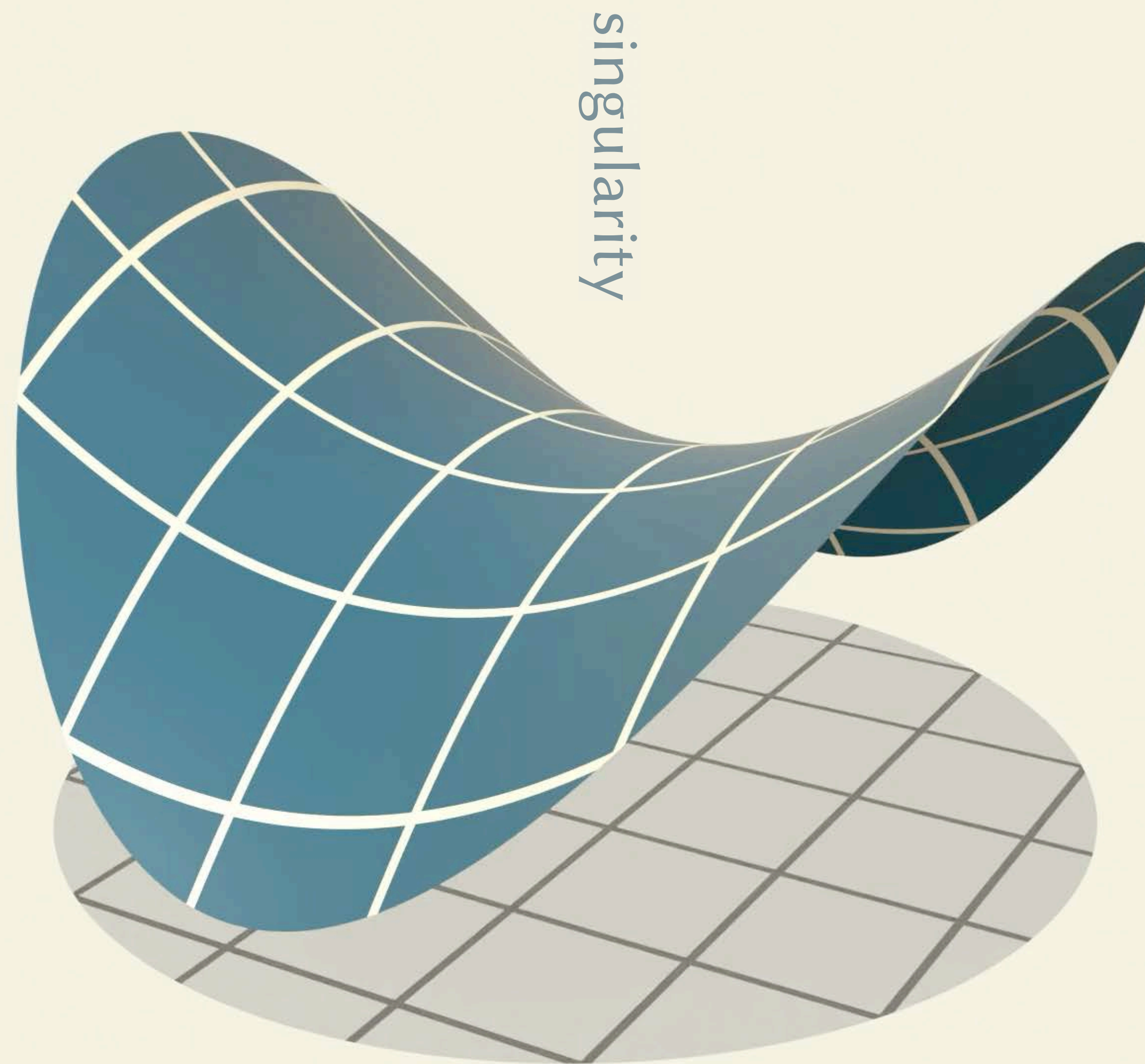
electrostatics

complex analysis

well-studied

Harmonic functions

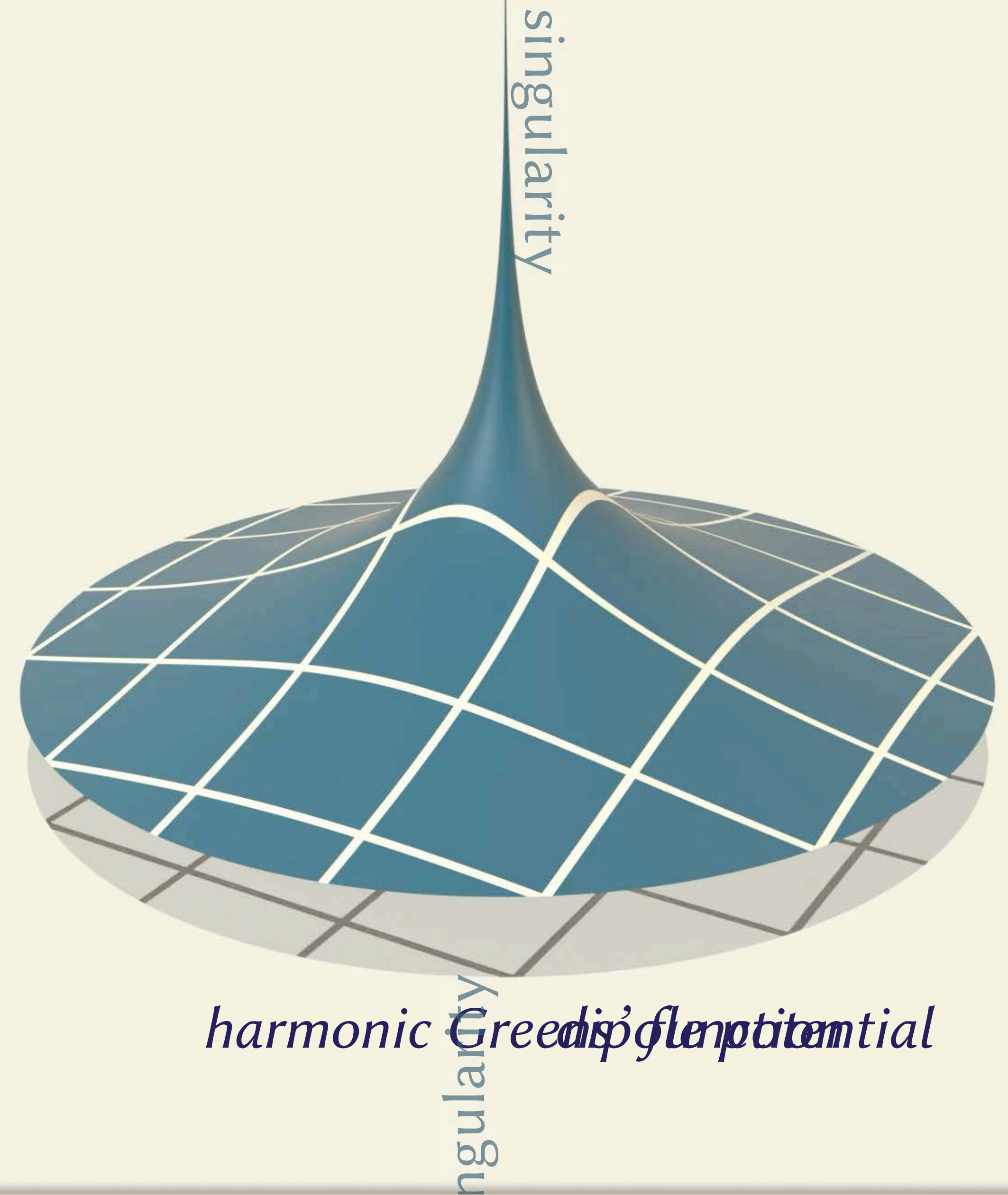
$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$$



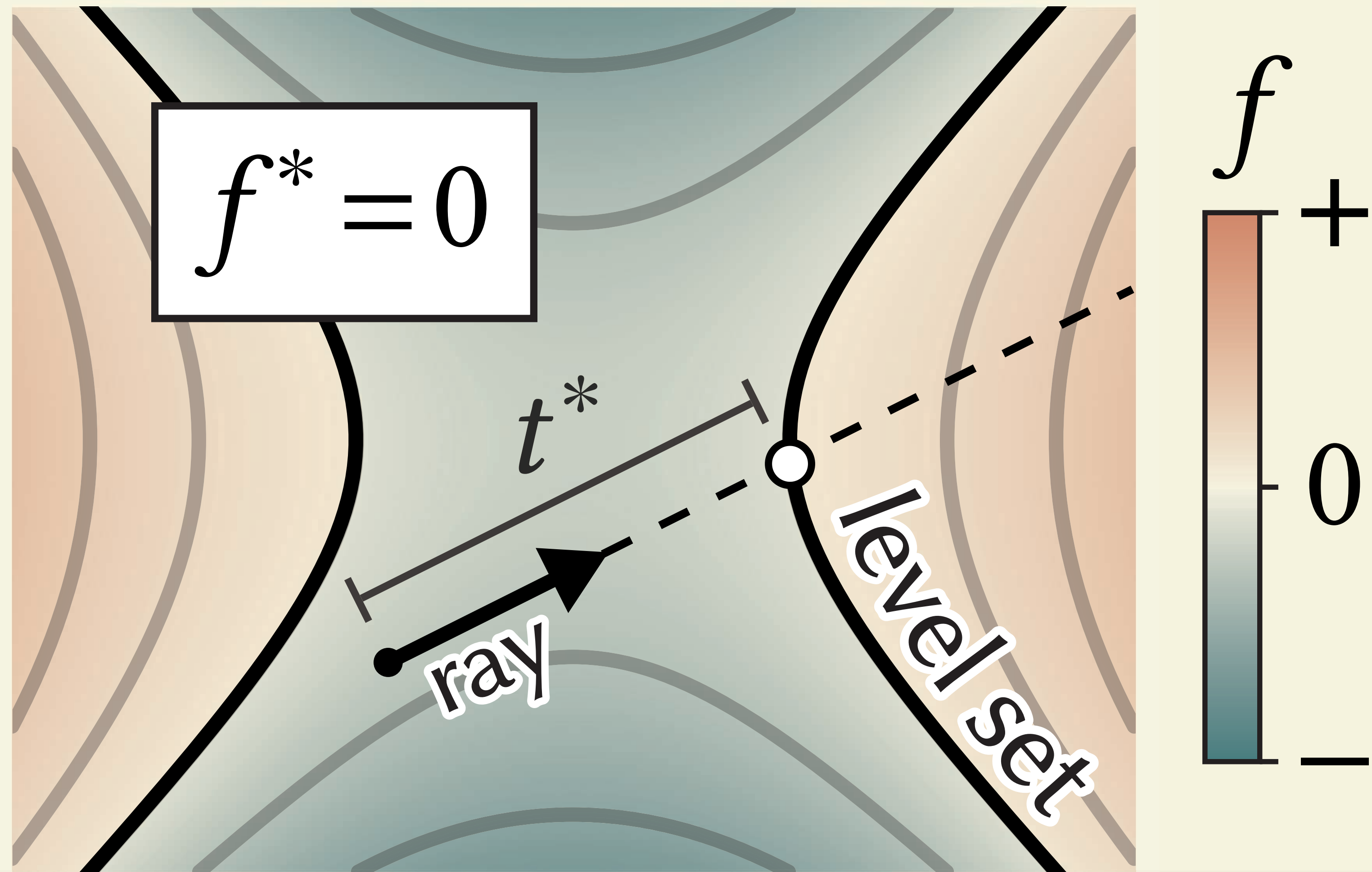
harmonic Green's function

Harmonic functions

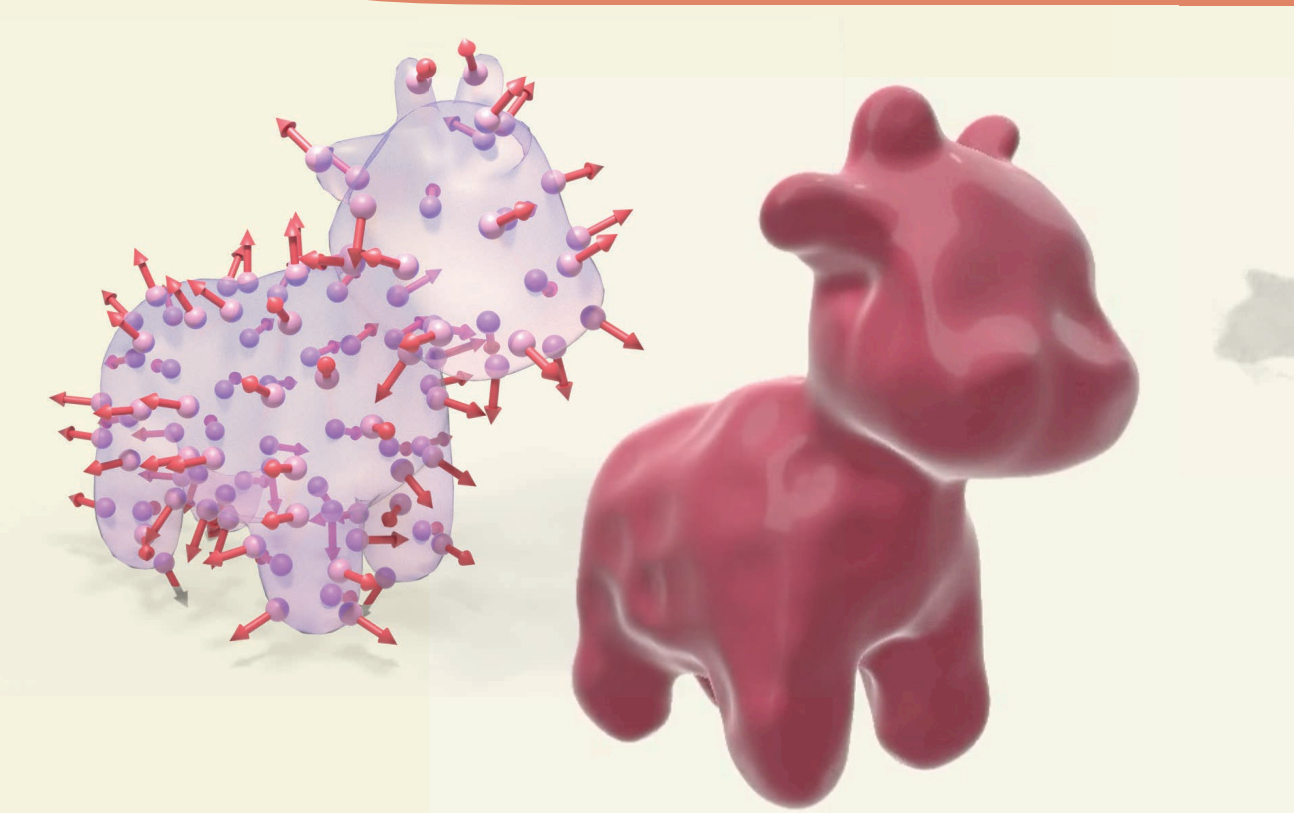
$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$$



Intersecting a ray with a level set



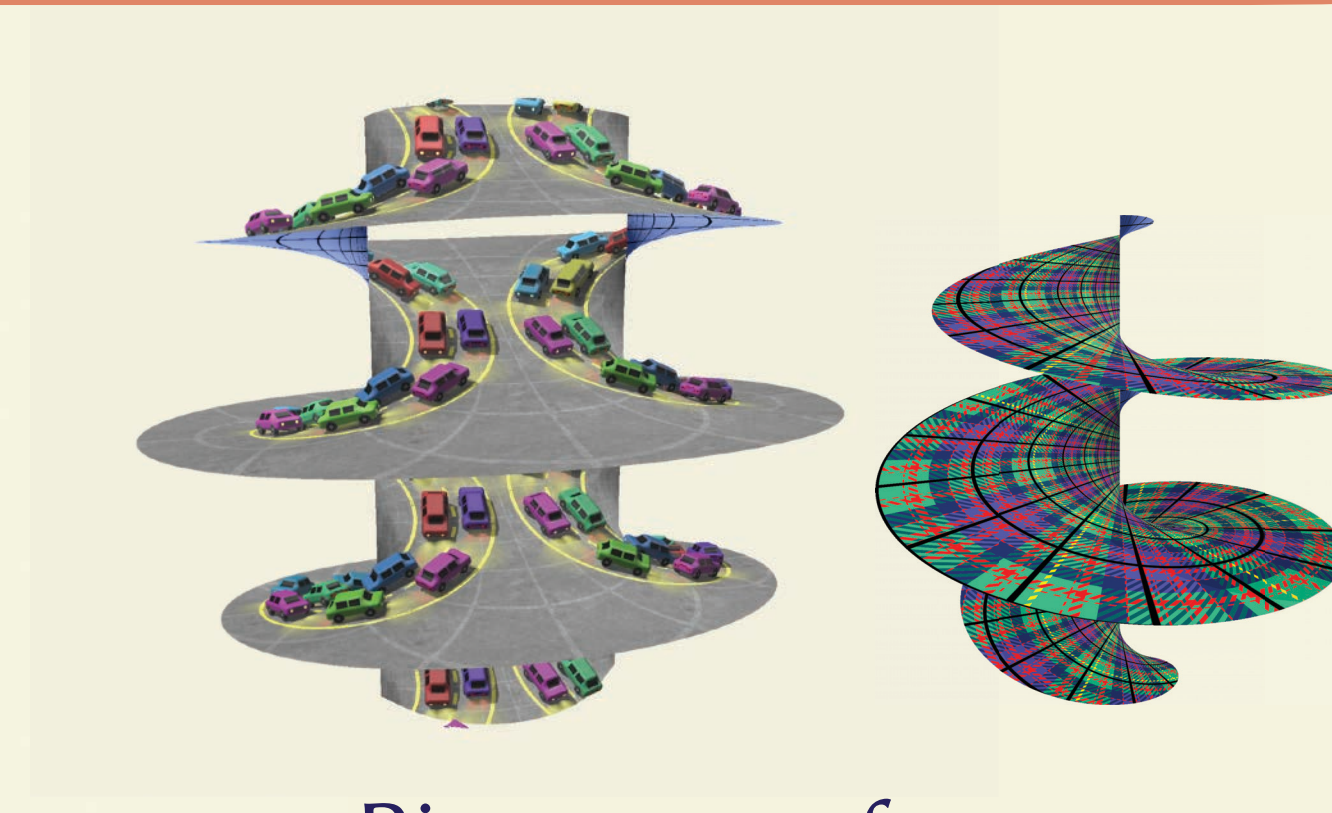
Level sets of harmonic functions show up everywhere



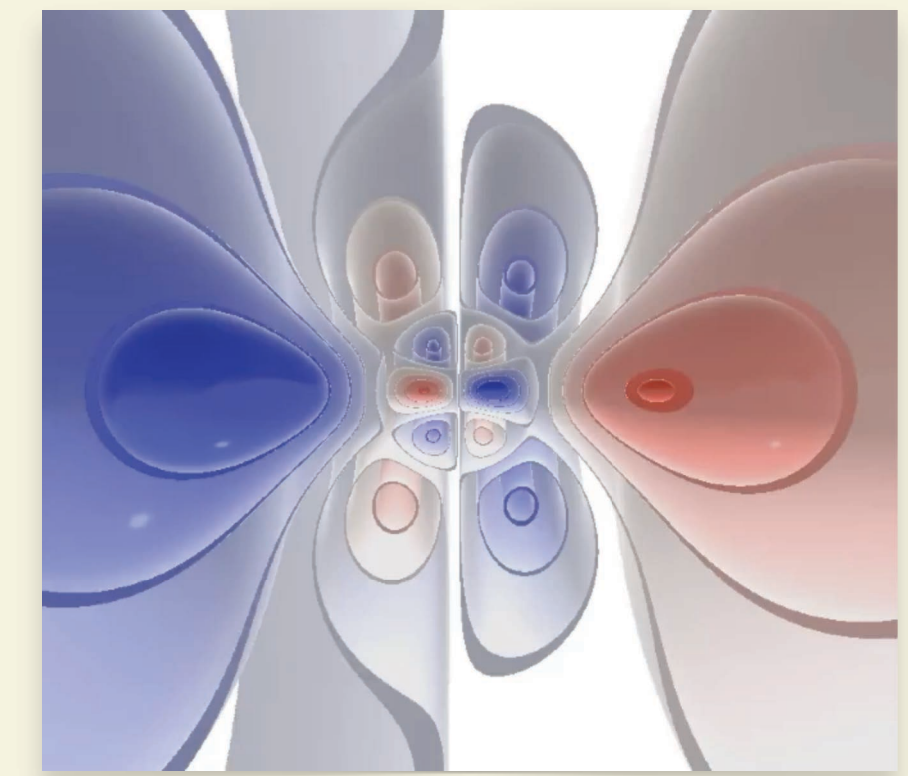
Poisson surface reconstruction
[Kazhdan *et al.* 2006]



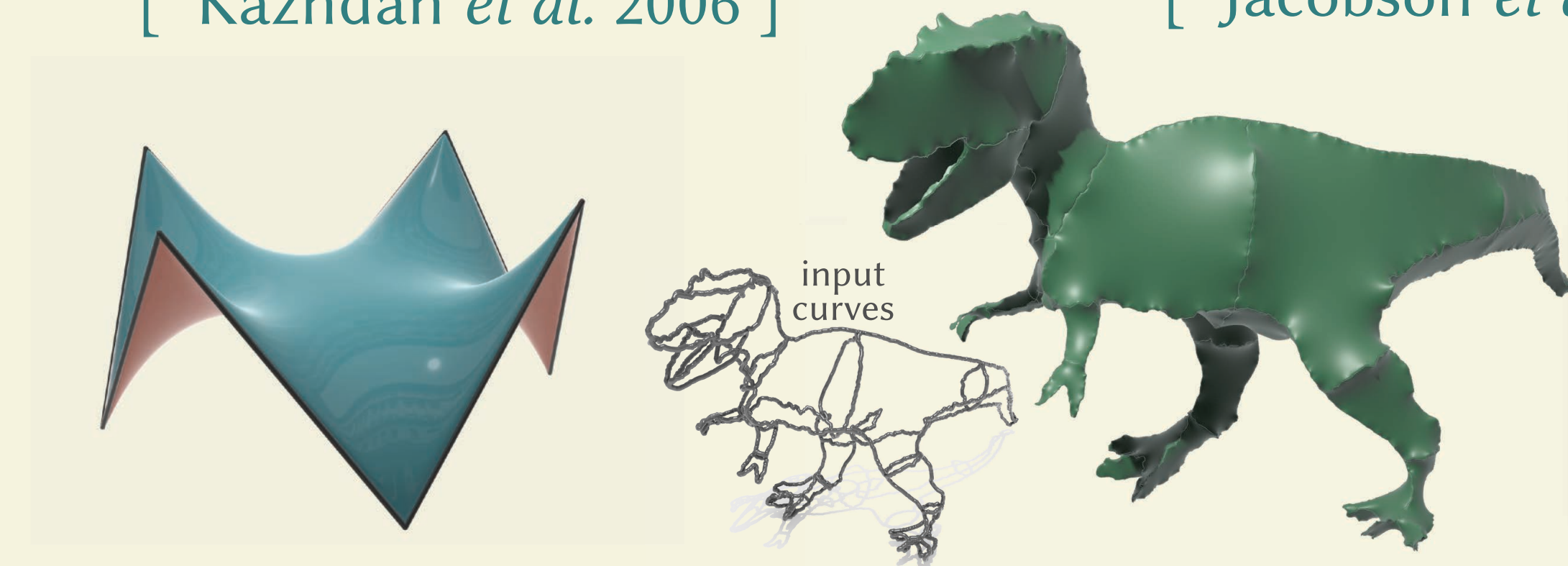
generalized winding numbers
[Jacobson *et al.* 2013]



Riemann surfaces
[Riemann 1851]

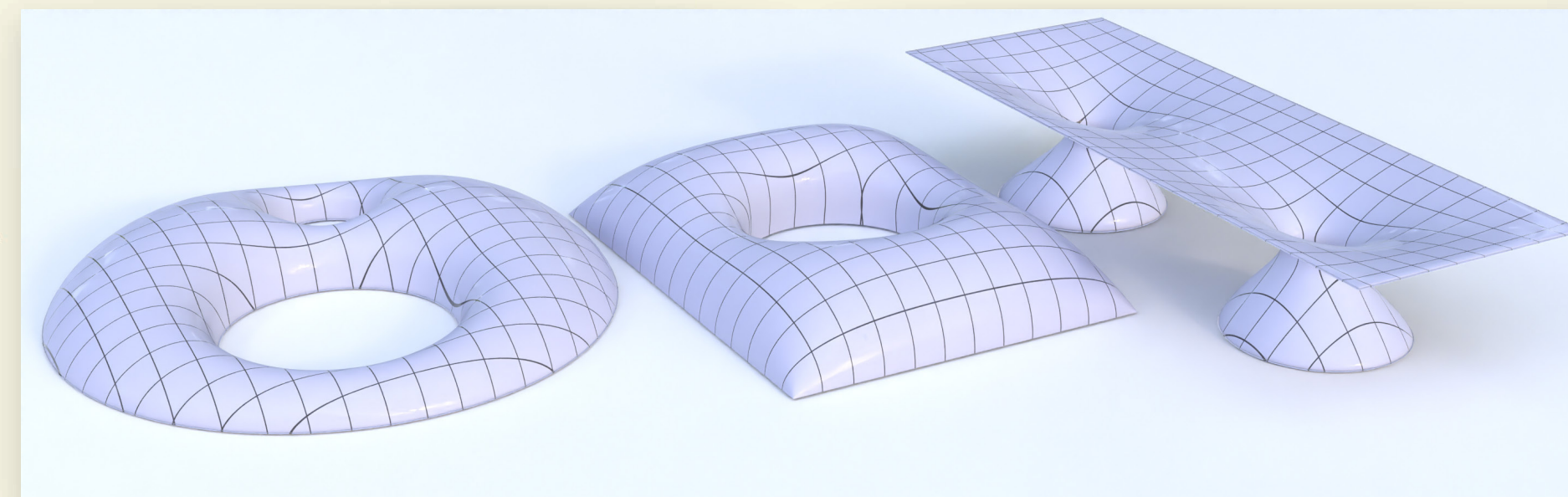


hyperspherical harmonics
[Fock 1935]

