

# Ray Tracing Harmonic Functions

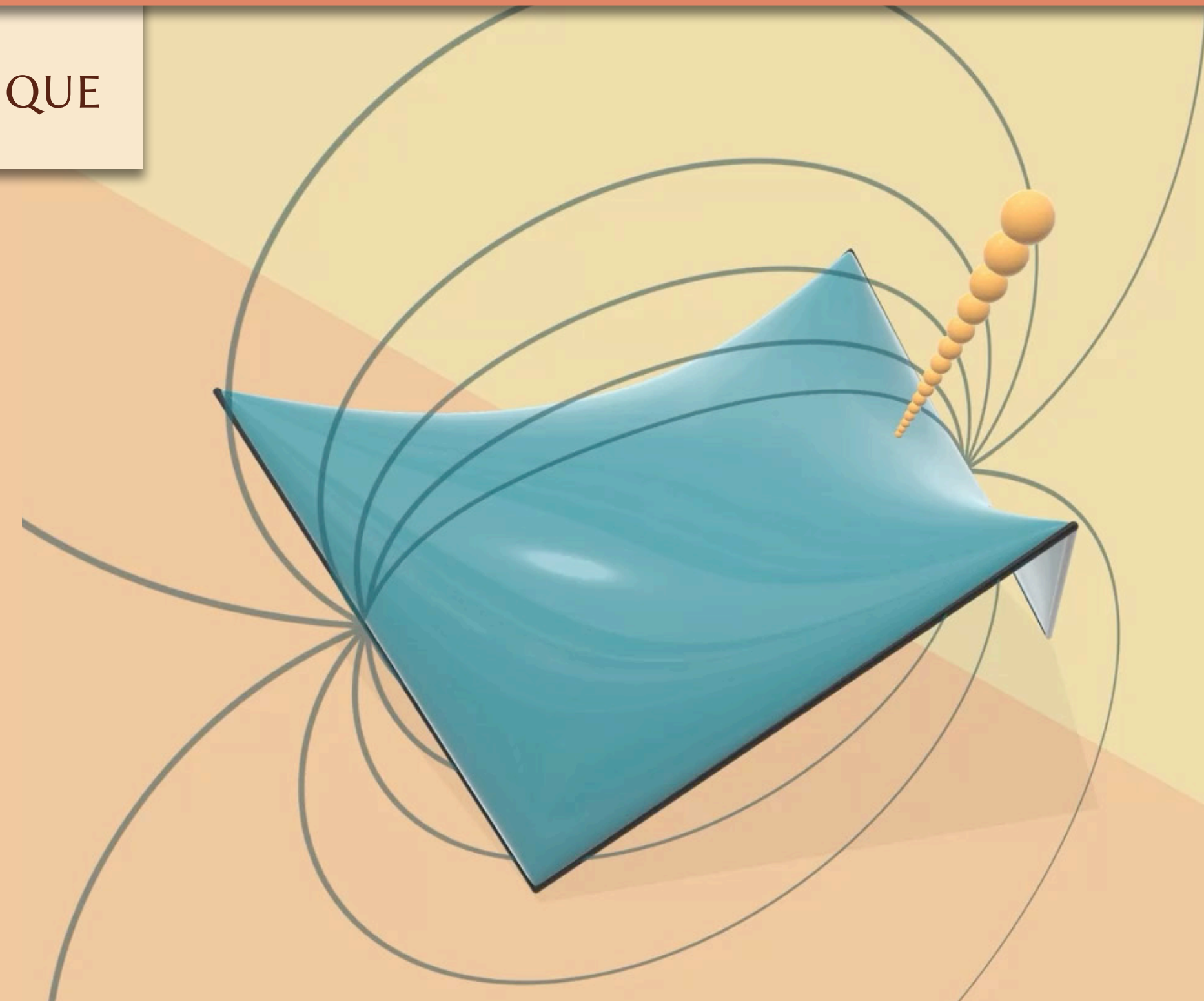
Mark Gillespie, INRIA / ÉCOLE POLYTECHNIQUE

*with*

Denise Yang, CARNEGIE MELLON UNIVERSITY  
AND PIXAR ANIMATION STUDIOS

Mario Botsch, TU DORTMUND UNIVERSITY

Keenan Crane, CARNEGIE MELLON UNIVERSITY



# Harmonic functions

special kind of function

$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$$

*heat transfer*

*gravitation*

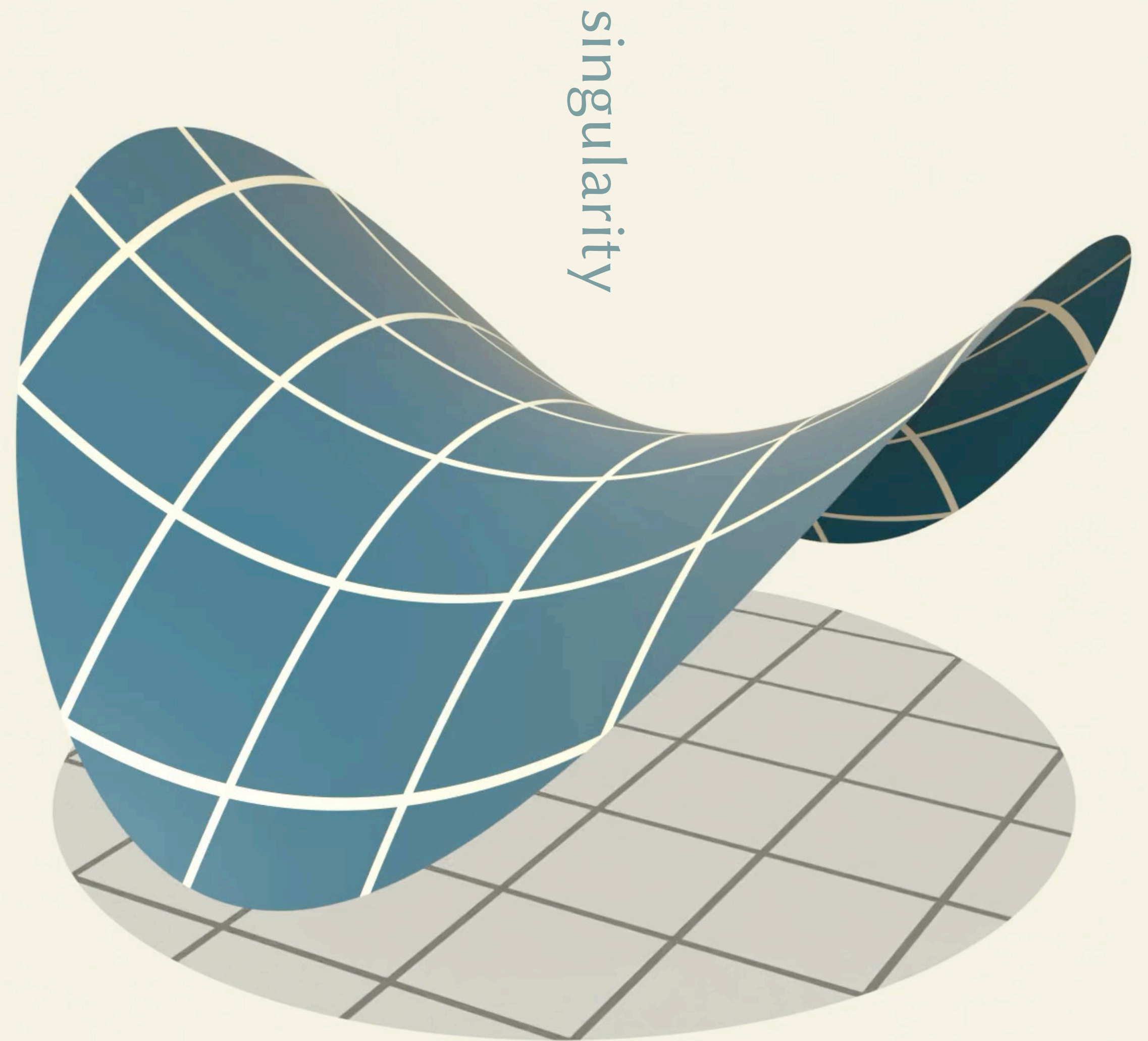
*electrostatics*

*complex analysis*

well-studied

# Harmonic functions

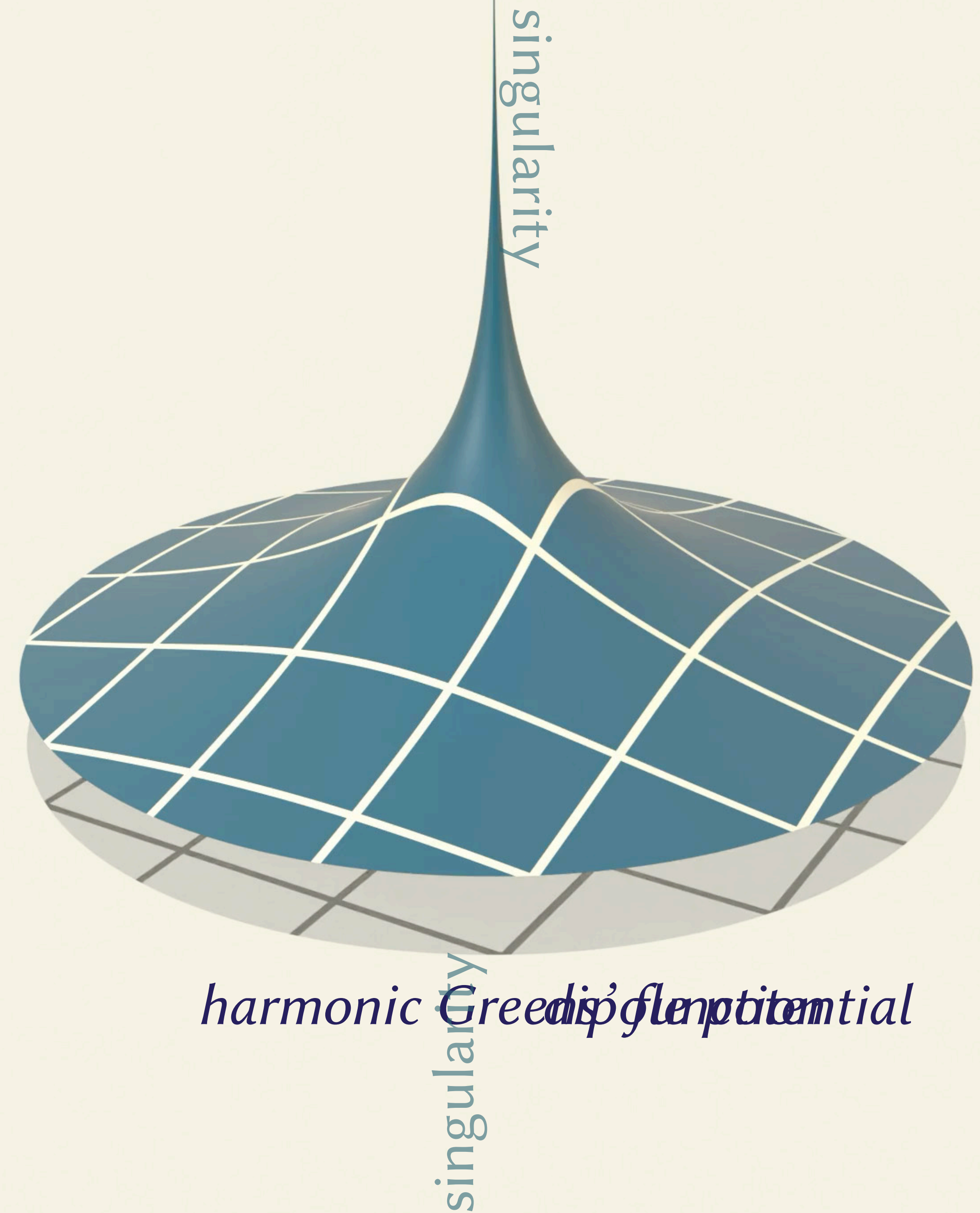
$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$$