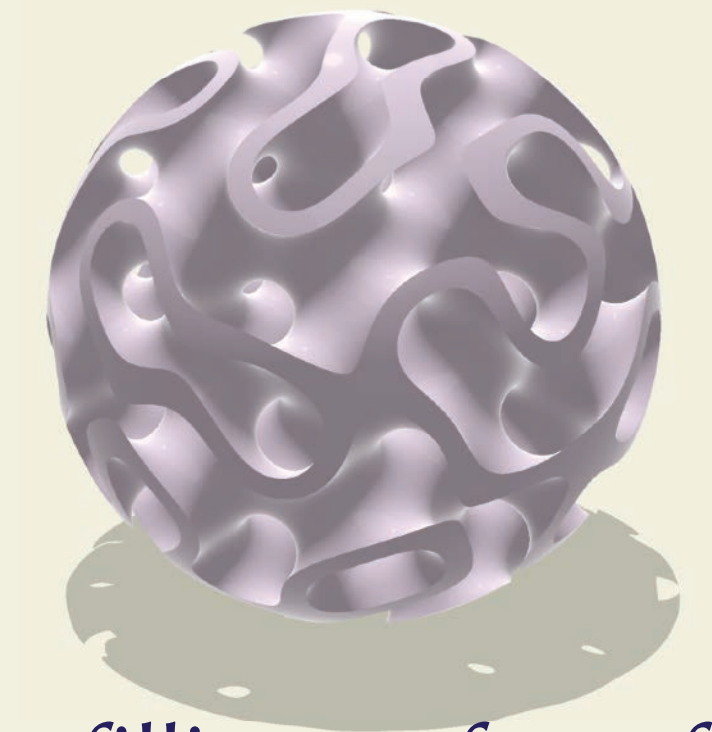


nonplanar polygons
[Maxwell 1873]

curve networks
[de Goes *et al.* 2011]



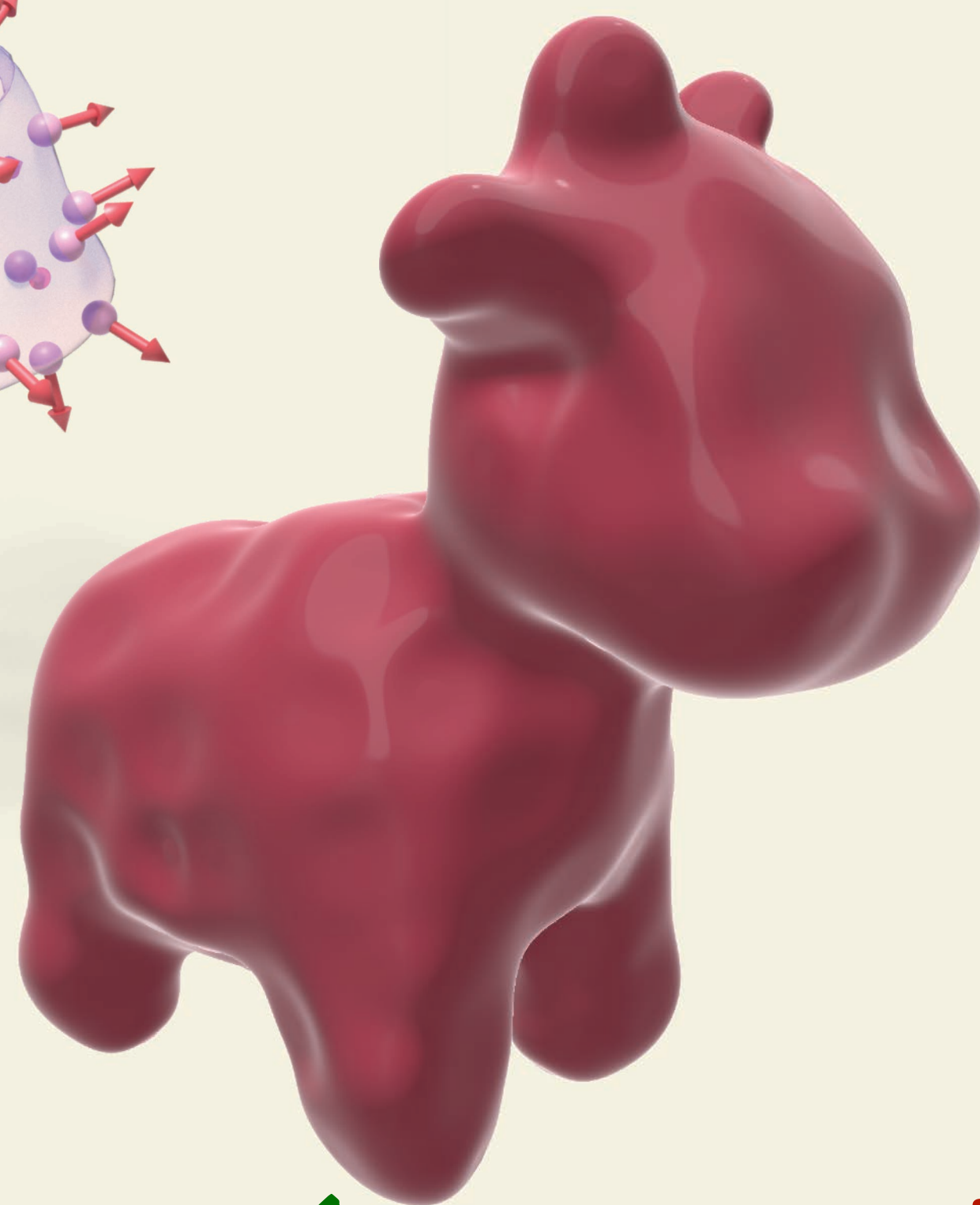
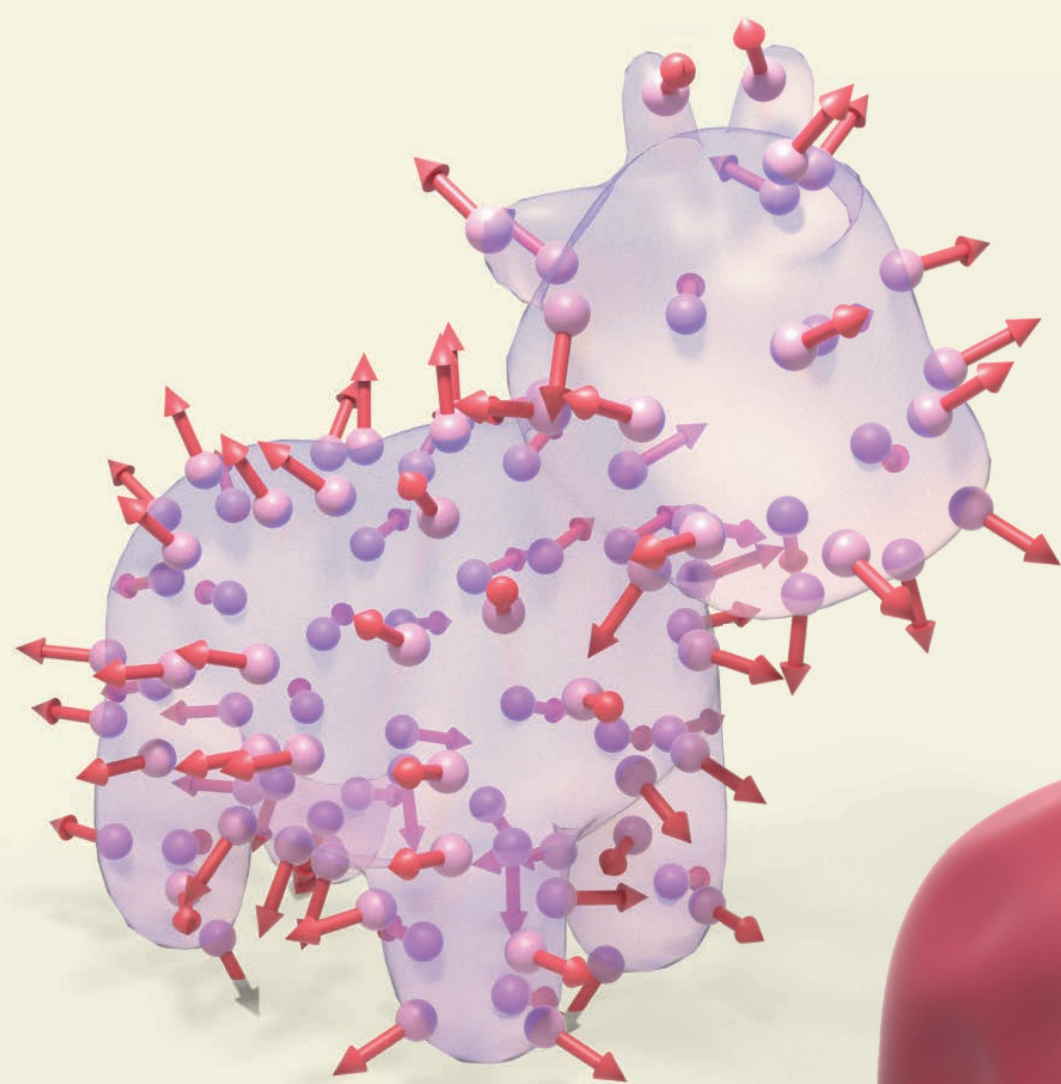
shell structures in architectural
geometry [Adiels *et al.* 2022]



space-filling surfaces for
digital fabrication

... but, they're hard to render with existing techniques

may have singularities



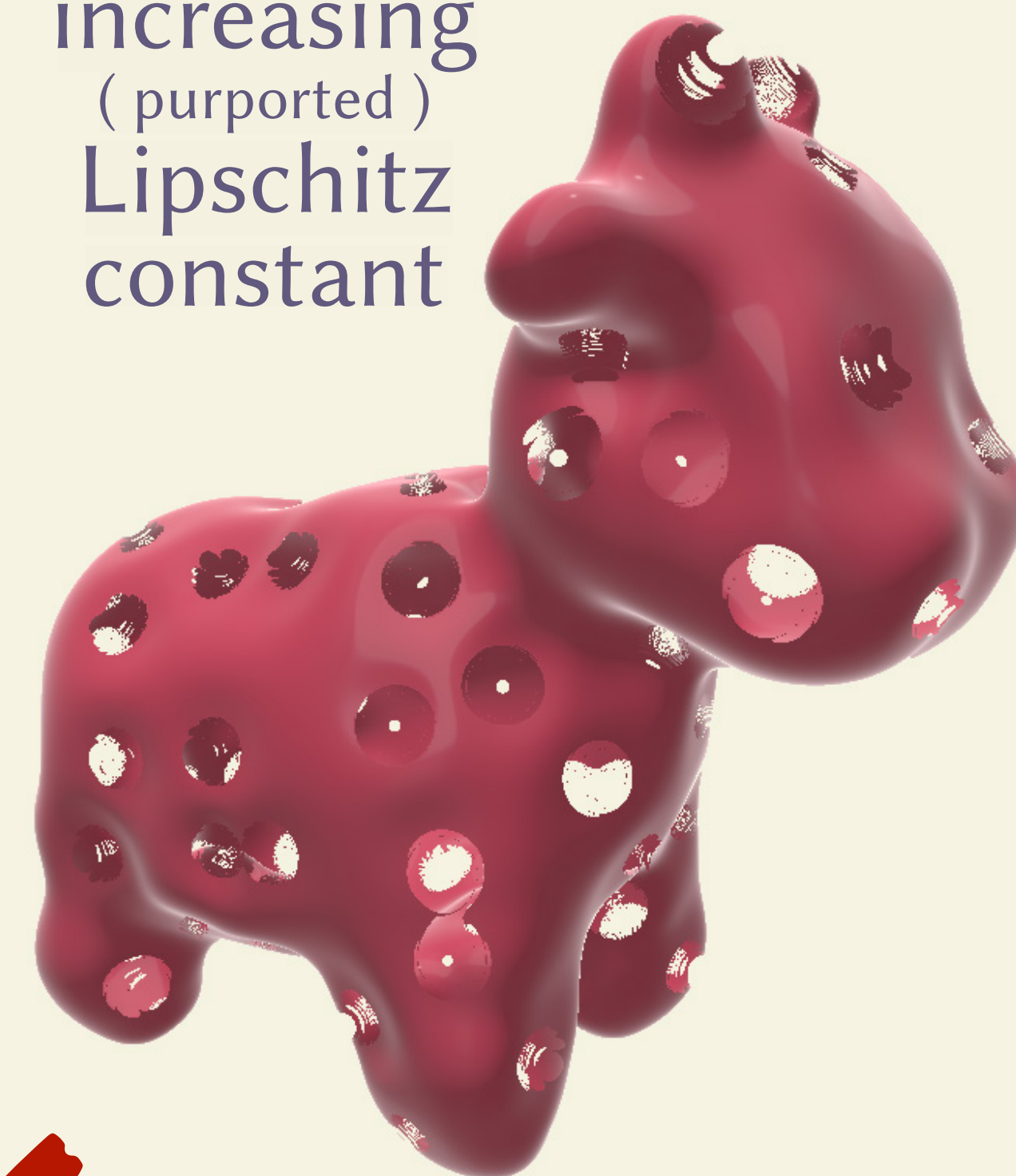
✓ ours

decreasing
step size



✗ ray marching
(with fixed step size)

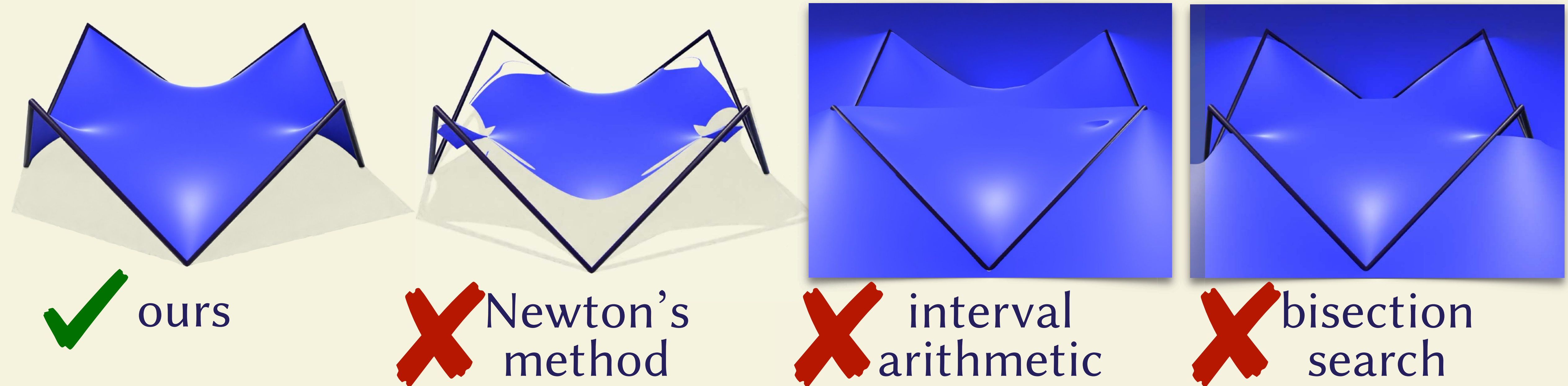
increasing
(purported)
Lipschitz
constant



✗ sphere tracing
(with purported Lipschitz constant)

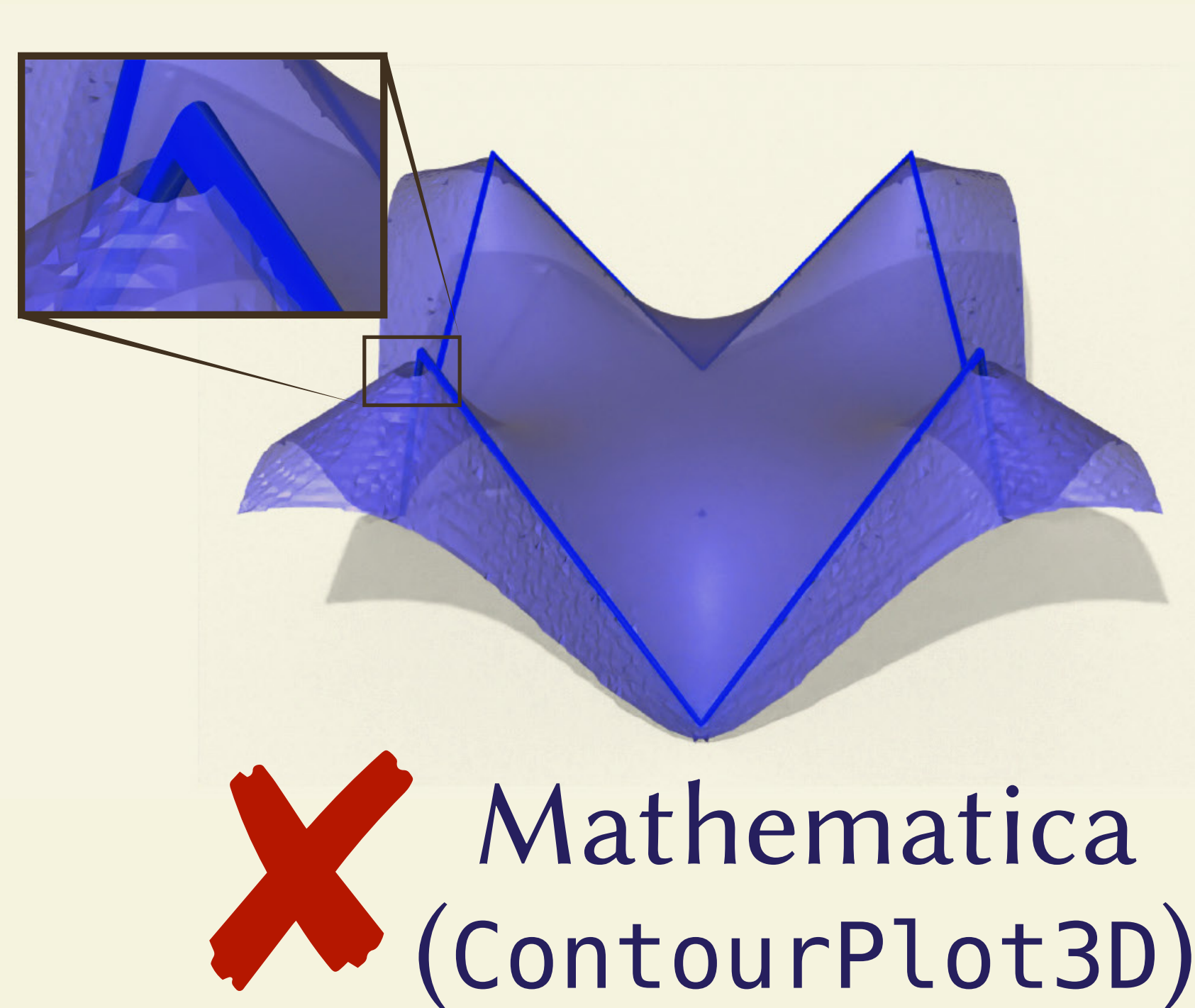
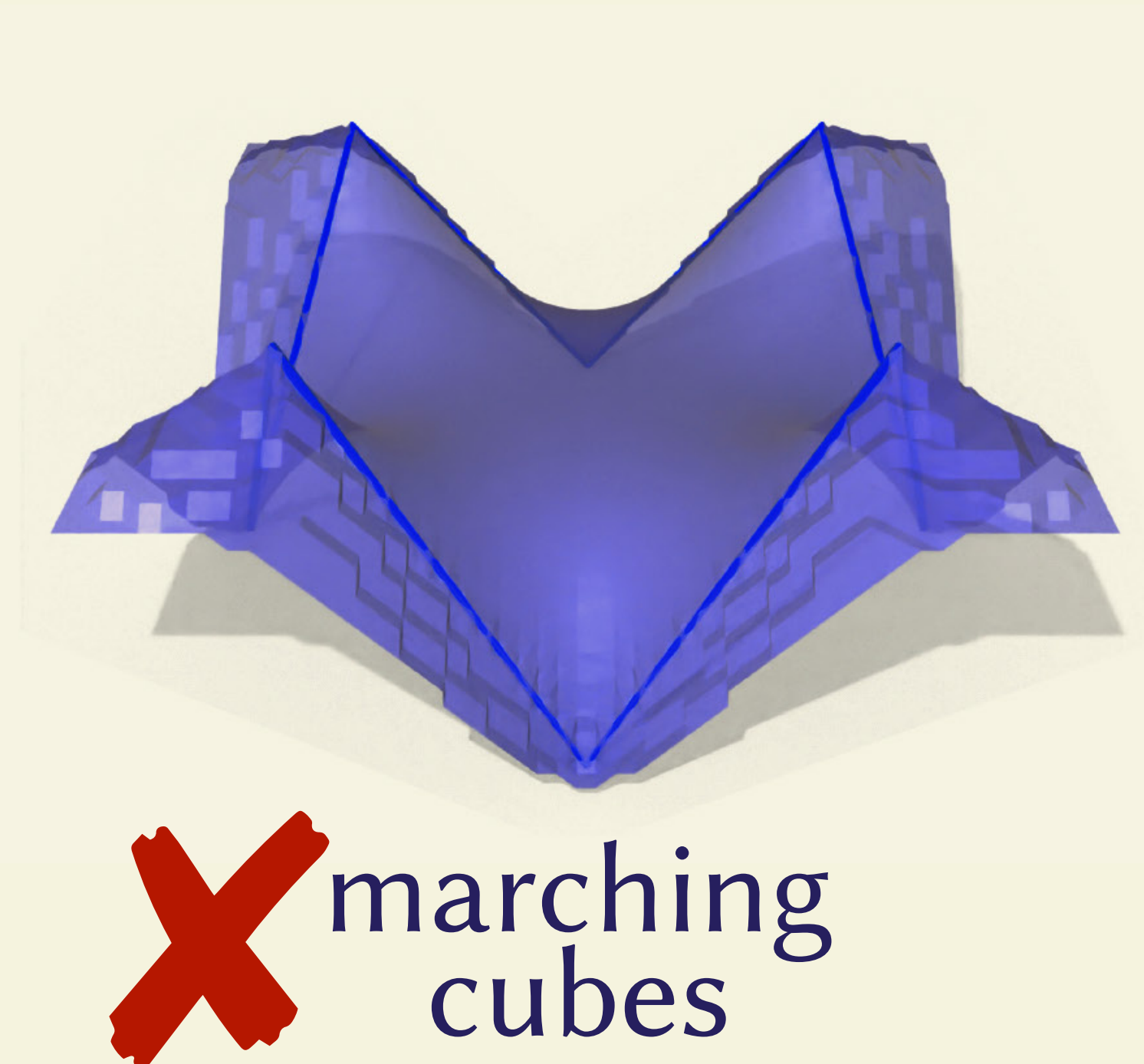
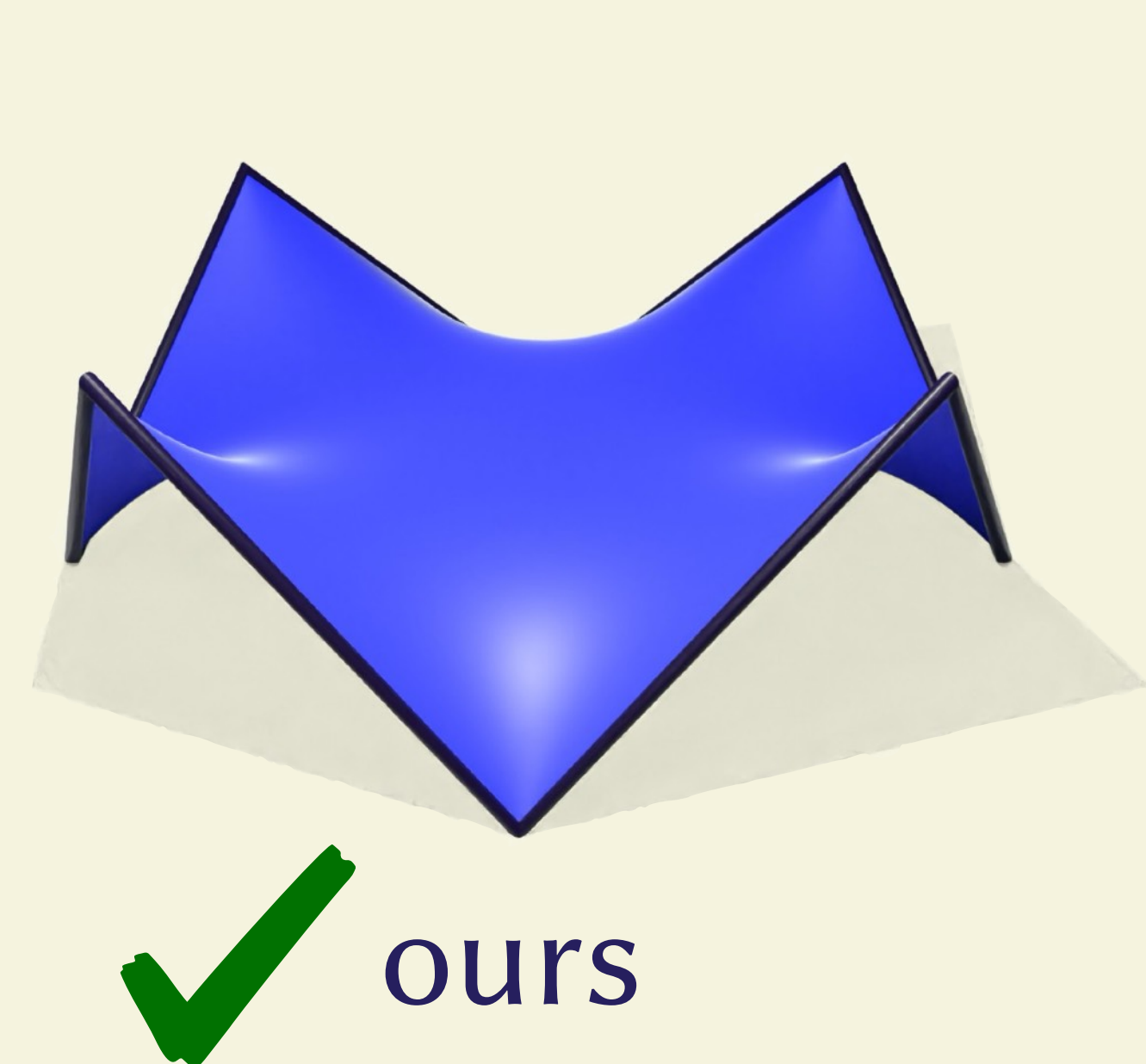
... but, they're hard to render with existing techniques

may have boundaries



... but, they're hard to render with existing techniques

may have boundaries



Sphere tracing

[Hart 1996]