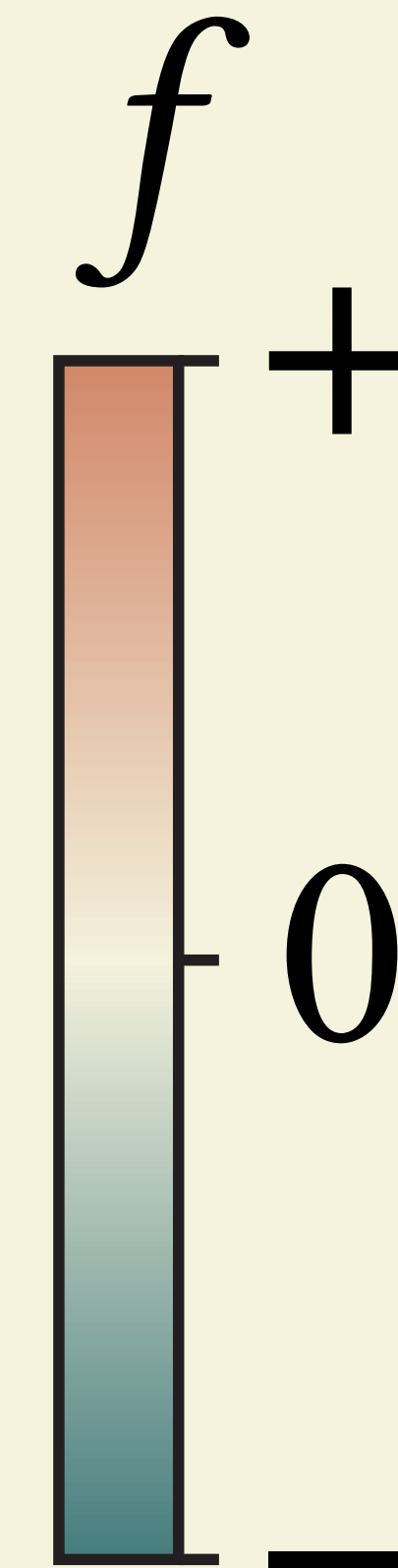
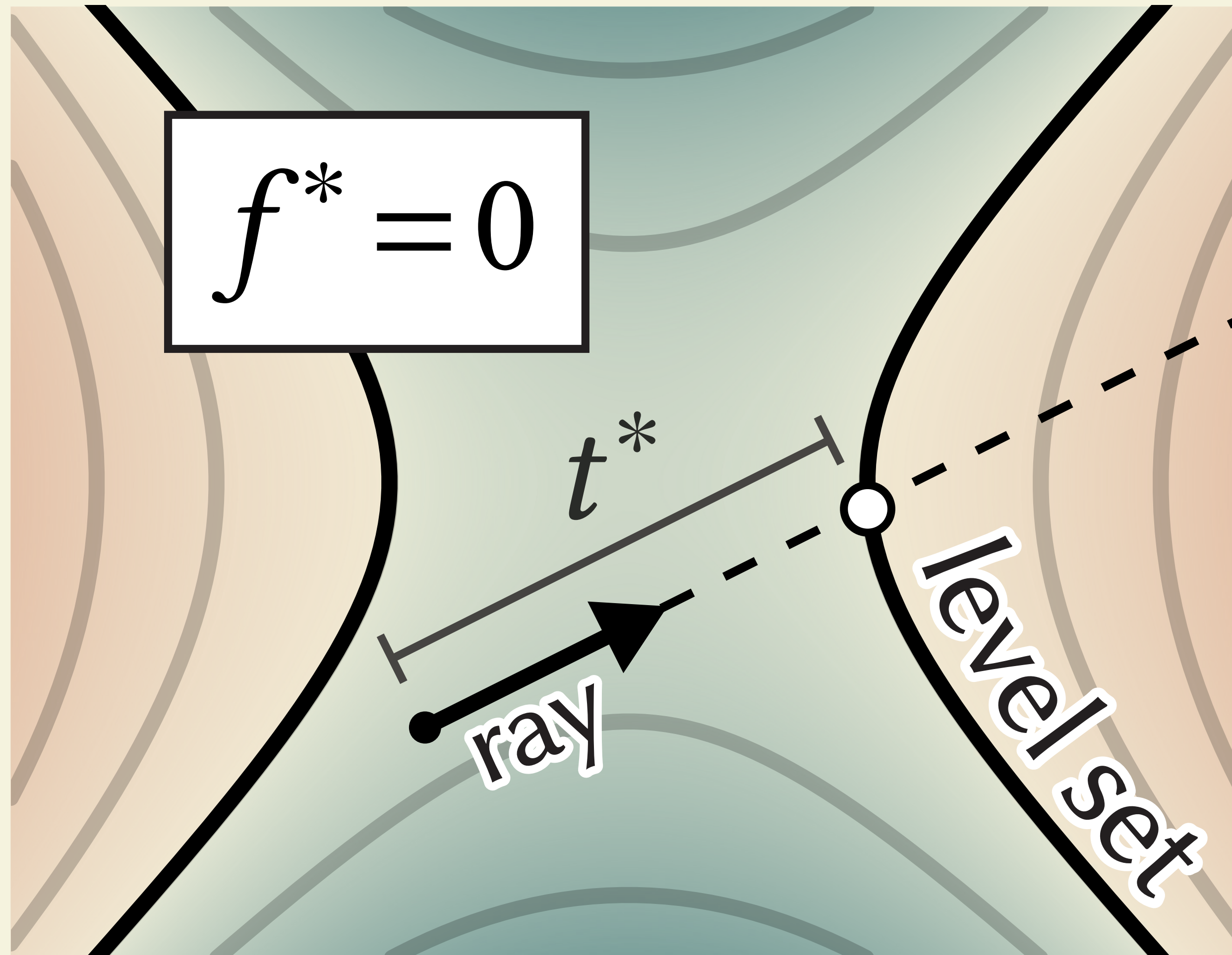
harmonic Green's function