$f(x)$ = distance to curve

compute intersections for *signed distance functions (SDFs)*

Sphere tracing

[Hart 1996]



compute intersections for *signed distance functions (SDFs)*

Sphere tracing: beyond SDFs

[Hart 1996]

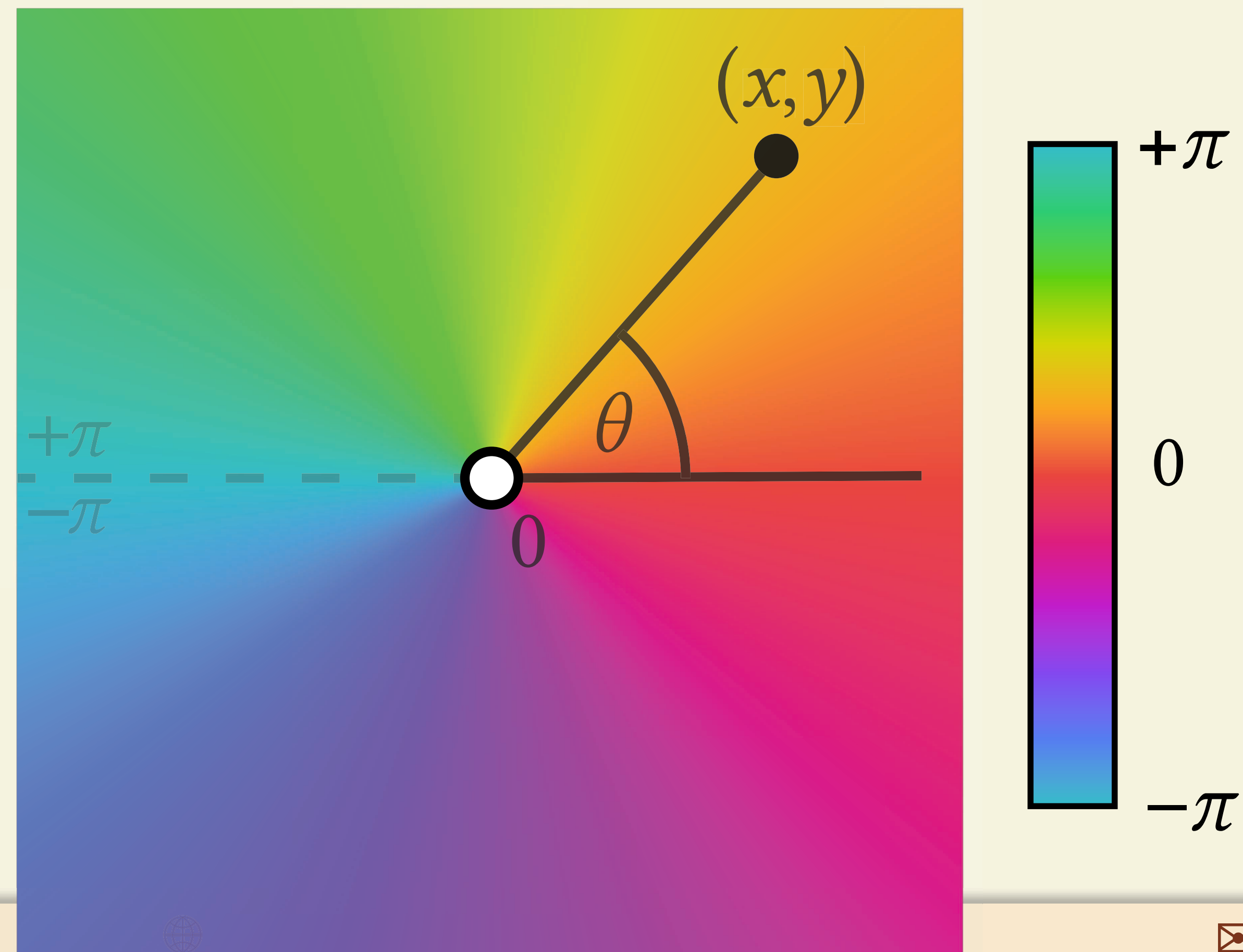
- Easy to generalize to *Lipschitz* functions:
(essentially, $|\nabla f| \leq L$)
- Important fact:
 $|f(x) - f(y)| \leq L|x - y|$
- provides a conservative bound on distance



[Inigo Quilez 2015]

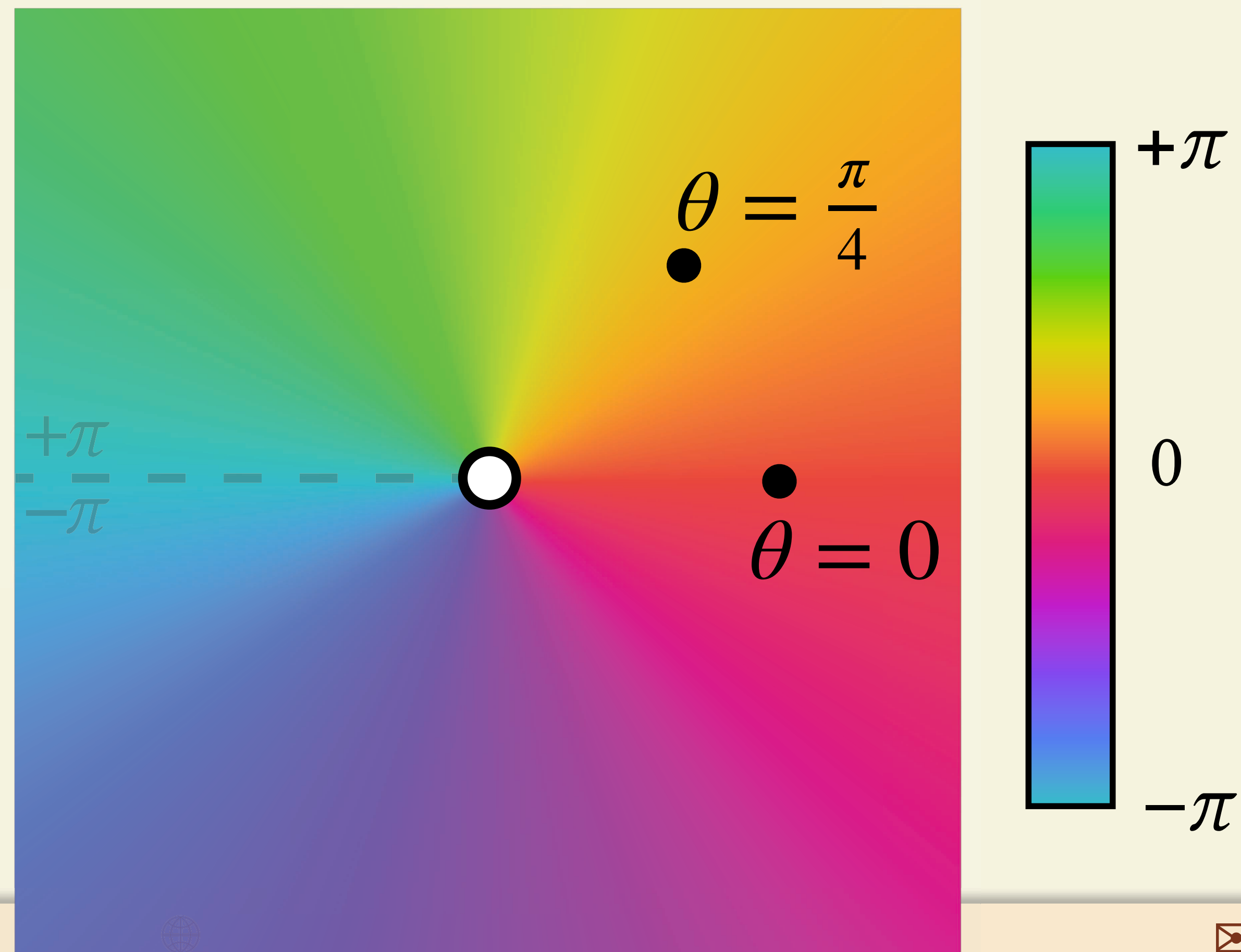
Problem: many harmonic functions are not Lipschitz

$$\theta(x, y) = \text{atan2}(y, x)$$



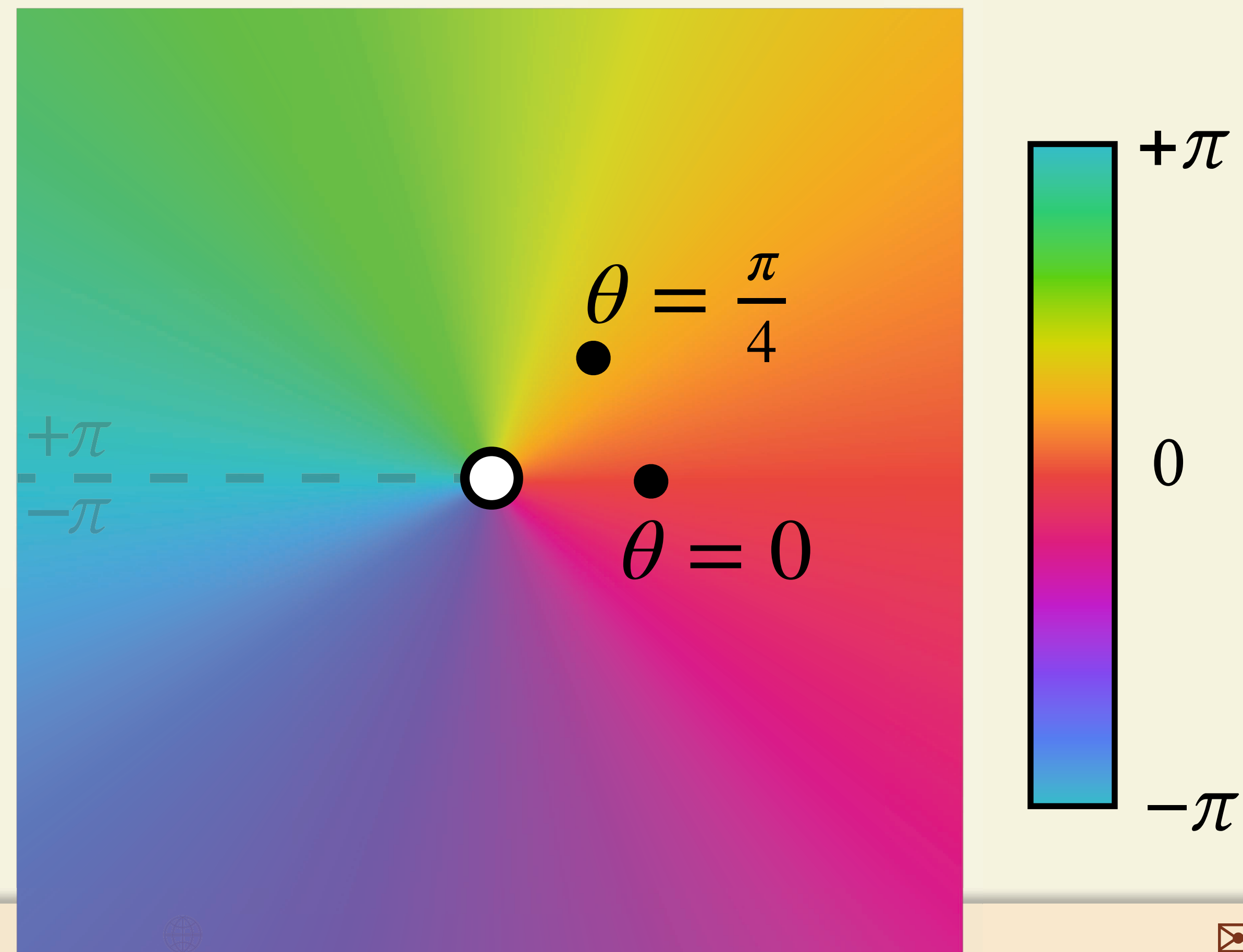
Problem: many harmonic functions are not Lipschitz

$$\theta(x, y) = \text{atan2}(y, x)$$



Problem: many harmonic functions are not Lipschitz

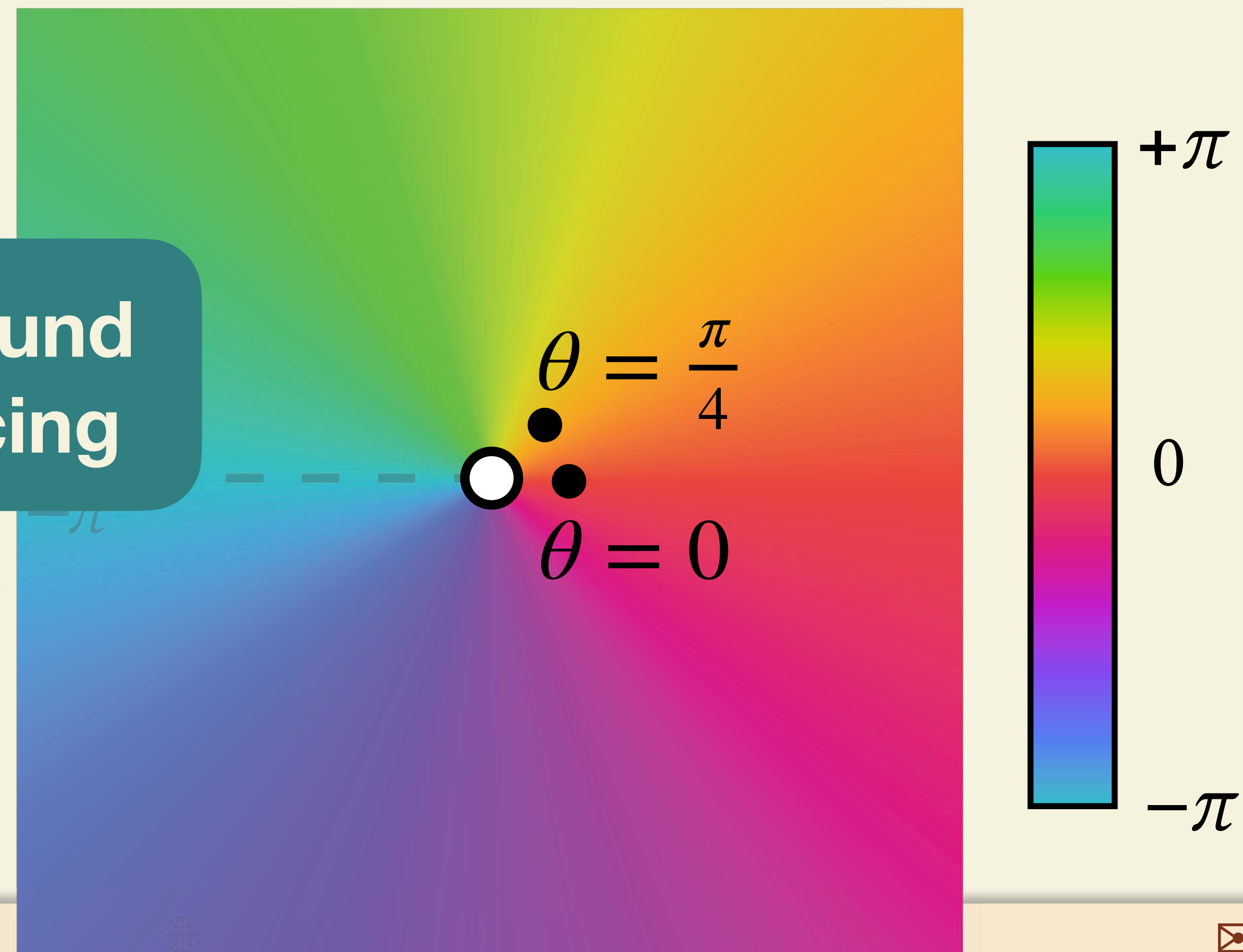
$$\theta(x, y) = \text{atan2}(y, x)$$



Problem: many harmonic functions are not Lipschitz

$$\theta(x, y) = \text{atan2}(y, x)$$

no distance bound
for sphere tracing



Main idea: get distance bounds from Harnack's inequality

Let f be a positive harmonic function on a ball:

$$\frac{1 - r/R}{(1 + r/R)^2} f(x) \leq f(y) \leq \frac{1 + r/R}{(1 - r/R)^2} f(x)$$

lower bound

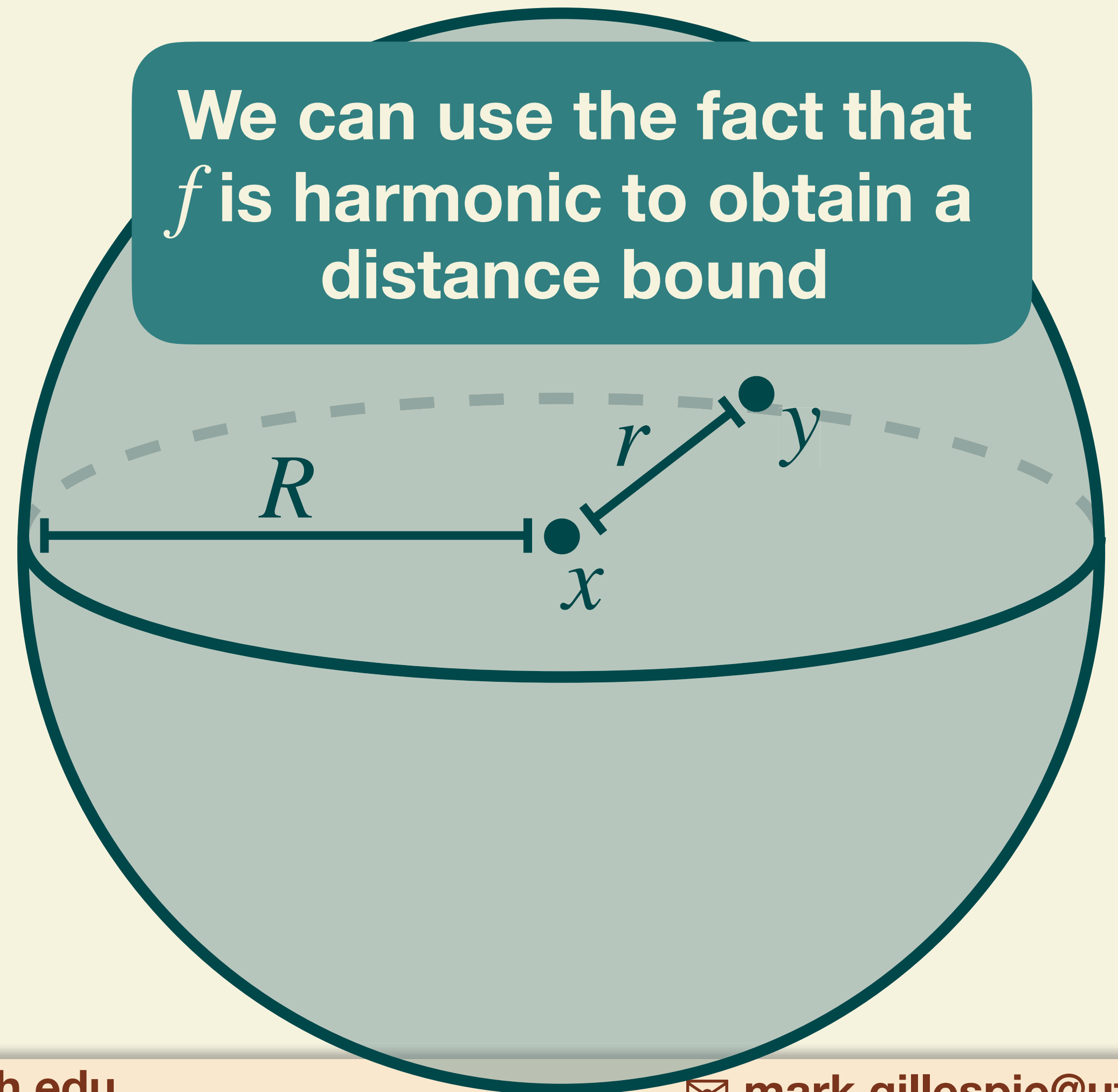
upper bound

always safe to take step of size

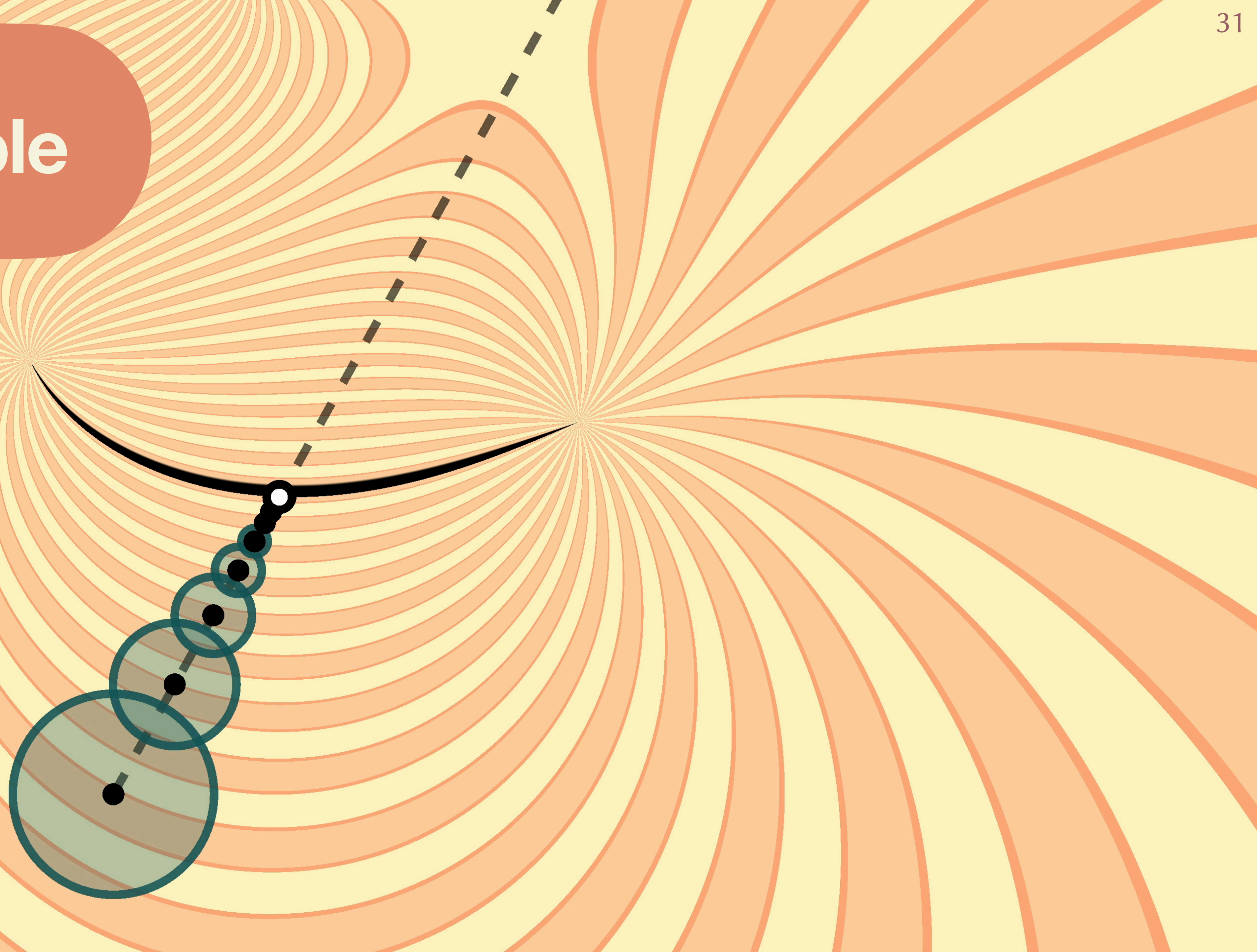
$$\frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$$

where $a = \frac{f(x)}{f^*}$

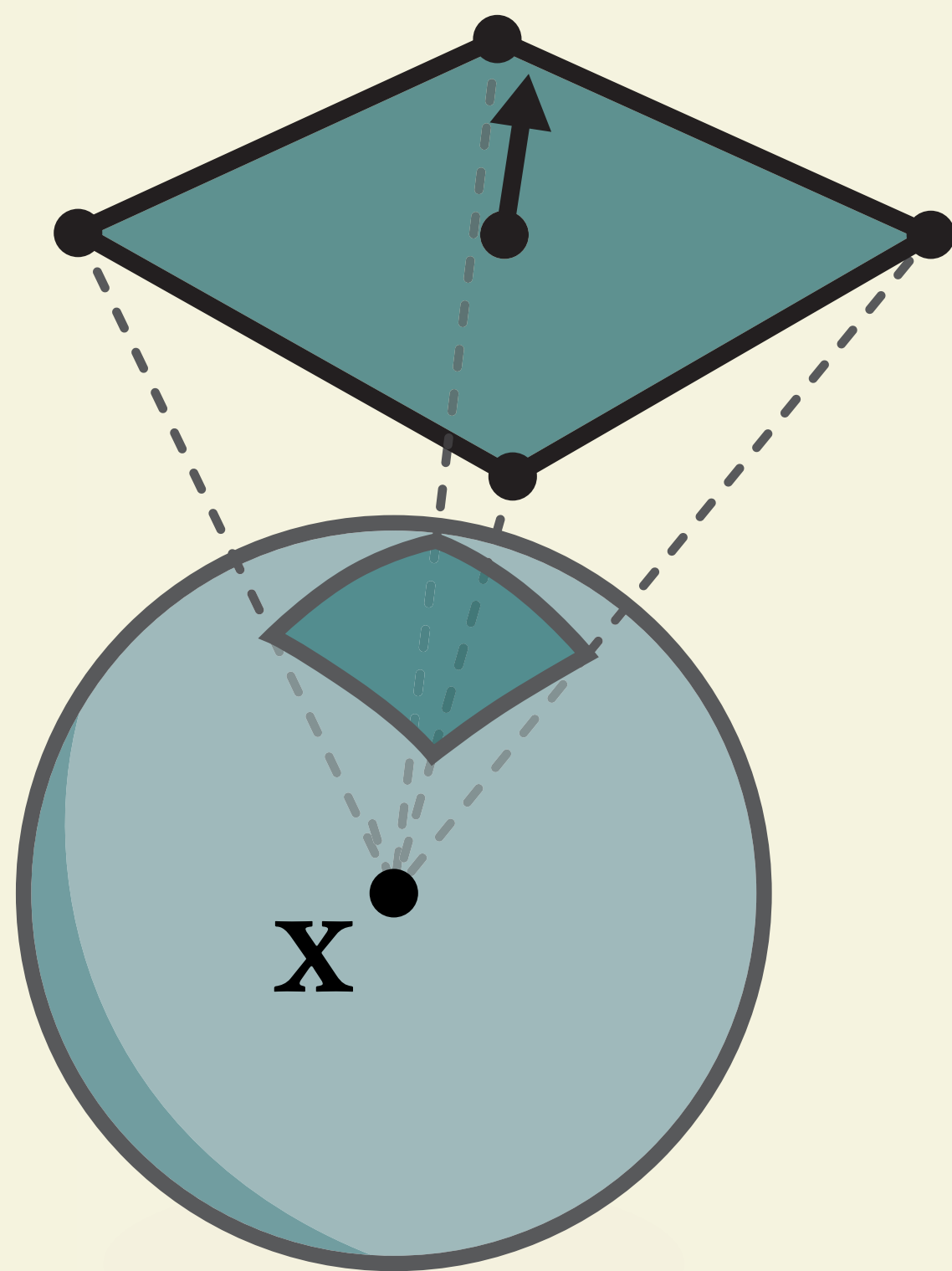
We can use the fact that f is harmonic to obtain a distance bound



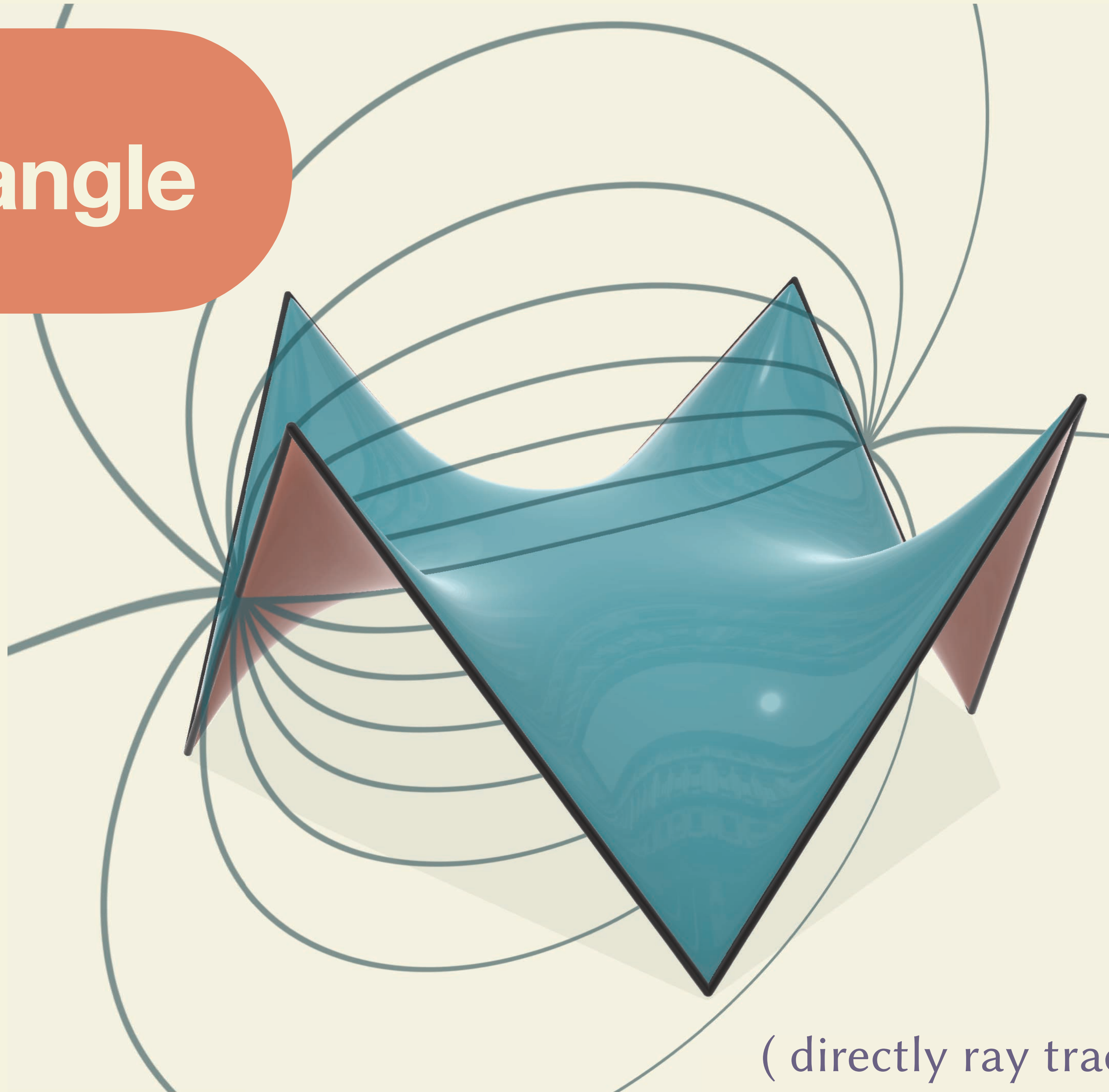
2D example



Signed solid angle

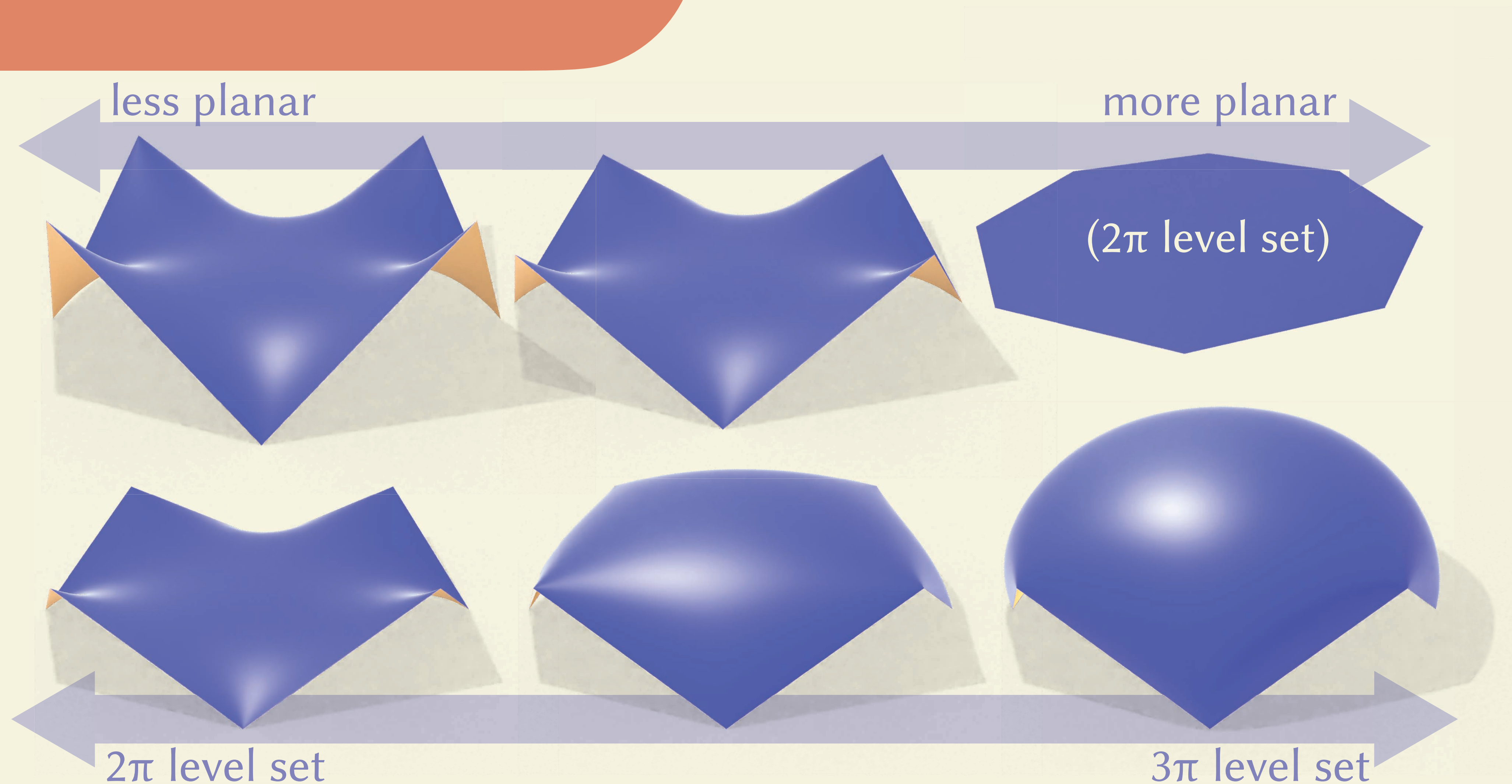


signed solid angle

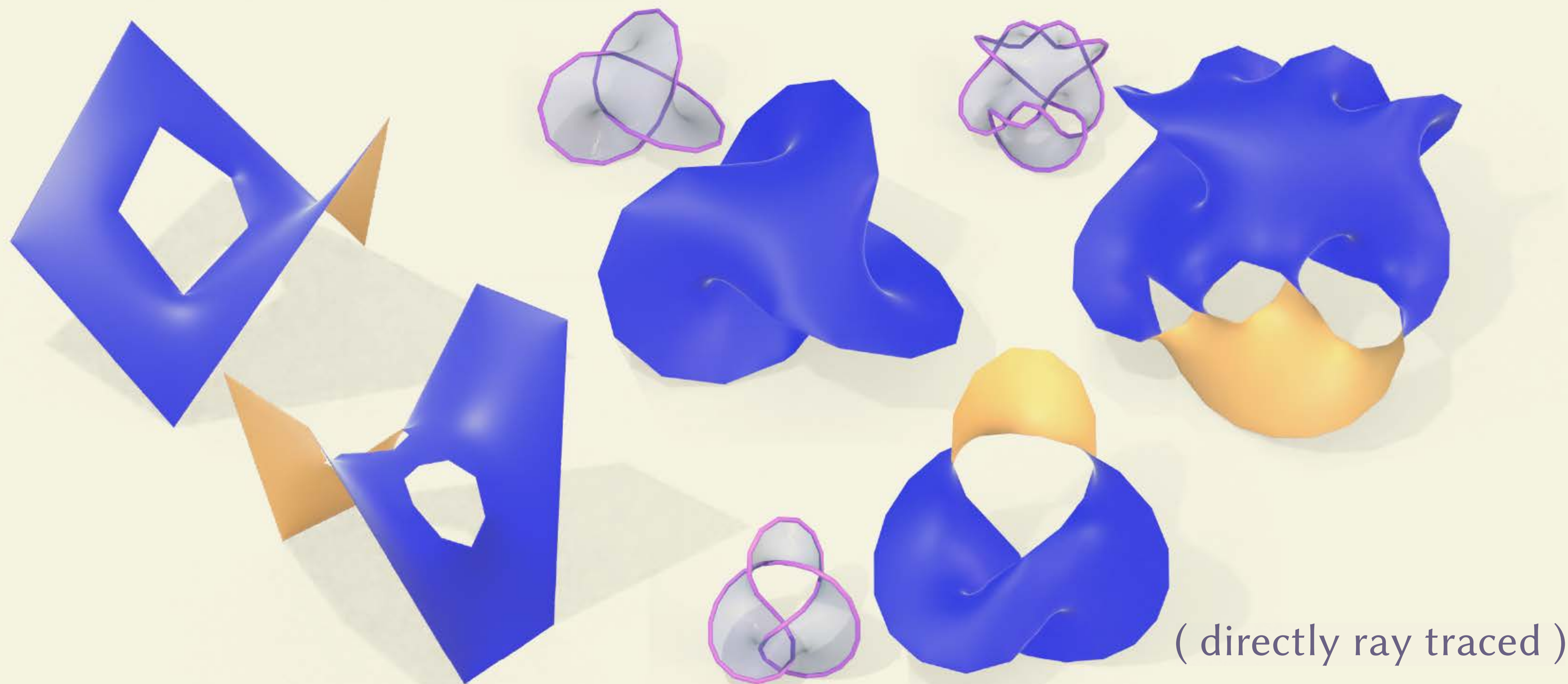


(directly ray traced)

Signed solid angle

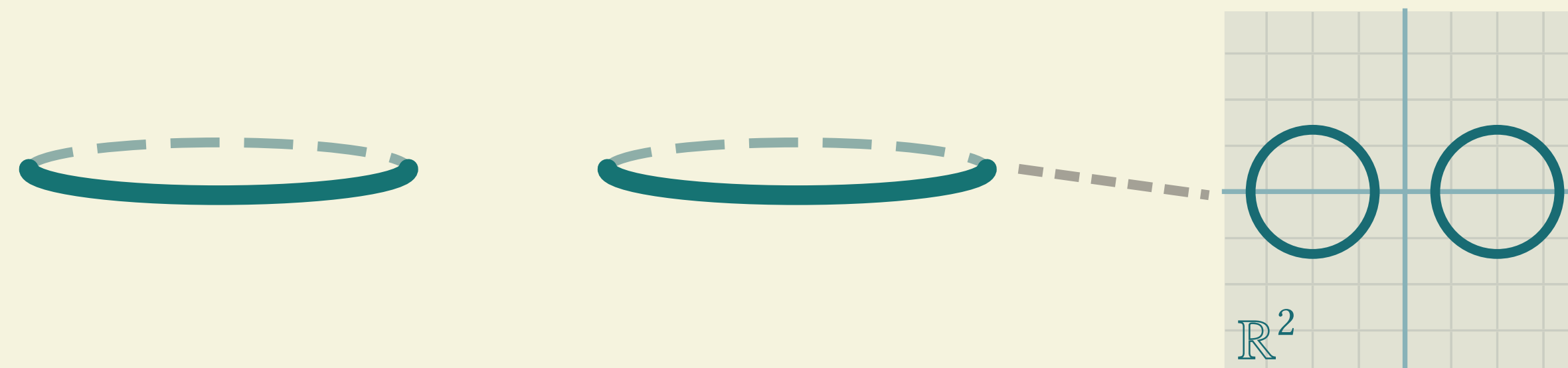
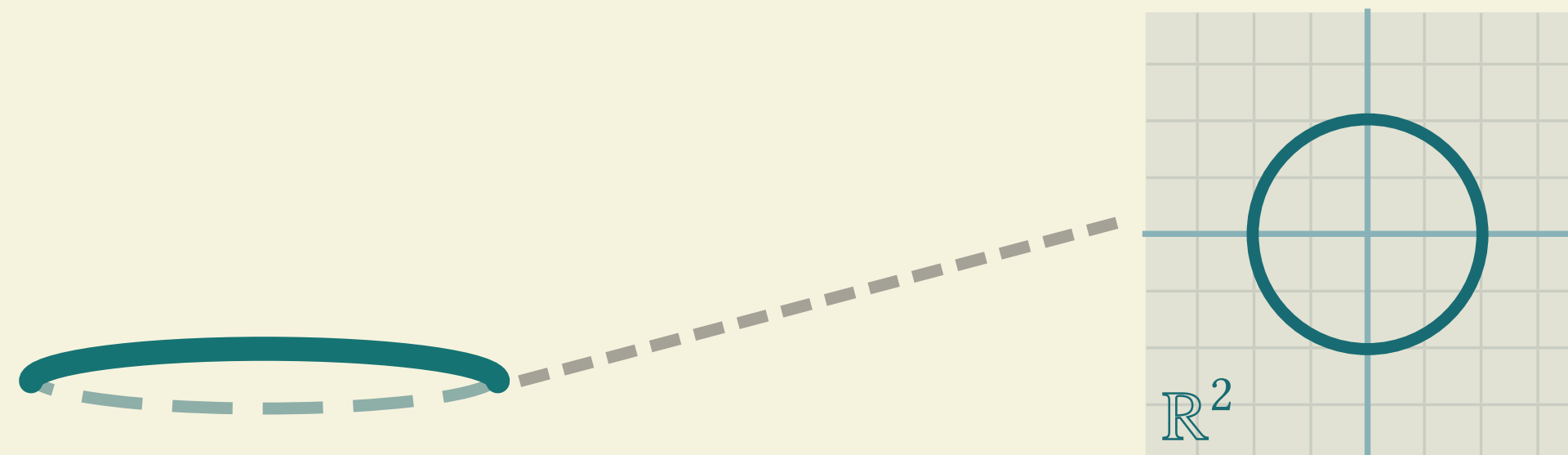


General nonplanar polygons

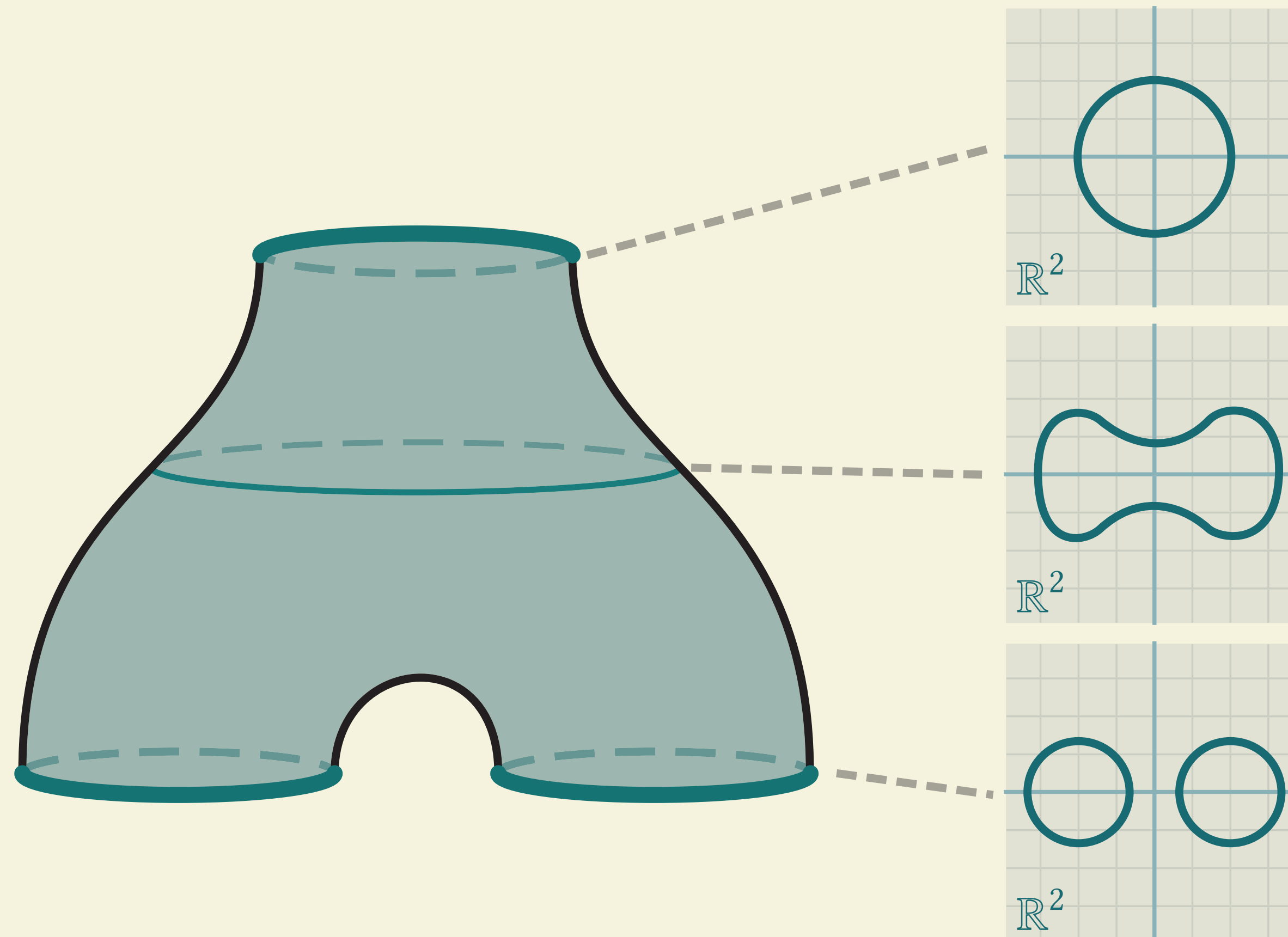


(directly ray traced)

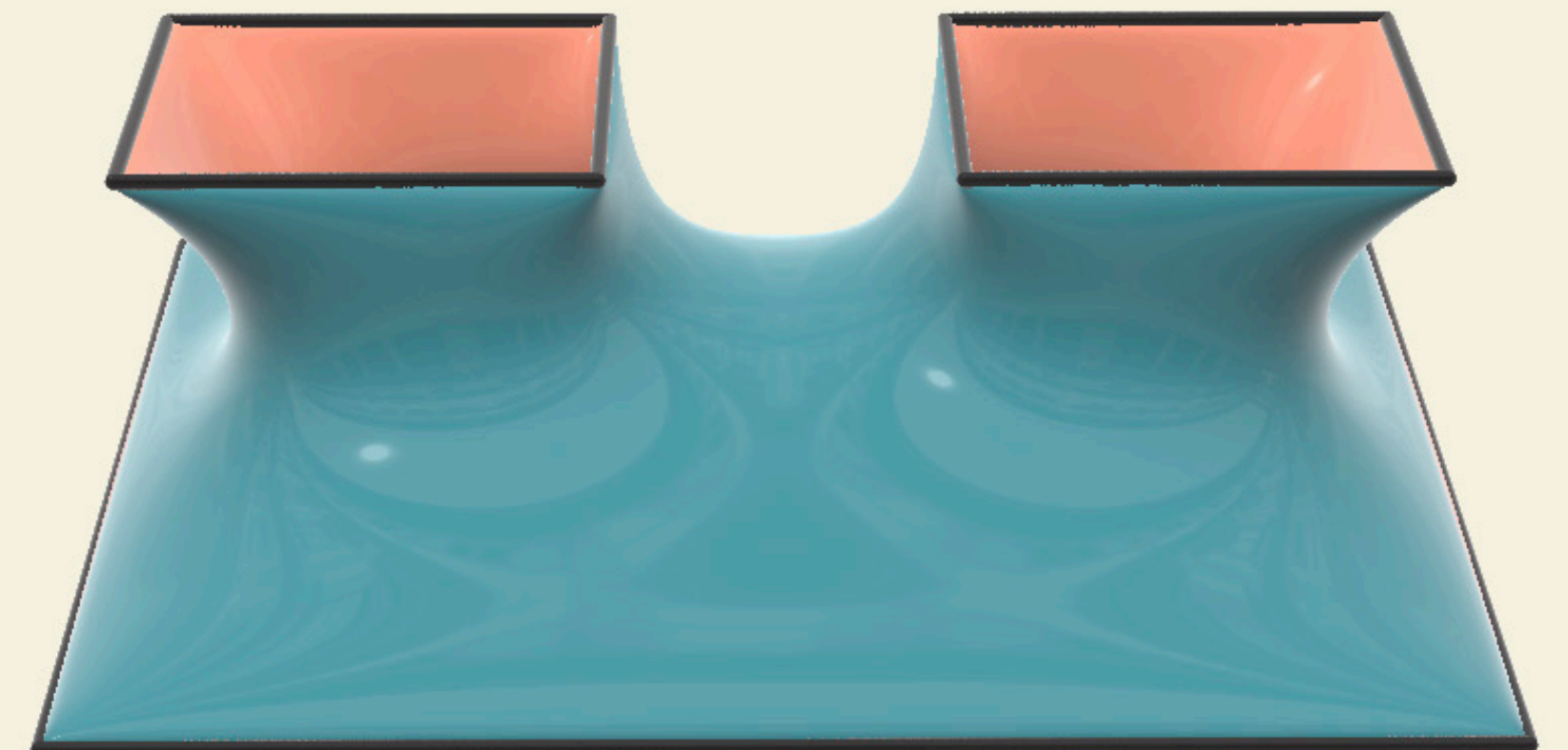
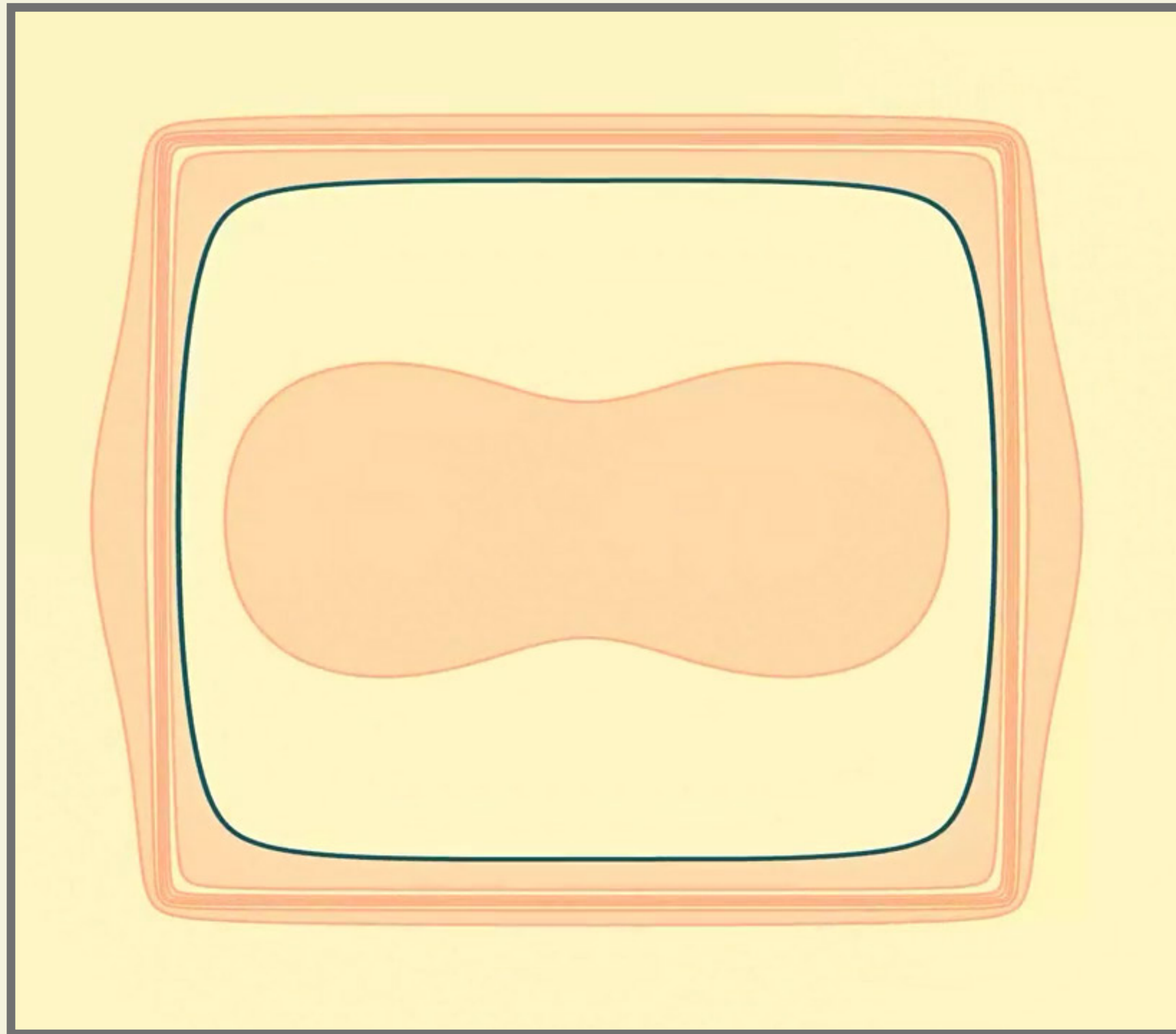
Shape Morphing with Solid Angle



Shape Morphing with Solid Angle

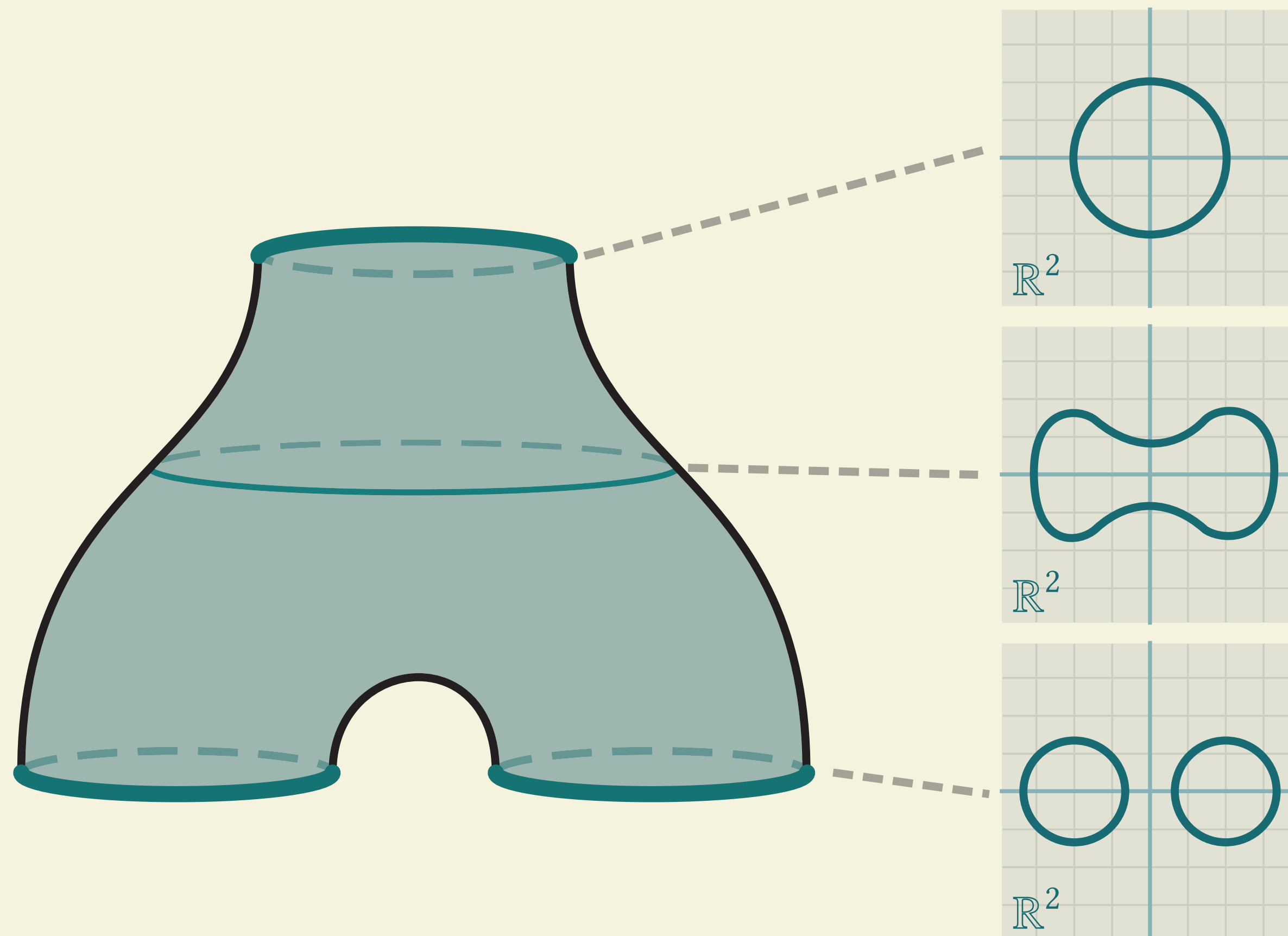


Morphing Polygons



Morphing 3D Surfaces

do the same, but one dimension up



Challenge: 4D solid angle

3D solid angle formula

[van Oosterom & Strackee 1983]

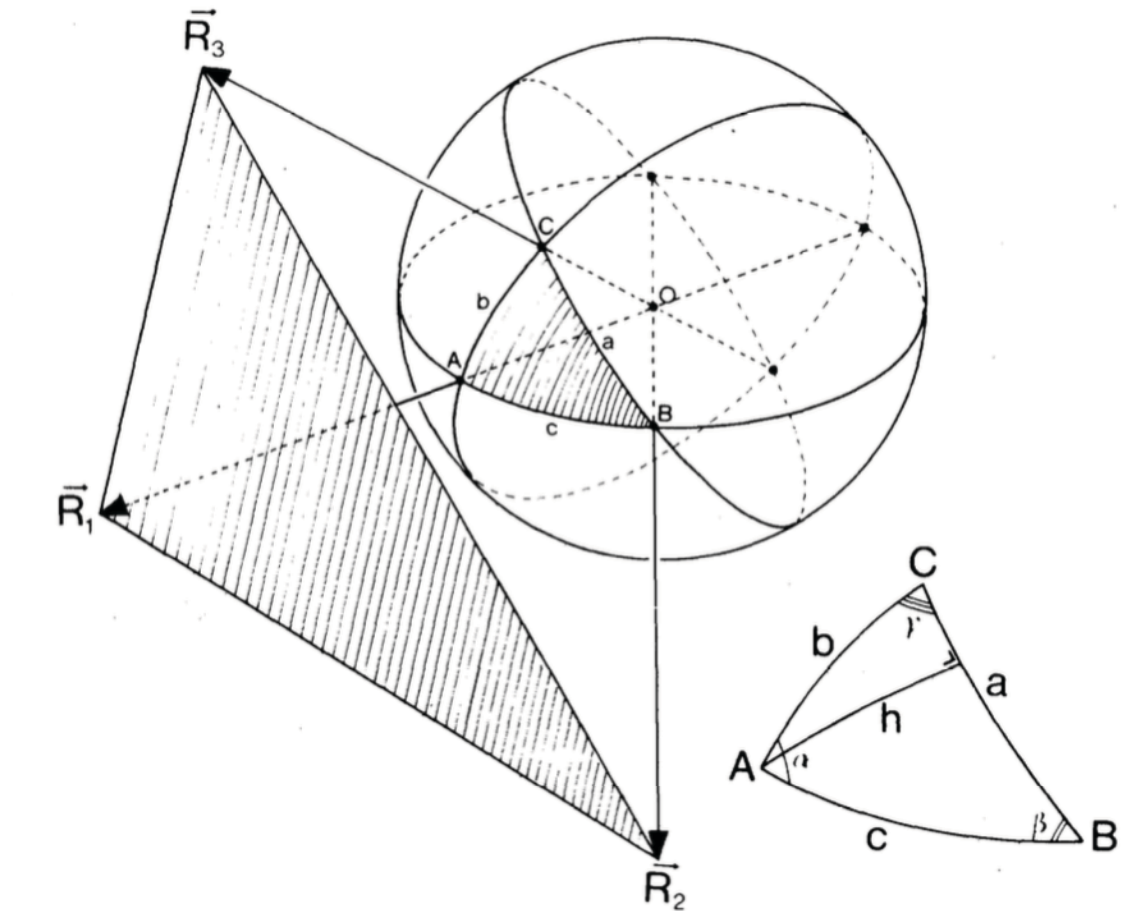


Fig. 1. Note: Characters with arrows appear boldface in text.

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{[\mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3]}{R_1 R_2 R_3 + (\mathbf{R}_1 \cdot \mathbf{R}_2) R_3 + (\mathbf{R}_1 \cdot \mathbf{R}_3) R_2 + (\mathbf{R}_2 \cdot \mathbf{R}_3) R_1}.$$

Theorem 2.2. Let $\Omega \subseteq \mathbb{R}^n$ be a solid angle spanned by unit vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, let V be the matrix whose i th column is $\mathbf{v}_i$, and let $\alpha_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$, as above. Let T_α be the following infinite multivariable Taylor series:

$$T_\alpha = \frac{|\det V|}{(4\pi)^{n/2}} \sum_{\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i < j} a_{ij}}}{\prod_{i < j} a_{ij}!} \prod_i \Gamma\left(\frac{1 + \sum_{m \neq i} a_{im}}{2}\right) \right] \alpha^{\mathbf{a}}.$$

The series T_α agrees with $\tilde{V}_\Omega$, the normalized measure of solid angle Ω , wherever T_α converges.

[Ribando 2006]

4D solid angle formula?

Voronoi Crochet Pattern Design



Happypotamus & Friends



[<http://fiekshaakselsenmeer.blogspot.com/2014/07/happypotamus-4.html>]



mrscuddles.blogspot.com

[<http://mrscuddles.blogspot.com/2014/04/puff-magic-stegosaurus.html>]

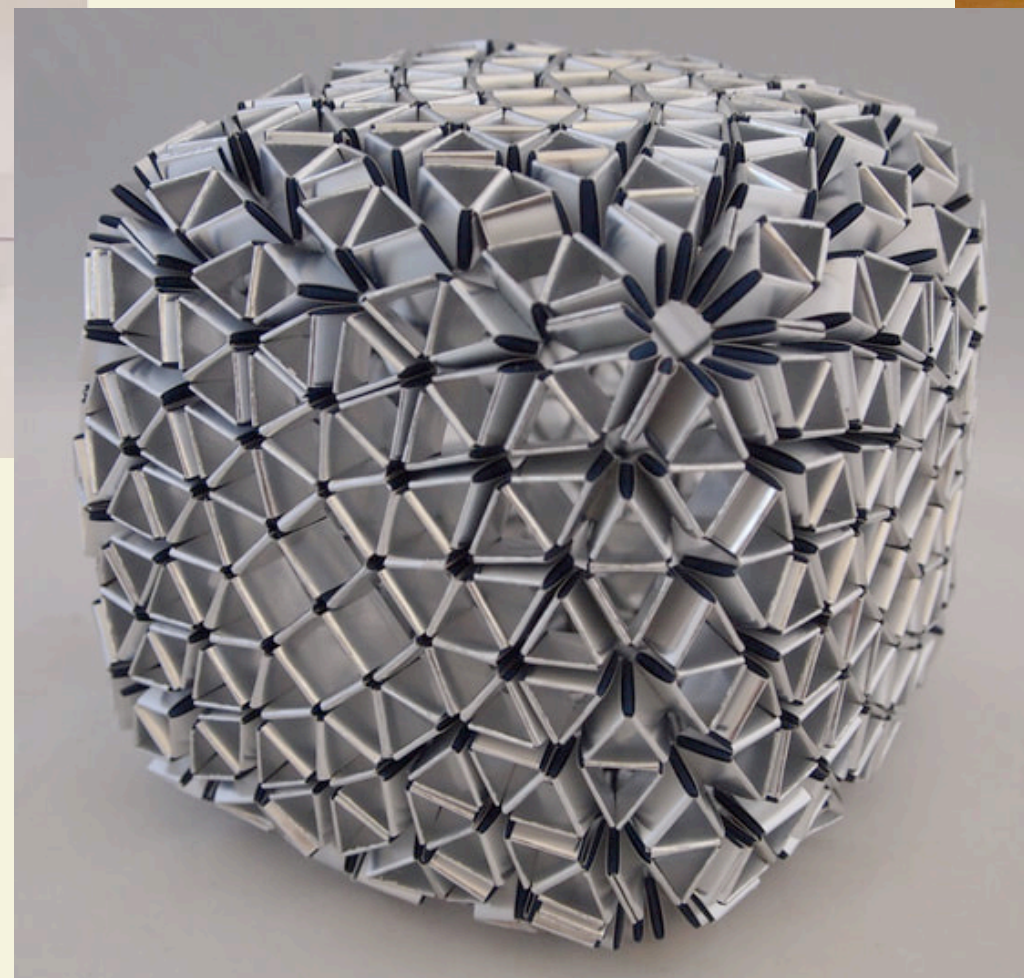


[Mrs Bs crochet African flower motif spider]

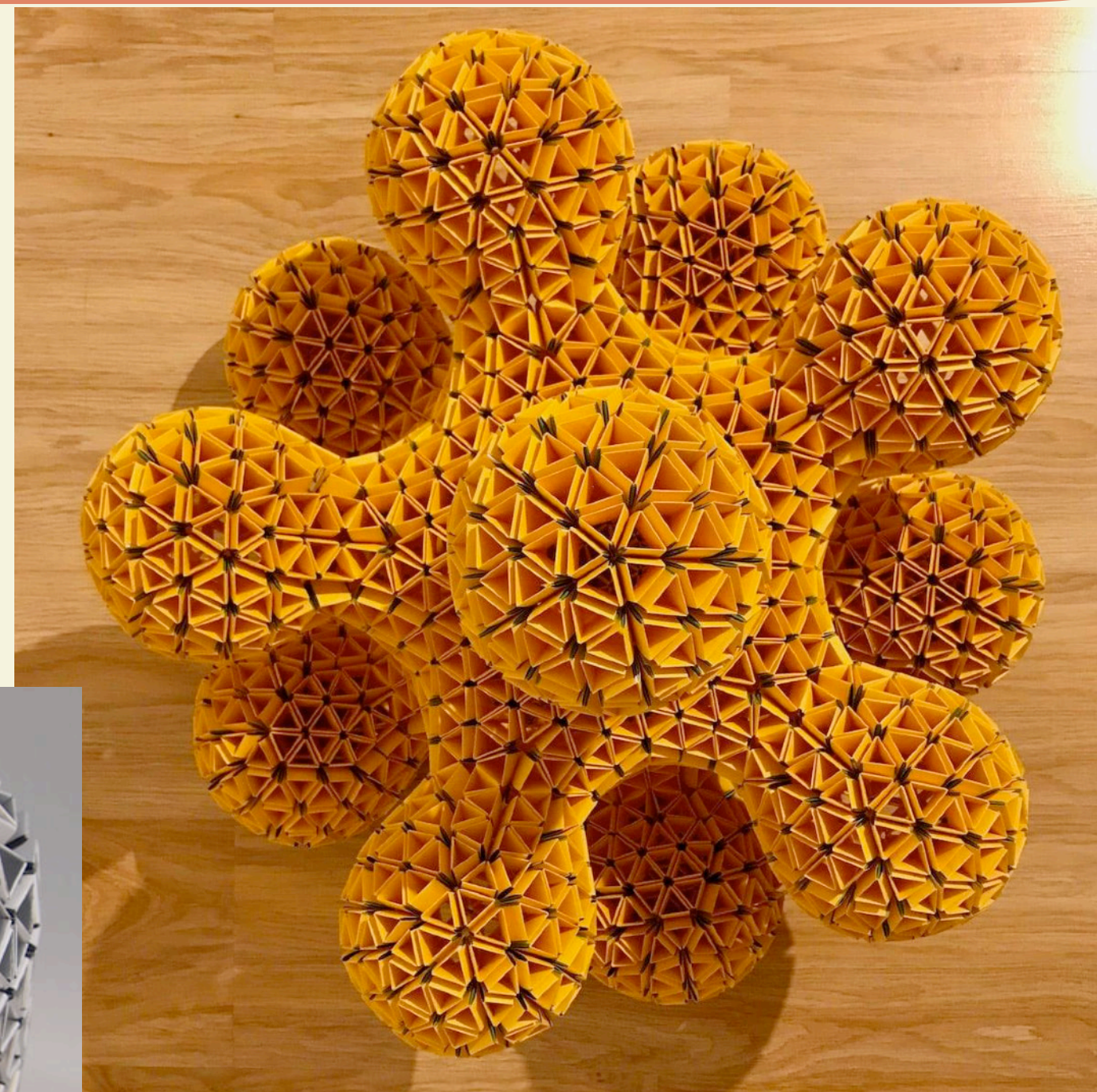
Fabrication via Regular Polygons



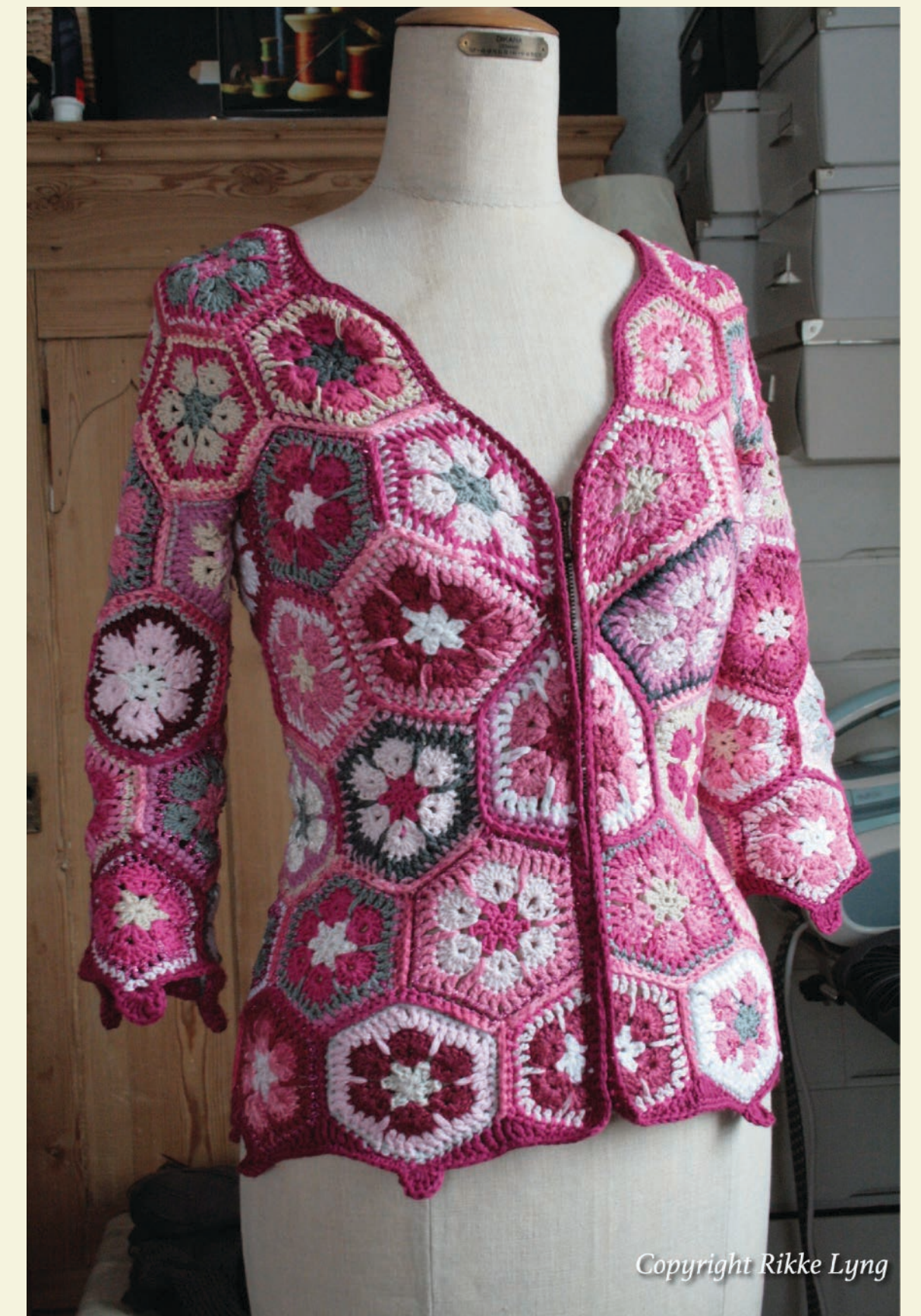
[<http://fionagillespie.com/origami>]



[<http://snaporigami.weebly.com/polyhedra>]



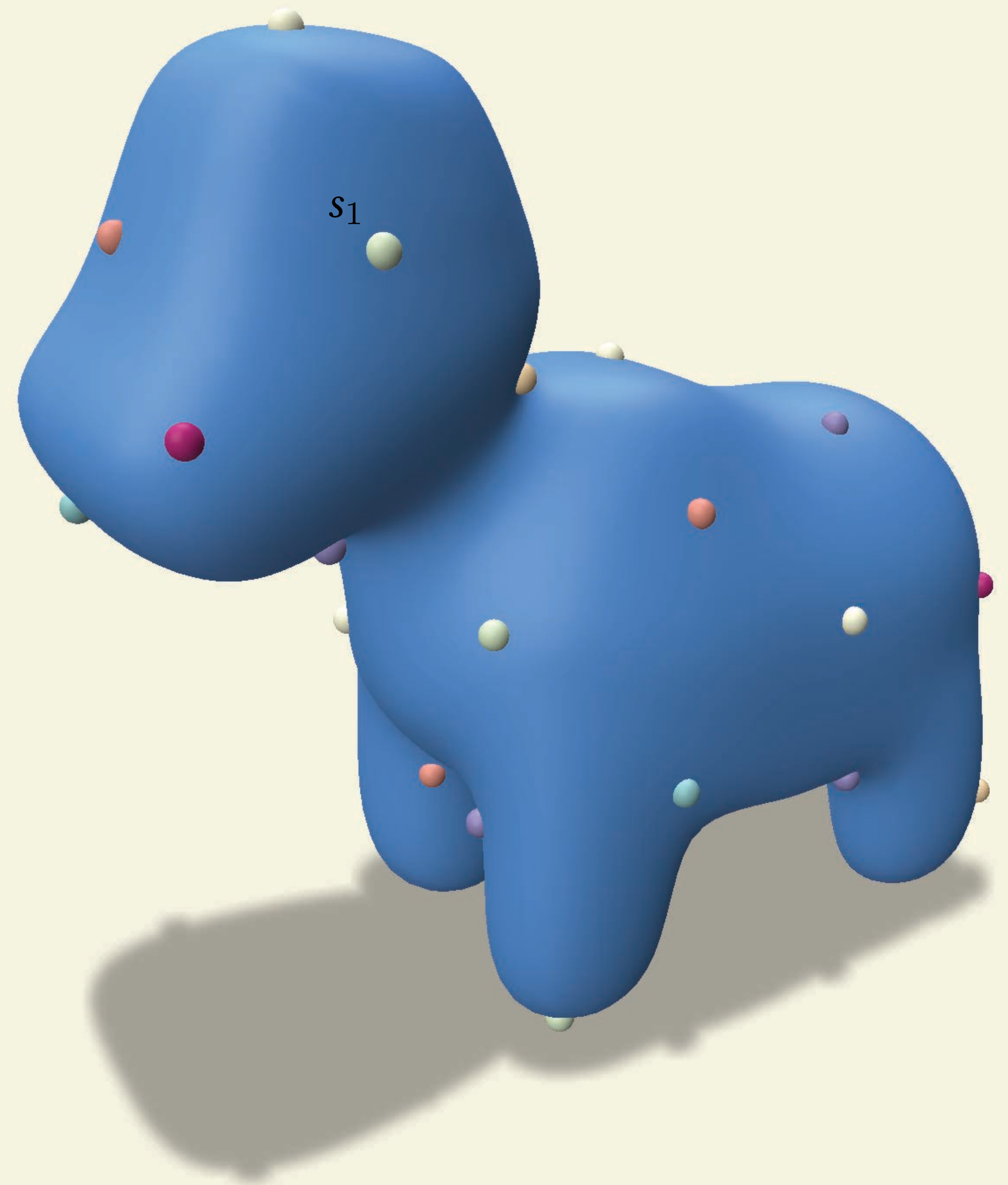
[<https://www.facebook.com/Snapology-origami-376055822981481/>]



[<https://www.pinterest.com/christensen1976/>]

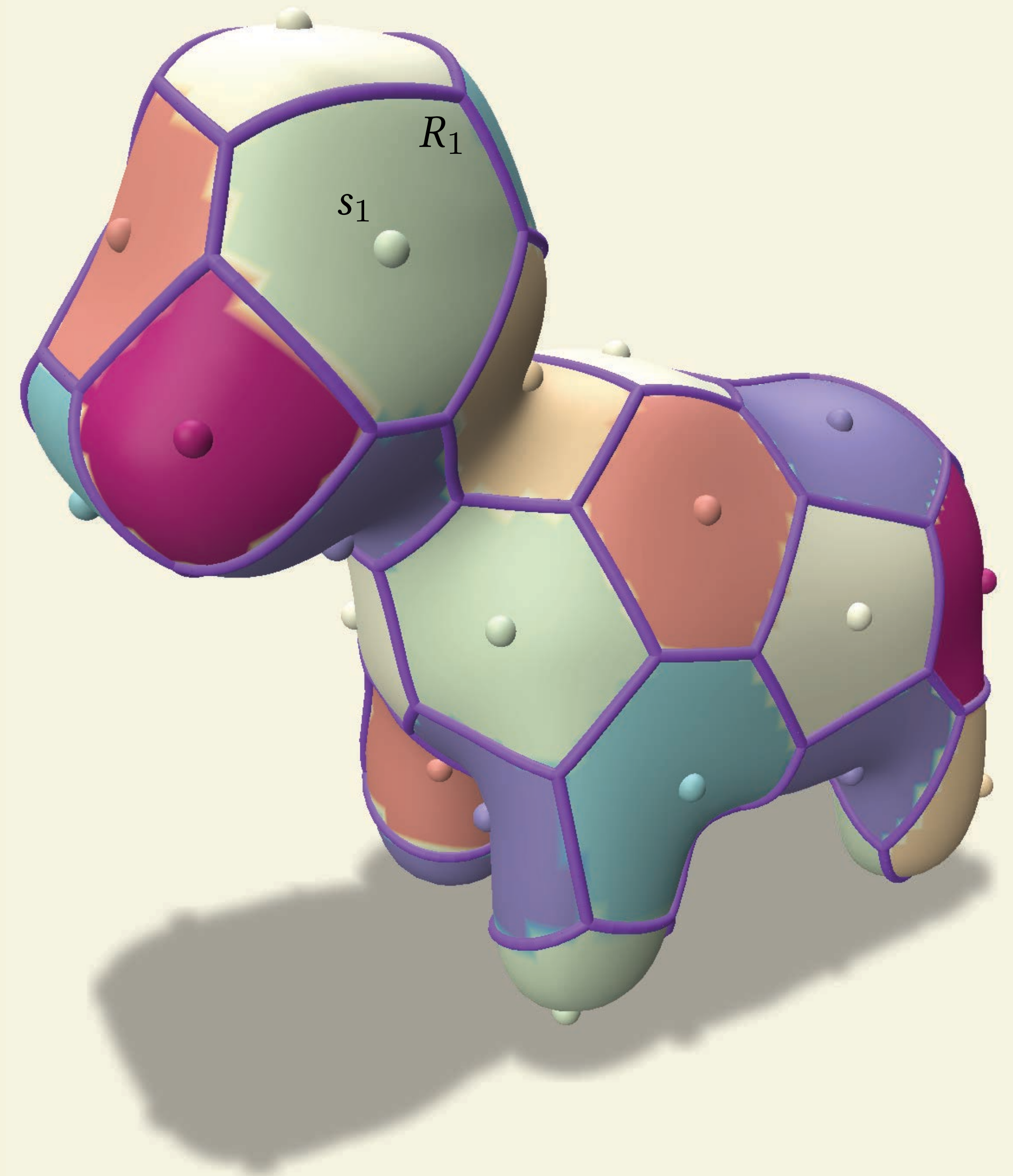
Voronoi Diagrams

- Start with a collection of sites s_i on the surface



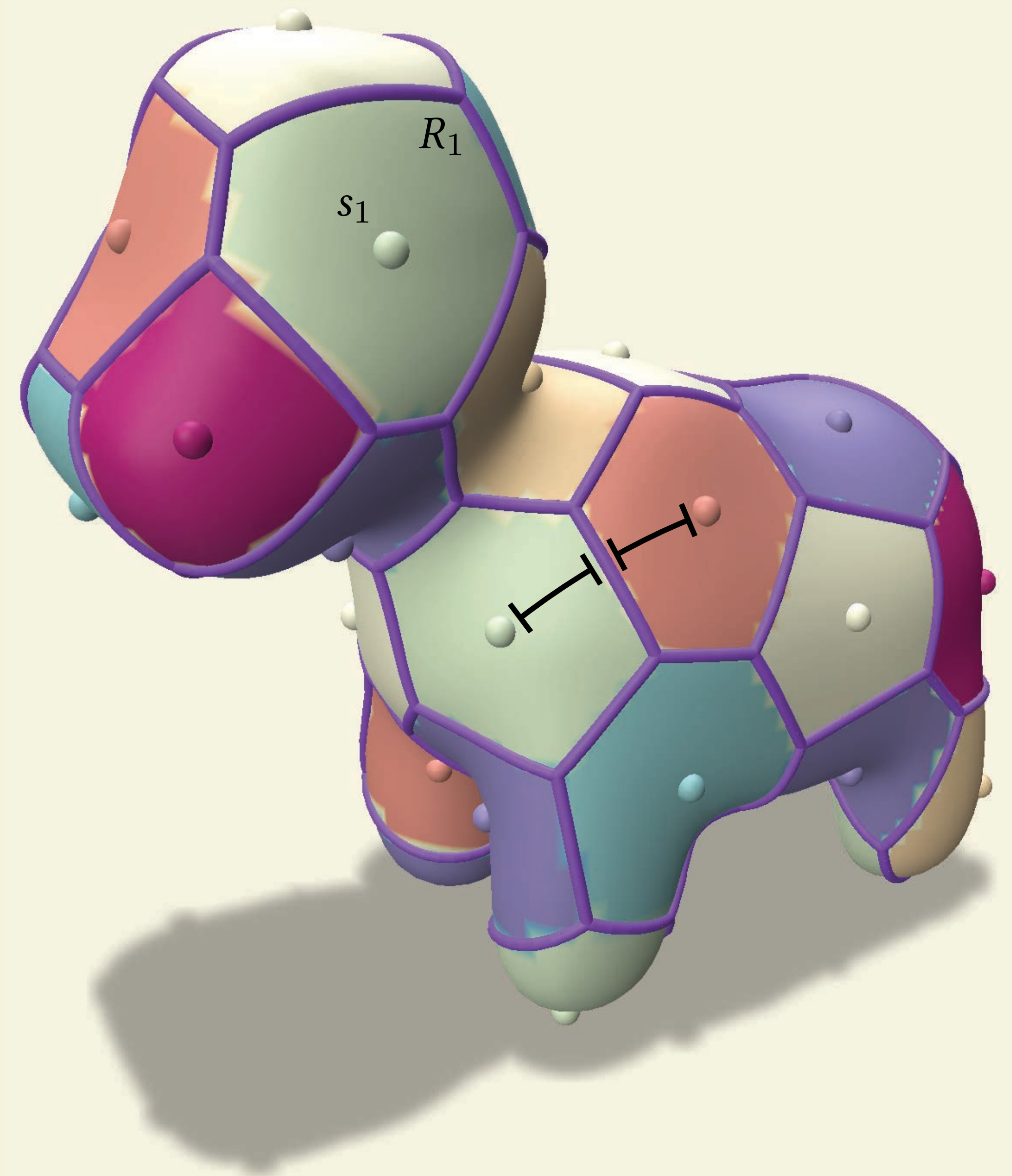
Voronoi Diagrams

- Start with a collection of sites s_i on the surface
- Define *Voronoi cells* R_i
 - R_i is the set of points closer to s_i than to any other site
- Decomposes the surface into polygons!

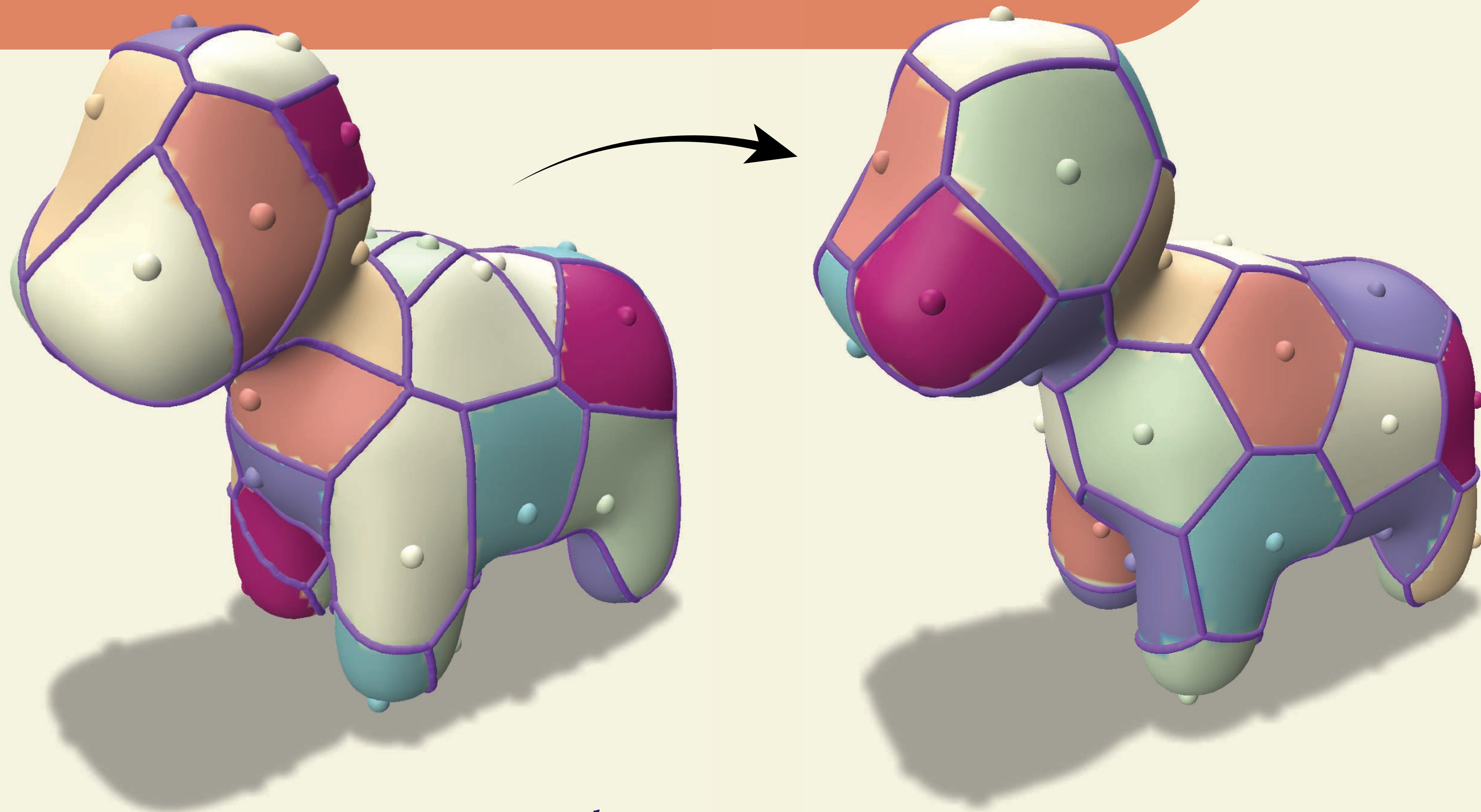


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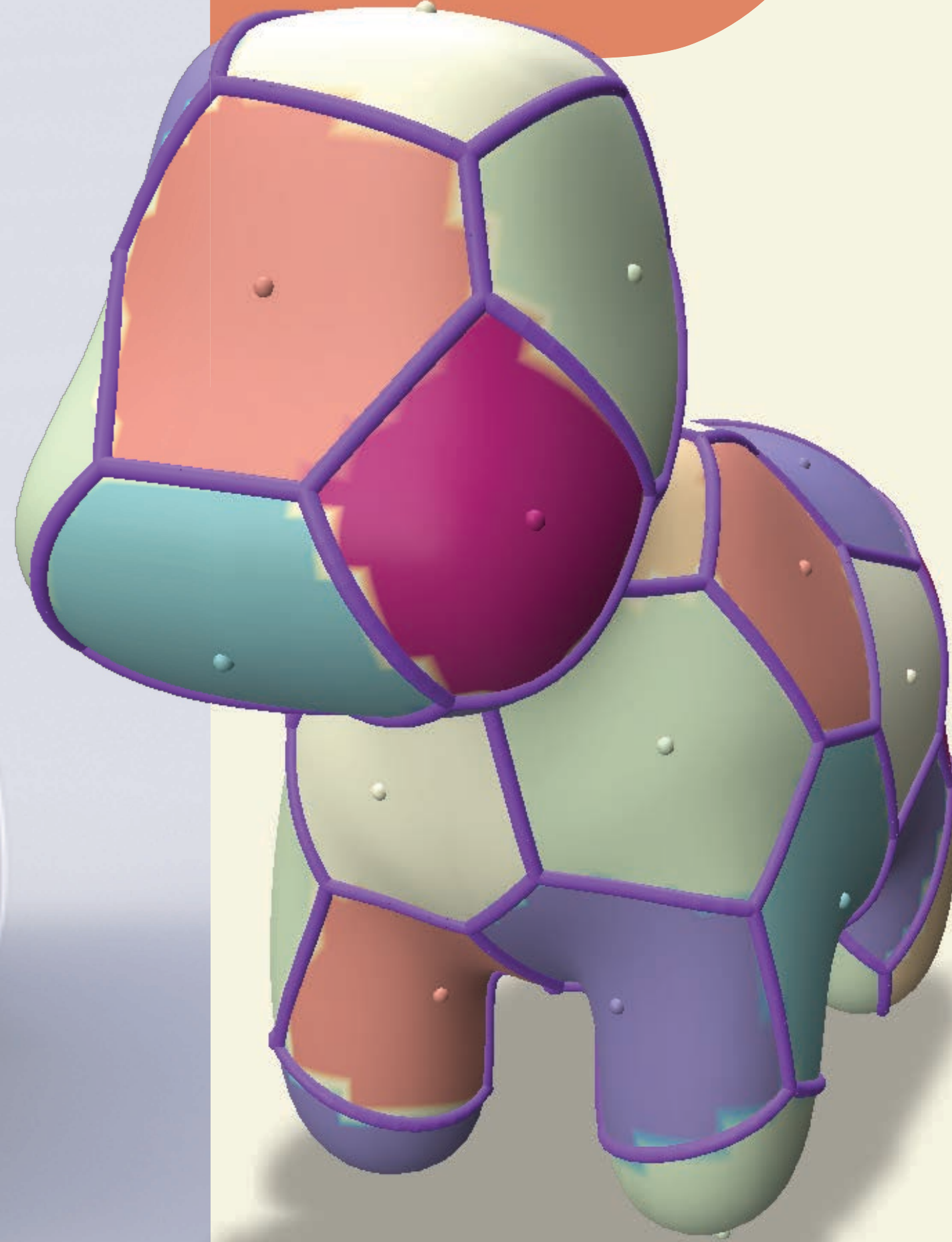


Improving Voronoi Cell Shape



1. Write down some energy
2. Run gradient descent

My Preliminary Result



My Preliminary Result



My Preliminary Result



Remaining Questions

- How do you pick/optimize the number of sites? (# sites determines polygon size)



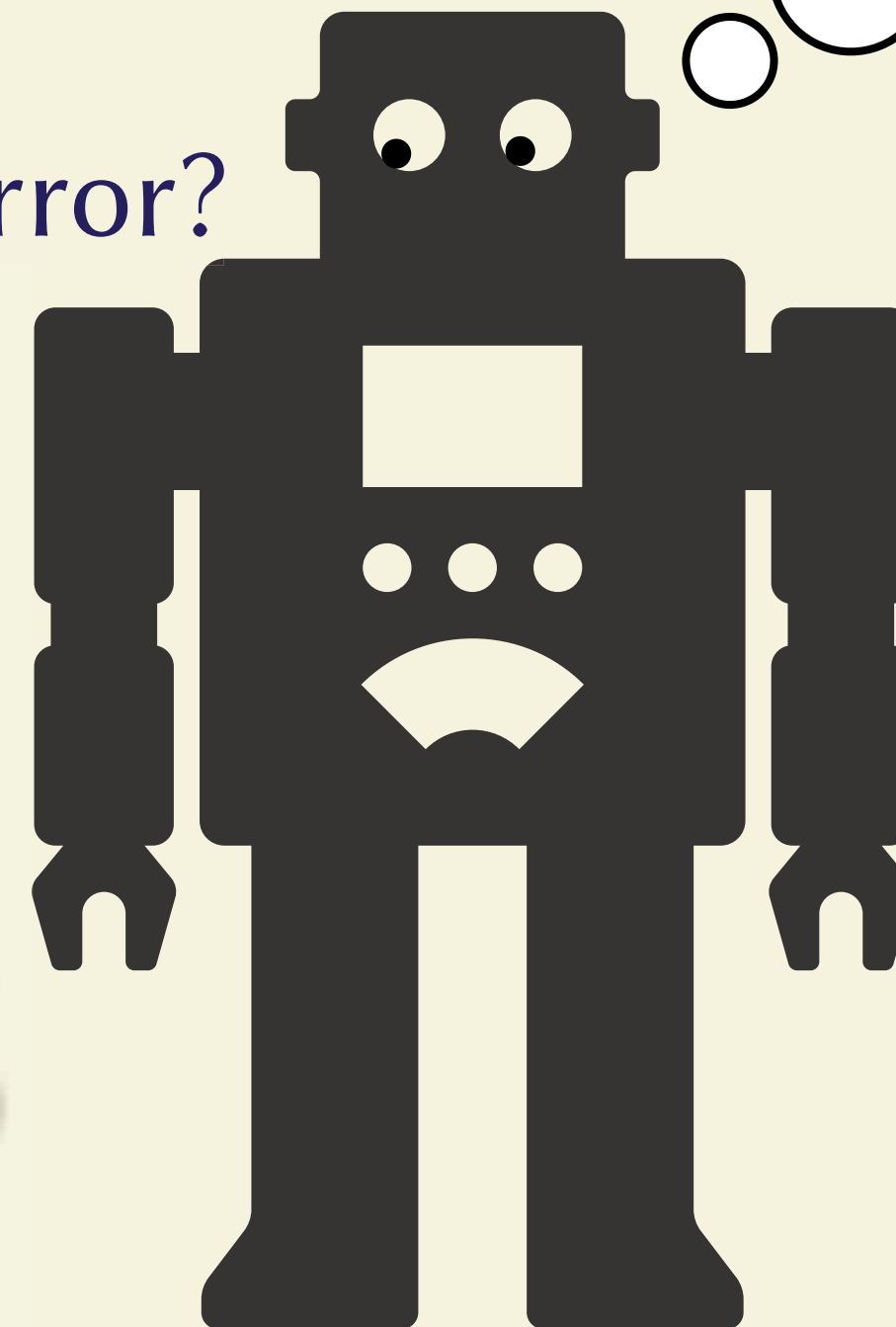
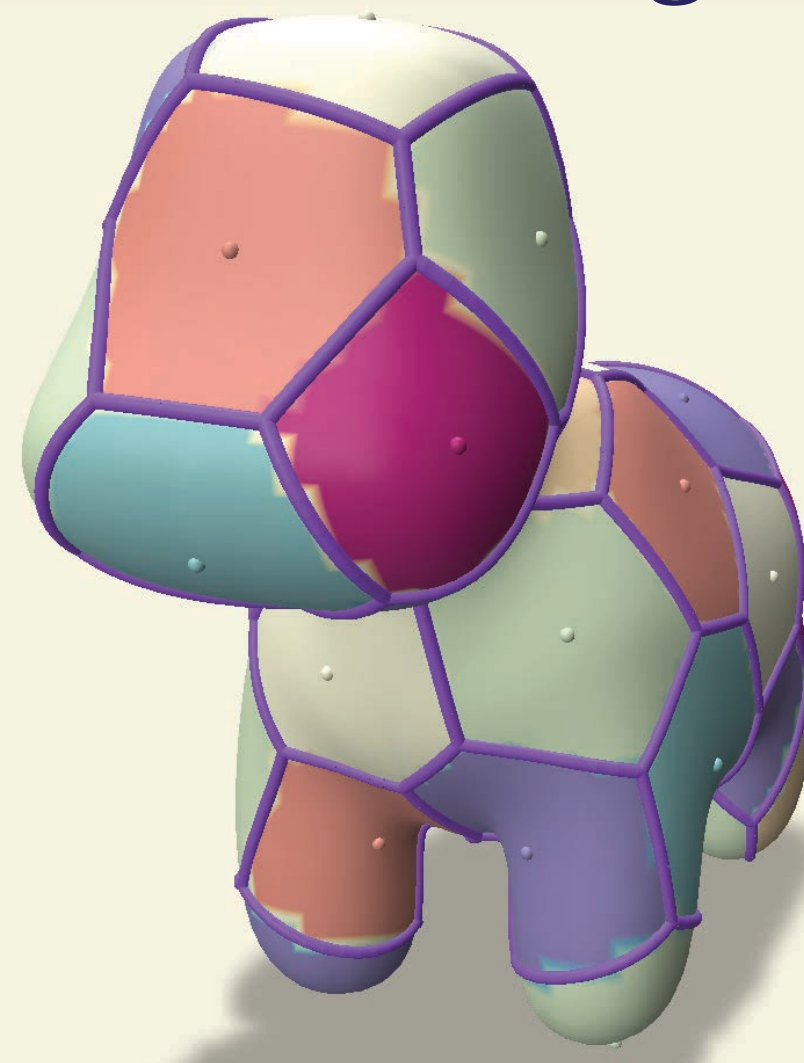
[<http://fiekshaakselsenmeer.blogspot.com/2014/07/happypotamus-4.html>]



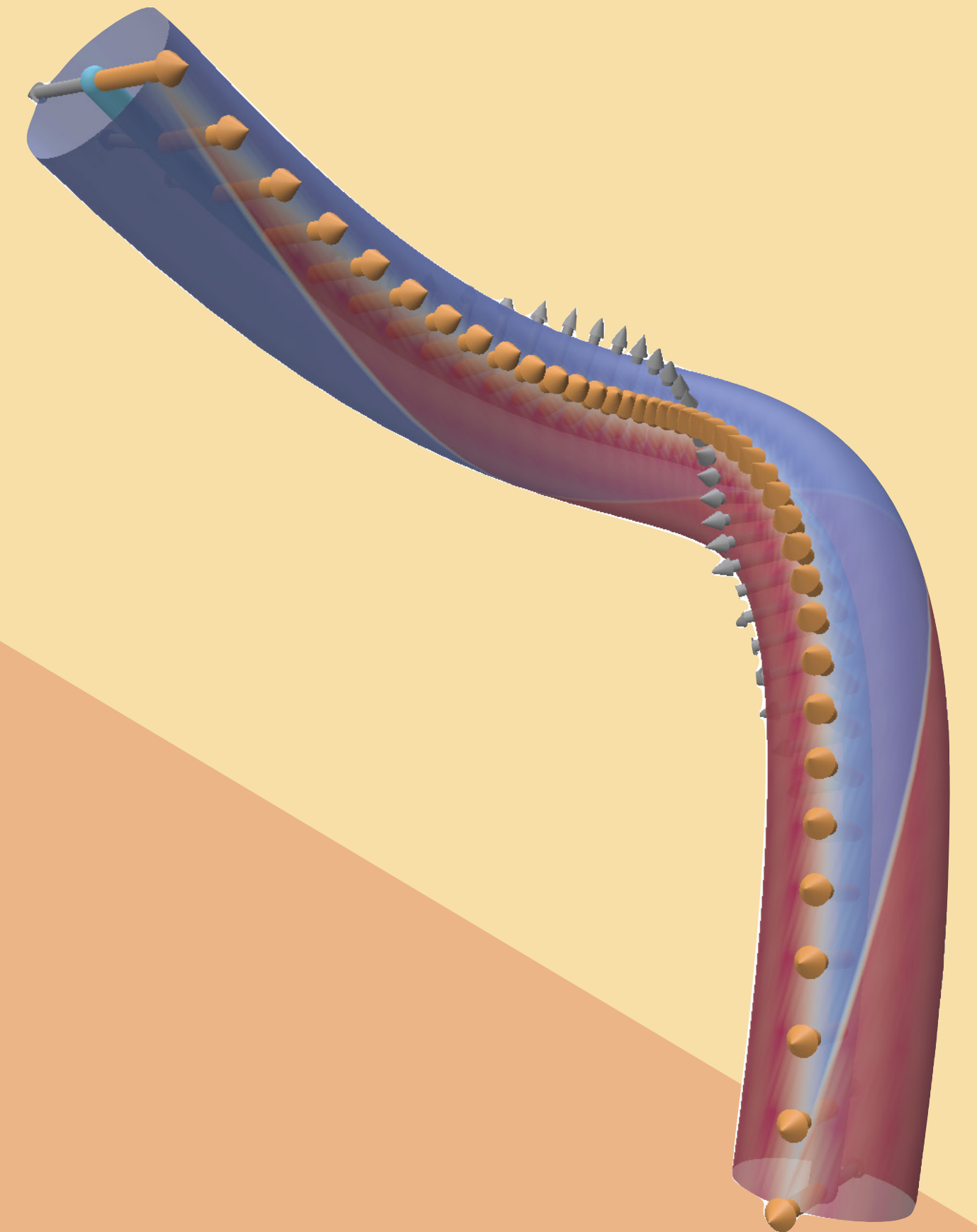
[Mrs Bs crochet African flower motif spider]

A Few Remaining Questions

- Is it feasible to simulate the output shape?
 - Is simulating at the polygon level sufficient?
 - Is there another good way of estimating error?

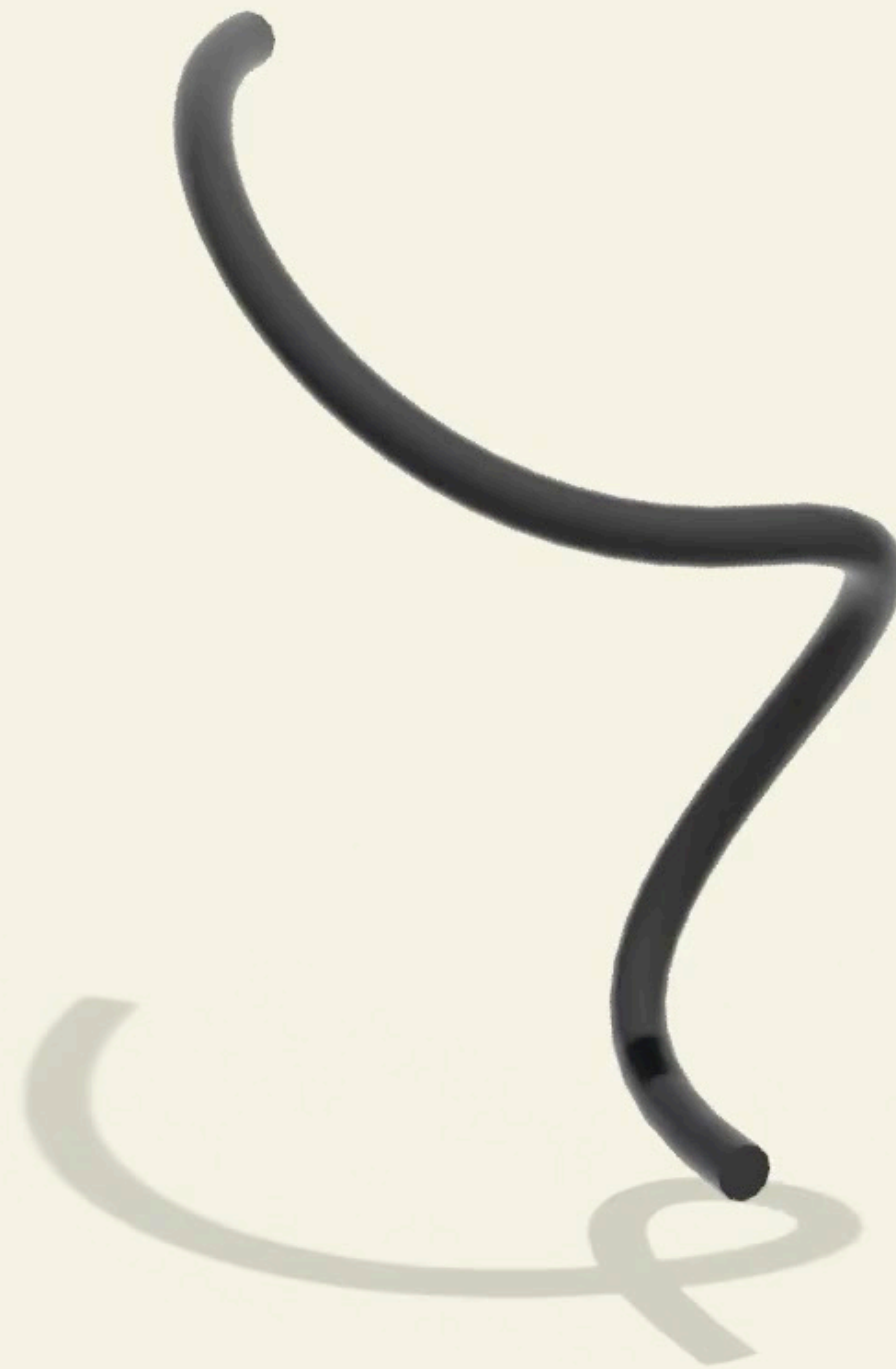


Implicit Curve Framing

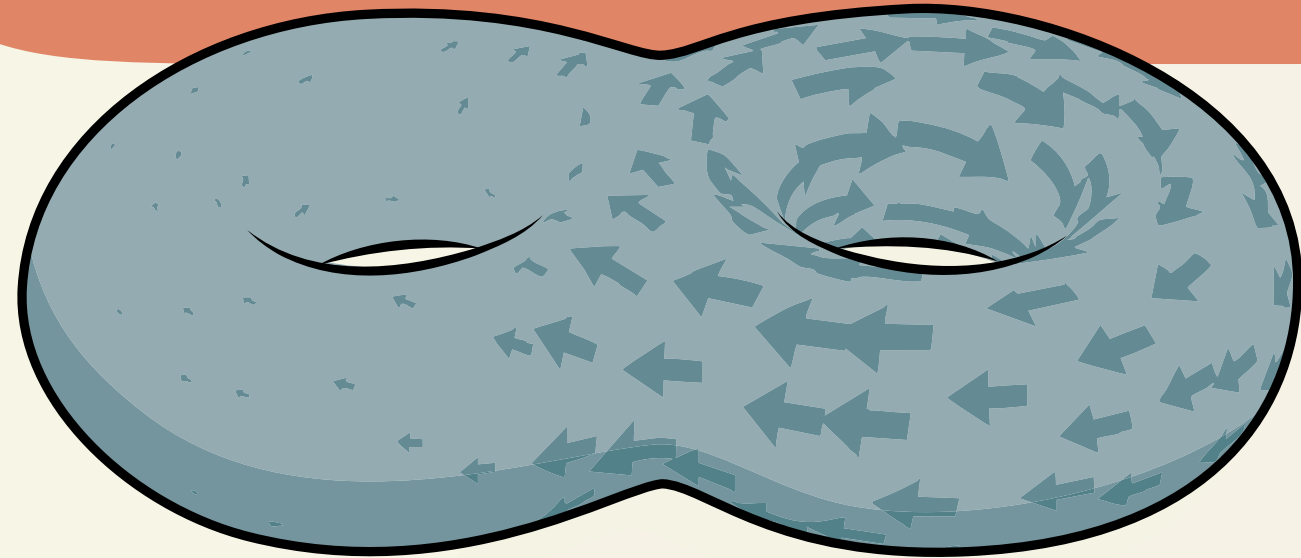


Framing a Curve

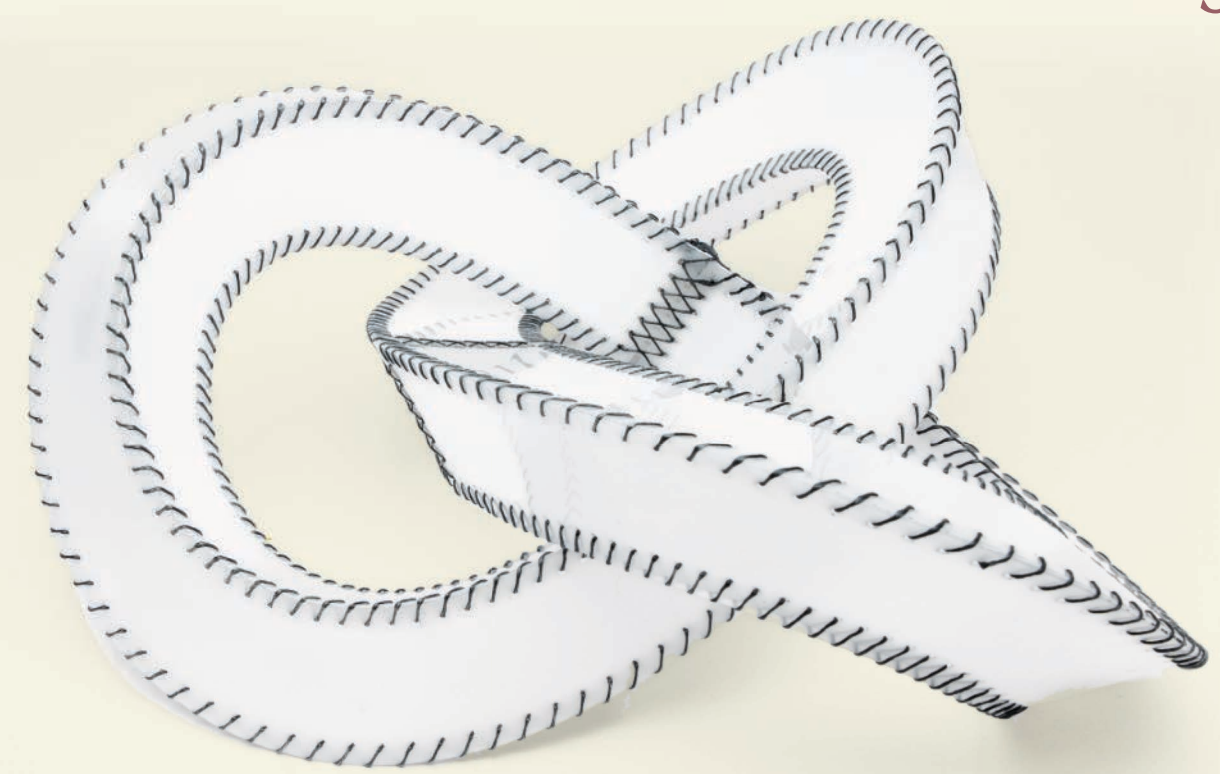
Framing a Curve



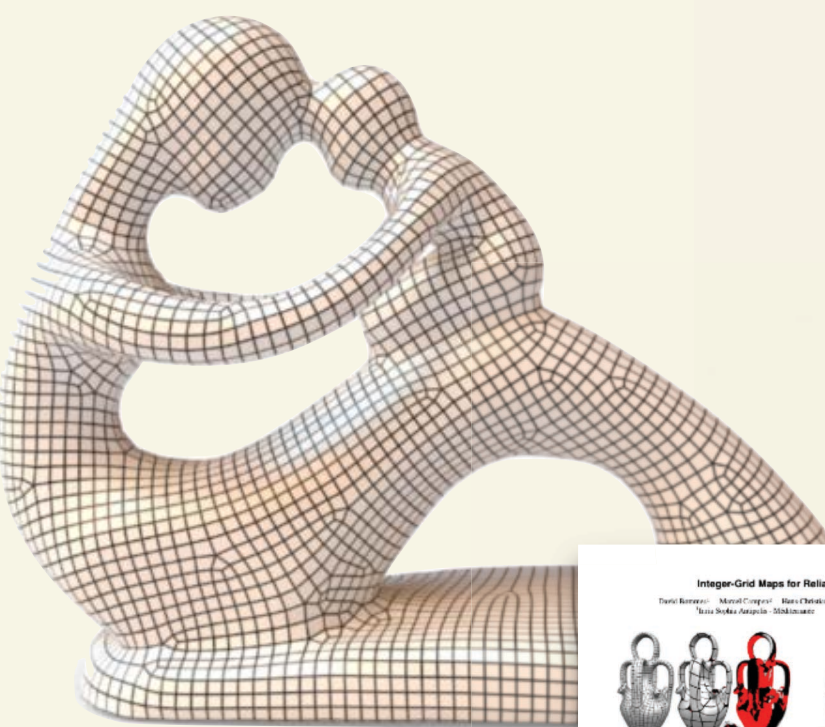
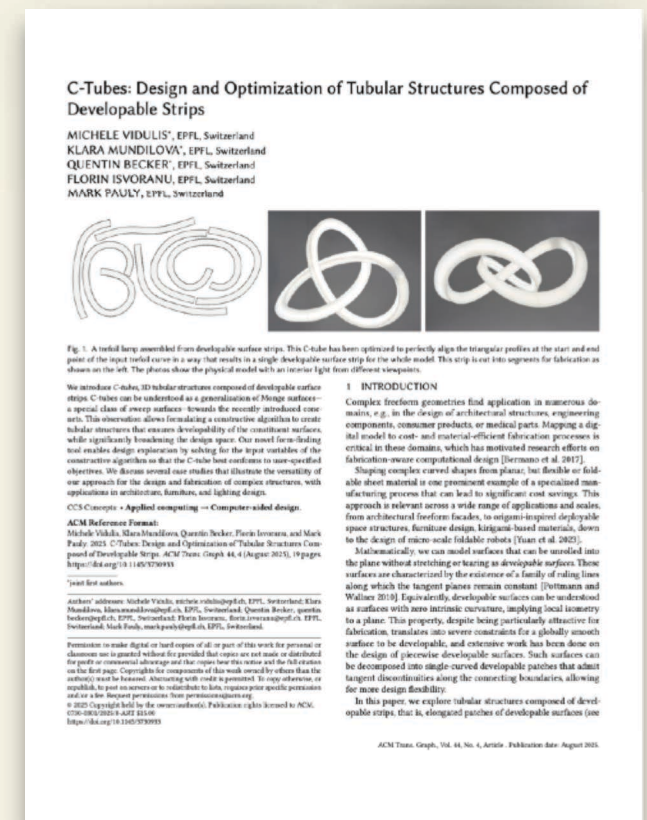
Framing a Curve



tangent vector fields on surfaces



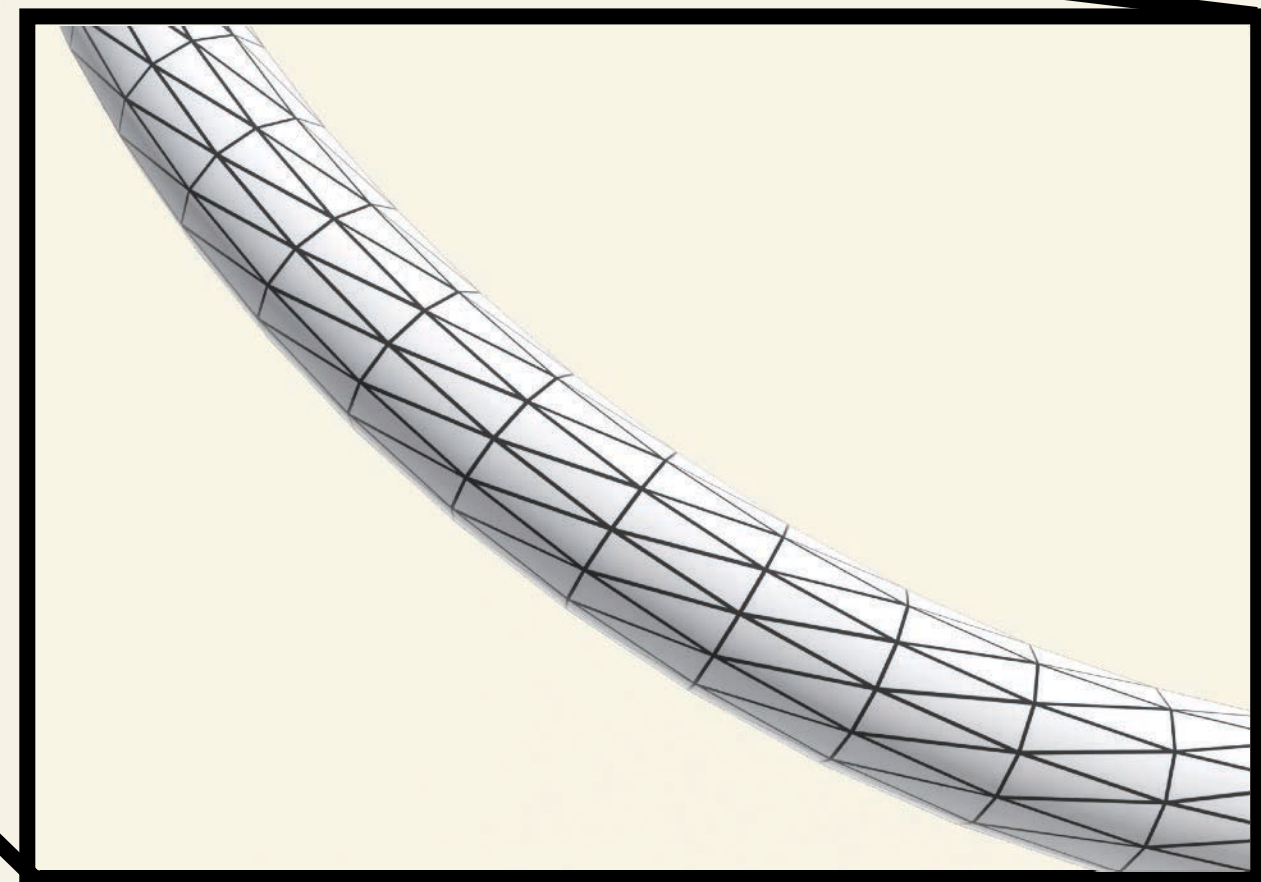
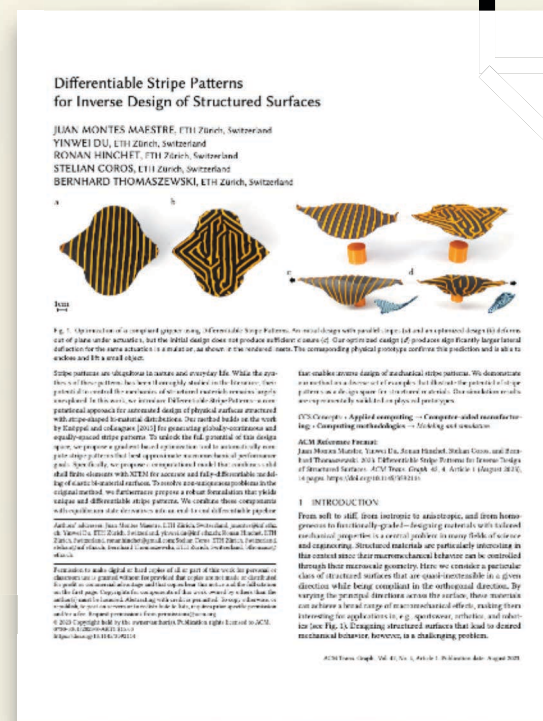
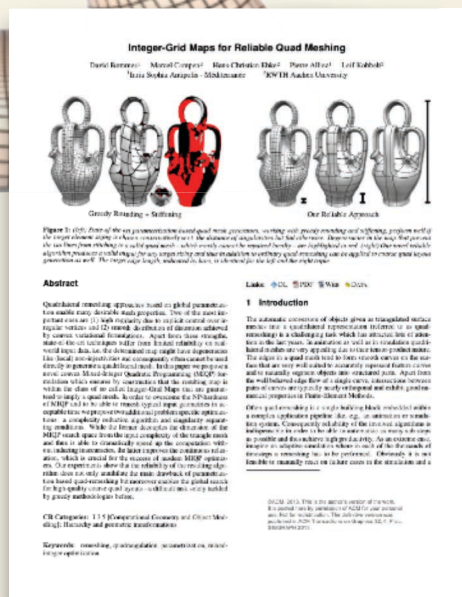
computational fabrication



remeshing



robot grippers

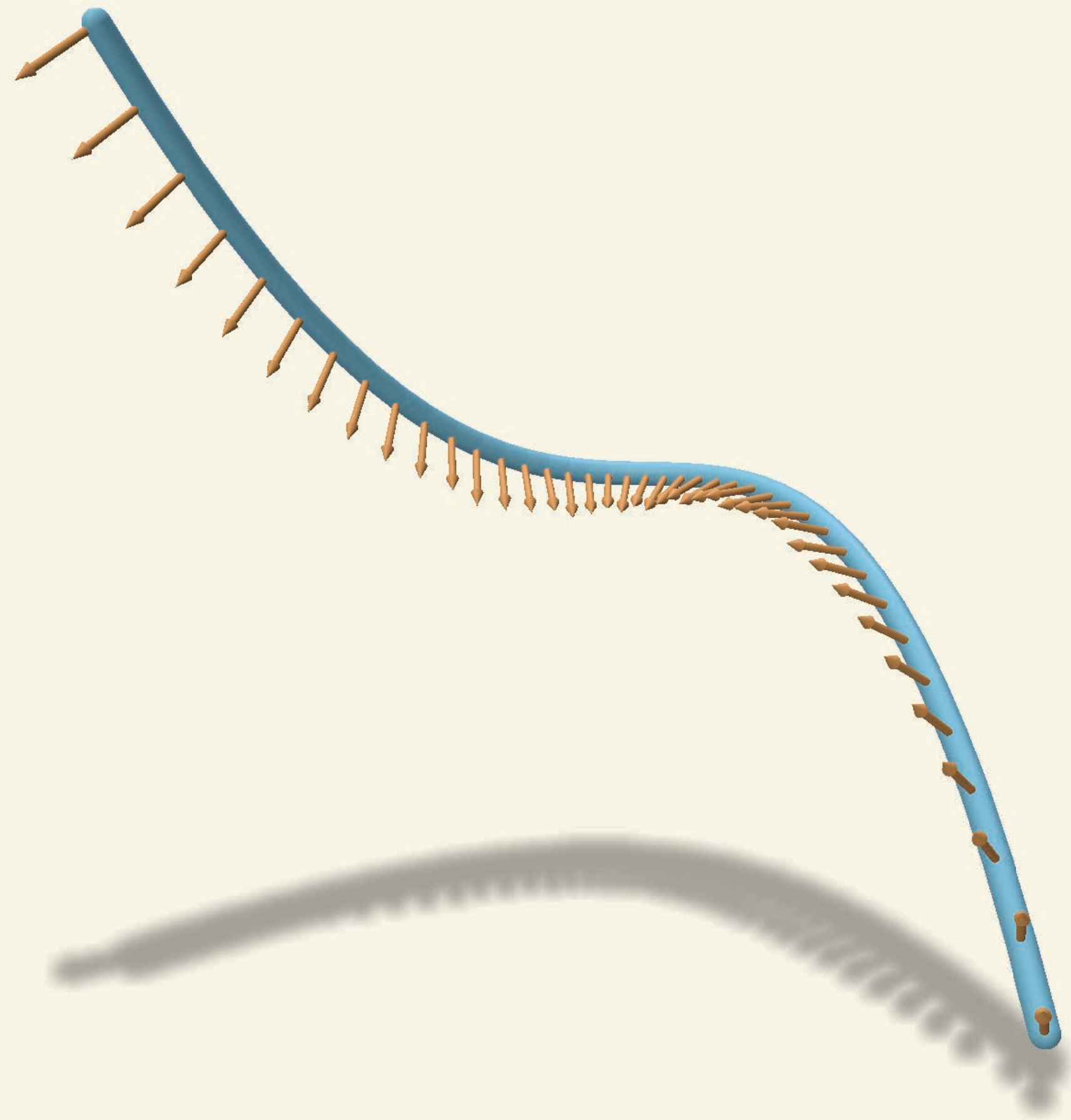


just rendering tubes

Implicit Curve Framing

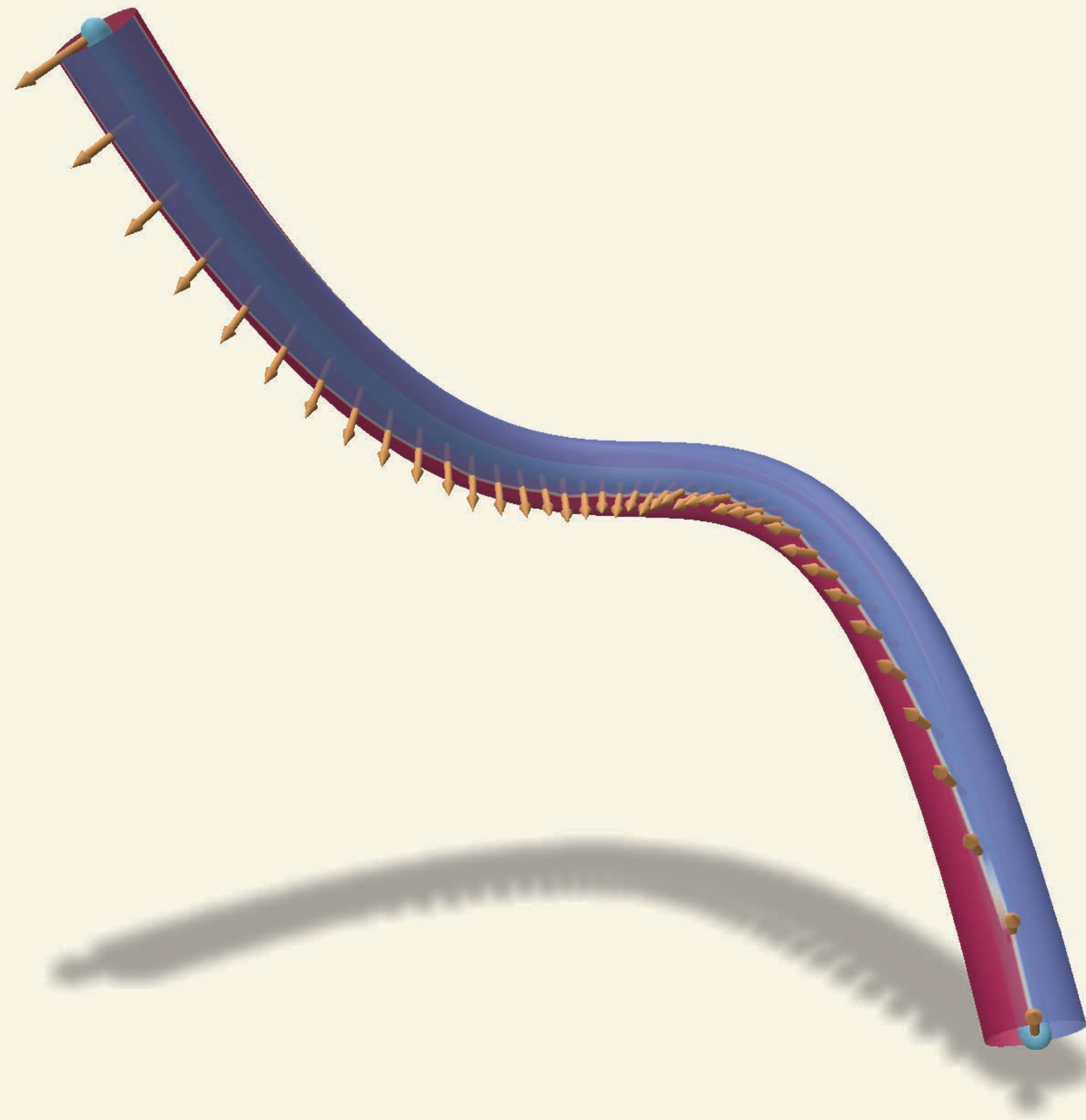


Implicit Curve Framing



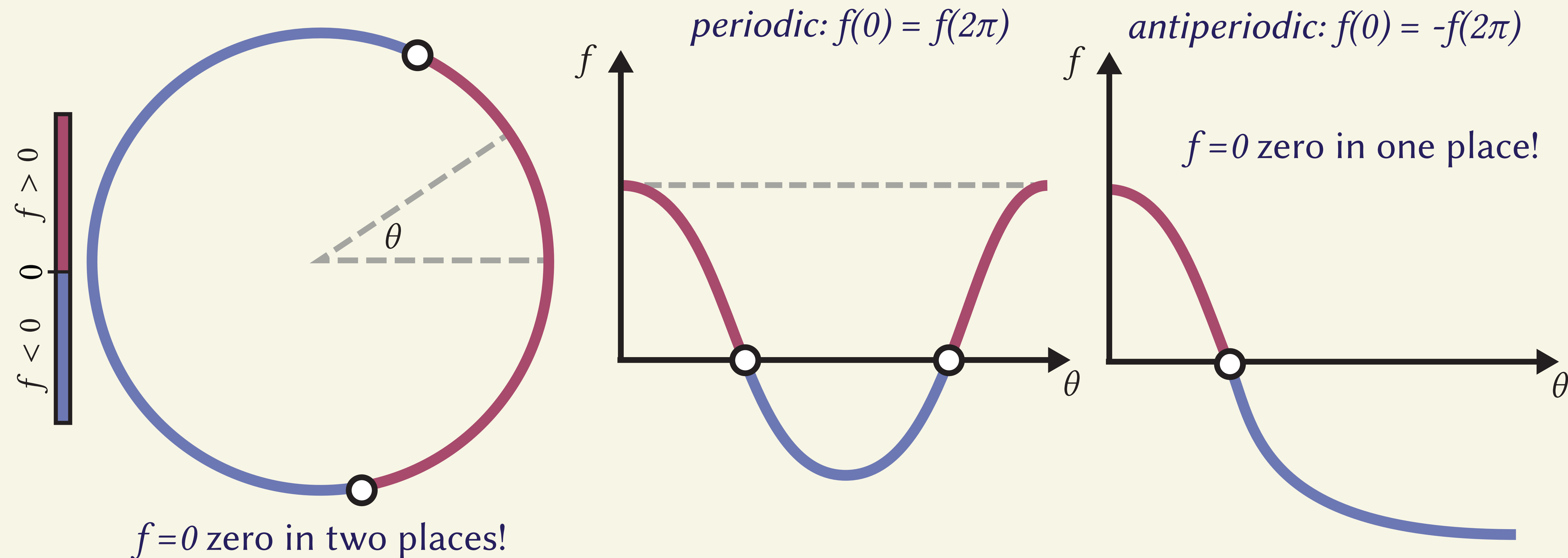
Implicit Curve Framing

But wait, what's happening on the back of the tube?



Functions on a Circle

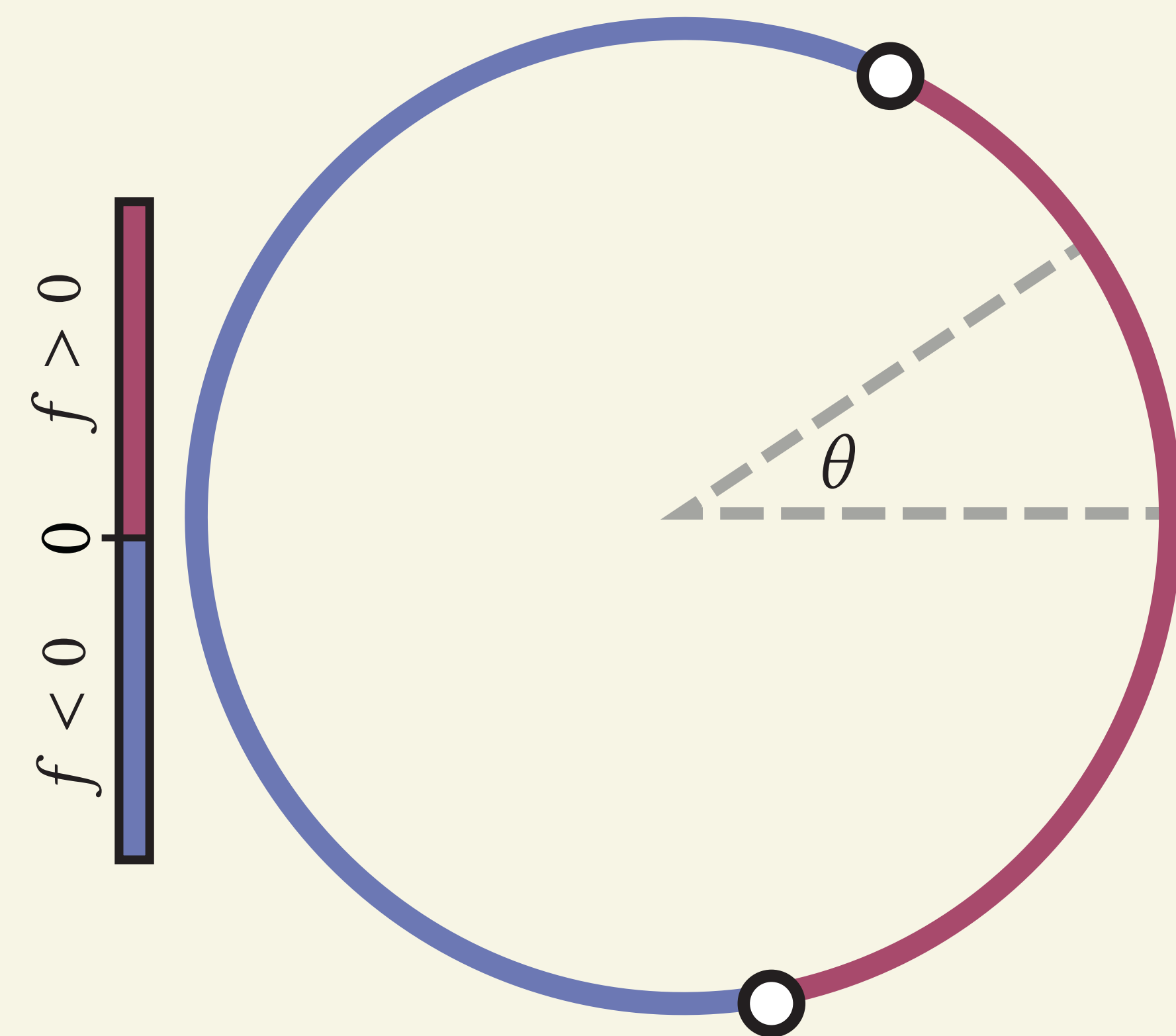
But wait, what's happening on the back of the tube?



$f=0$ zero in two places!

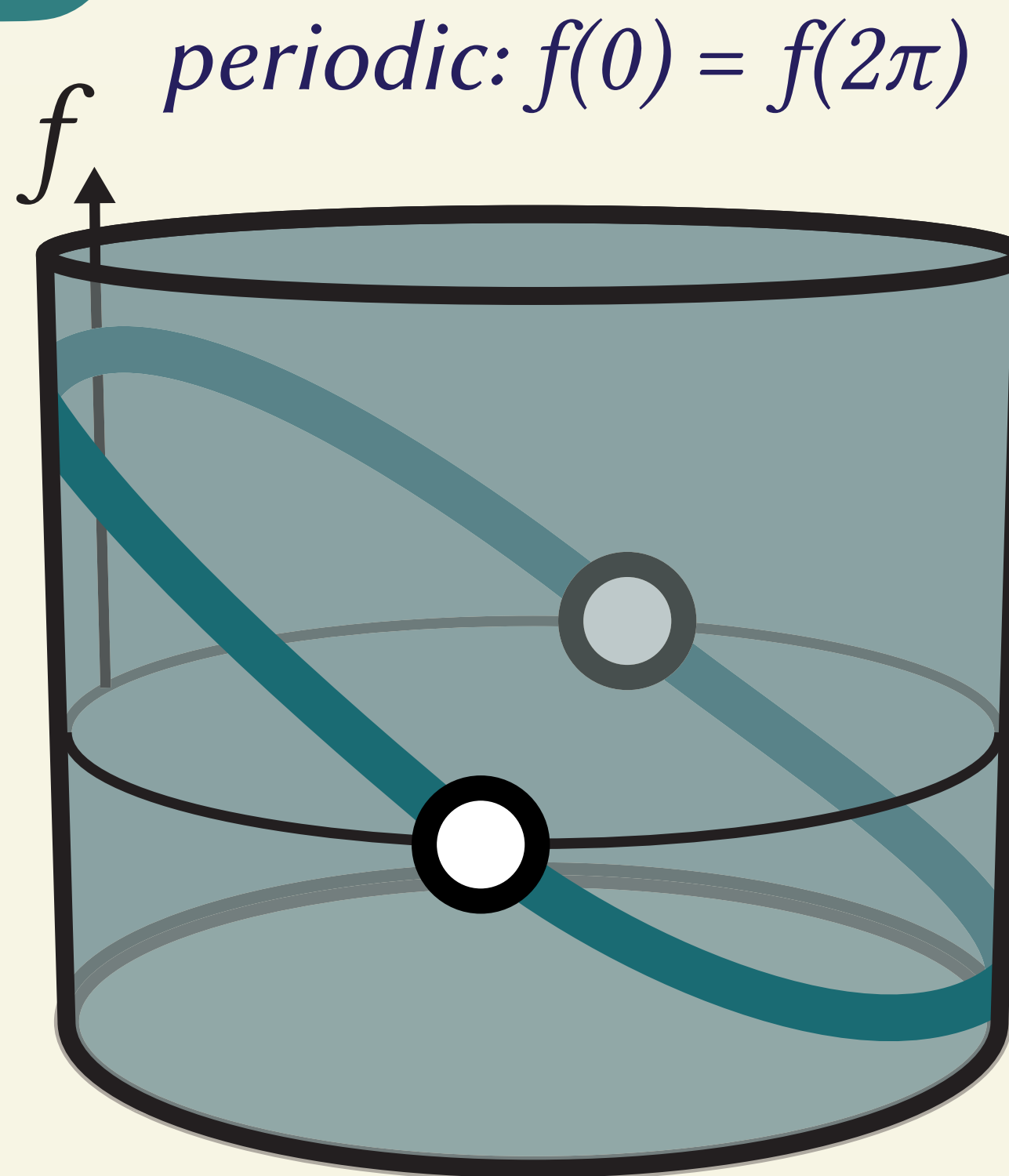
Functions on a Circle

topological picture

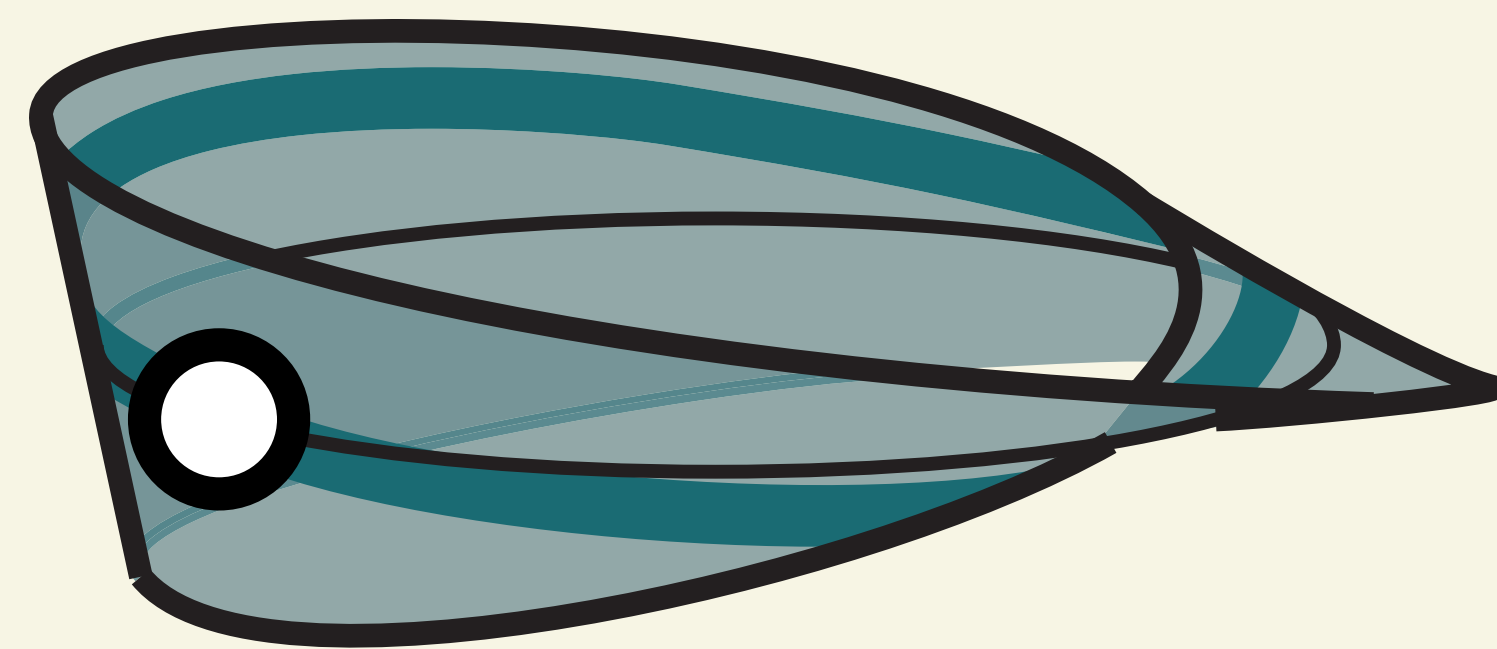


$f=0$ zero in two places!

But wait, what's happening on the back of the tube?



antiperiodic: $f(0) = -f(2\pi)$

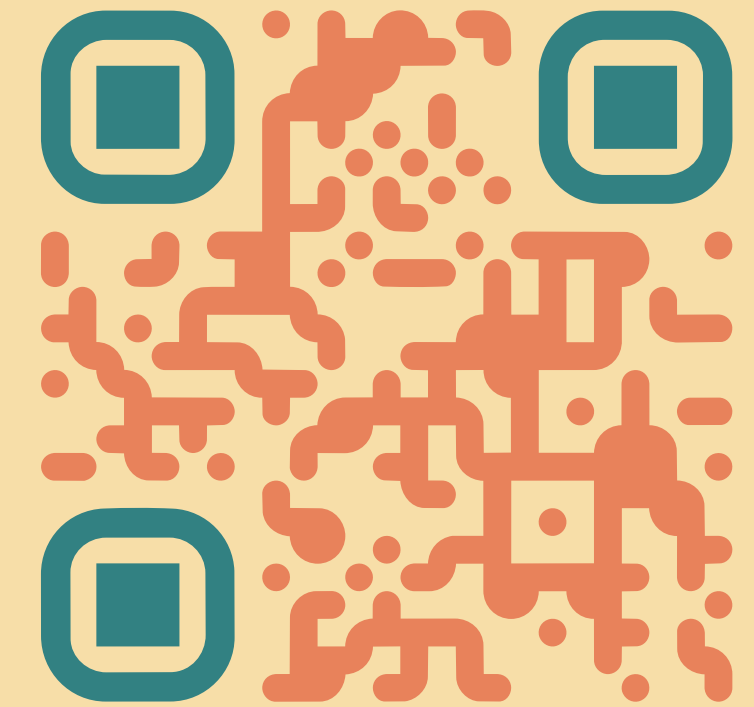
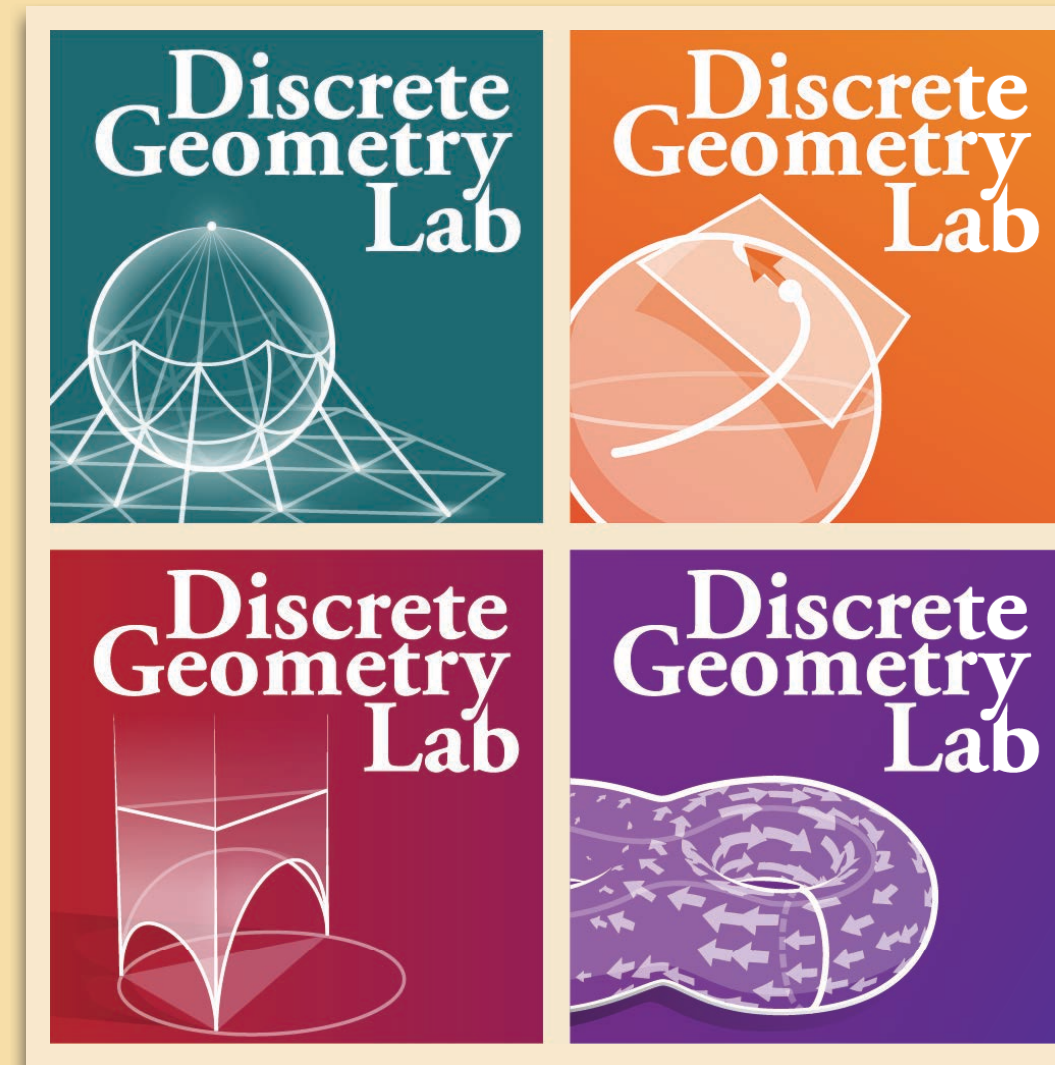


Thanks for listening

please reach out if you want to talk more!

✉ mark.gillespie@utah.edu

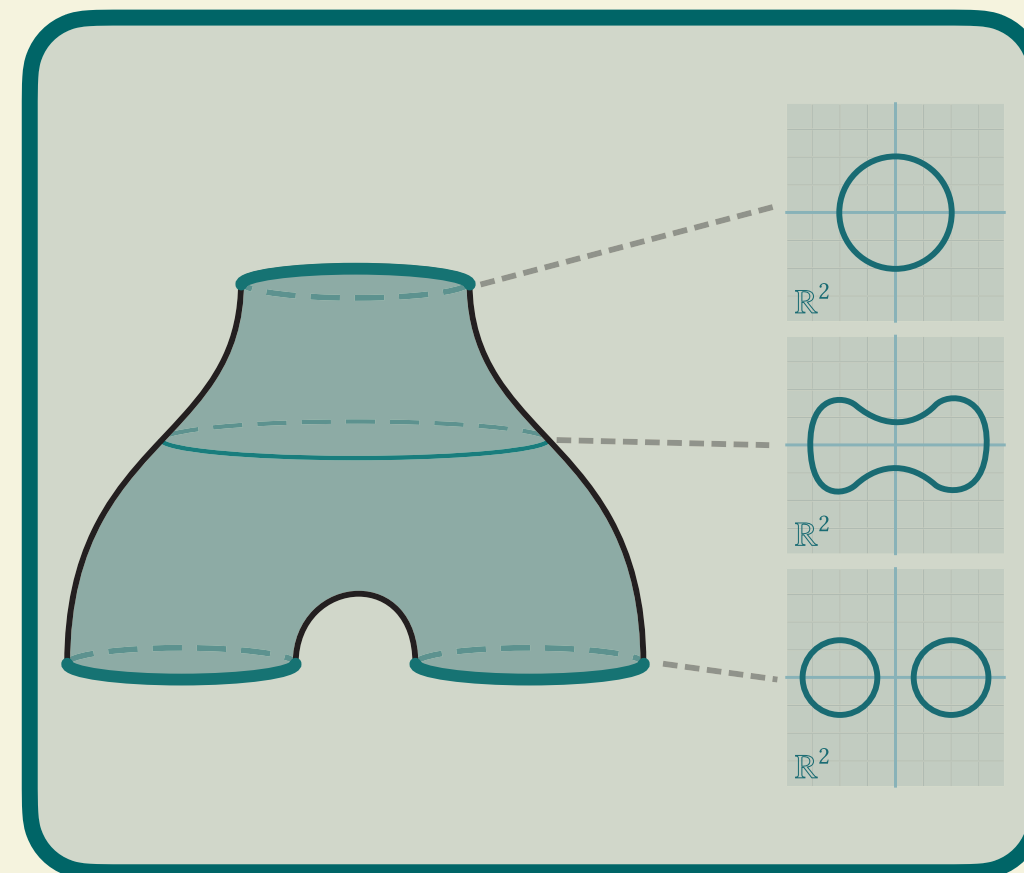
🌐 geometry.cs.utah.edu



I. SHAPE CORRESPONDENCE



II. SURFACE MORPHING VIA 4D RAY TRACING



III. VORONOI CROCHET PATTERN DESIGN



IV. IMPLICIT CURVE FRAMING