# Harmonic functions

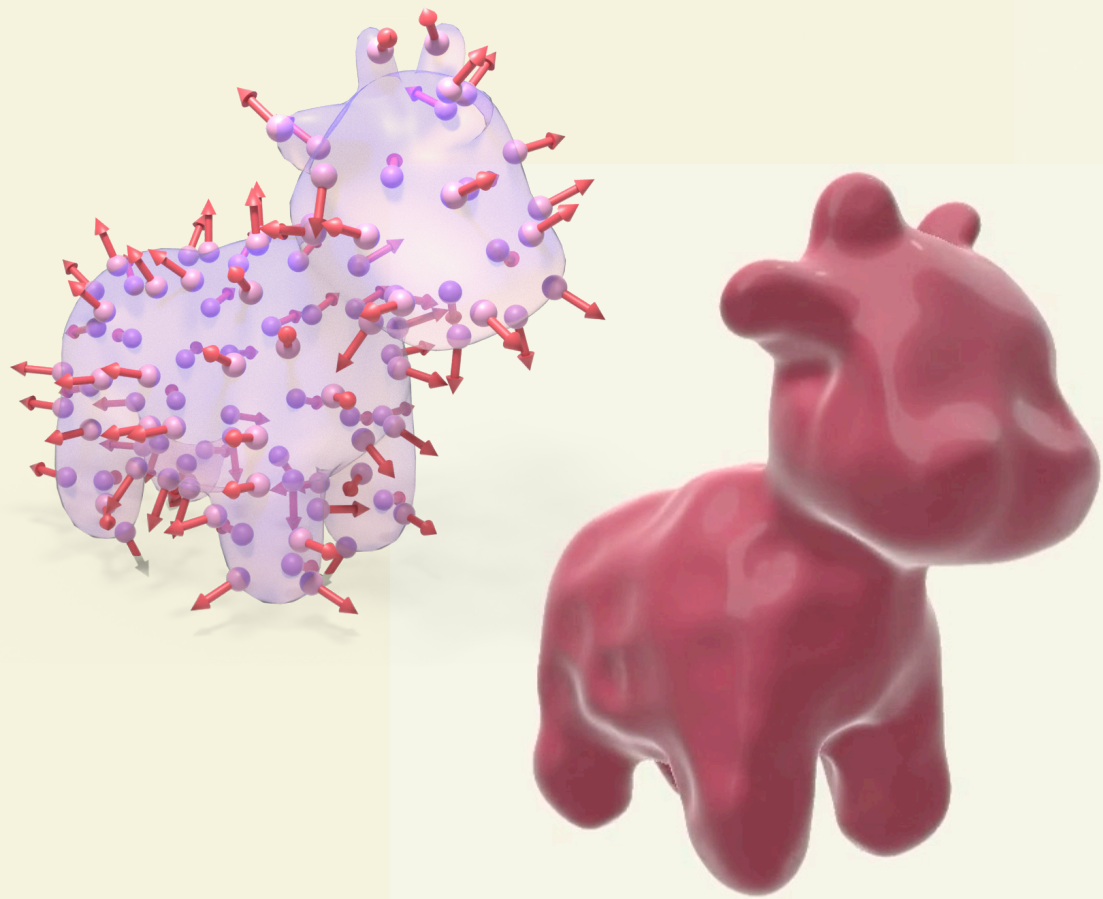
$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$$



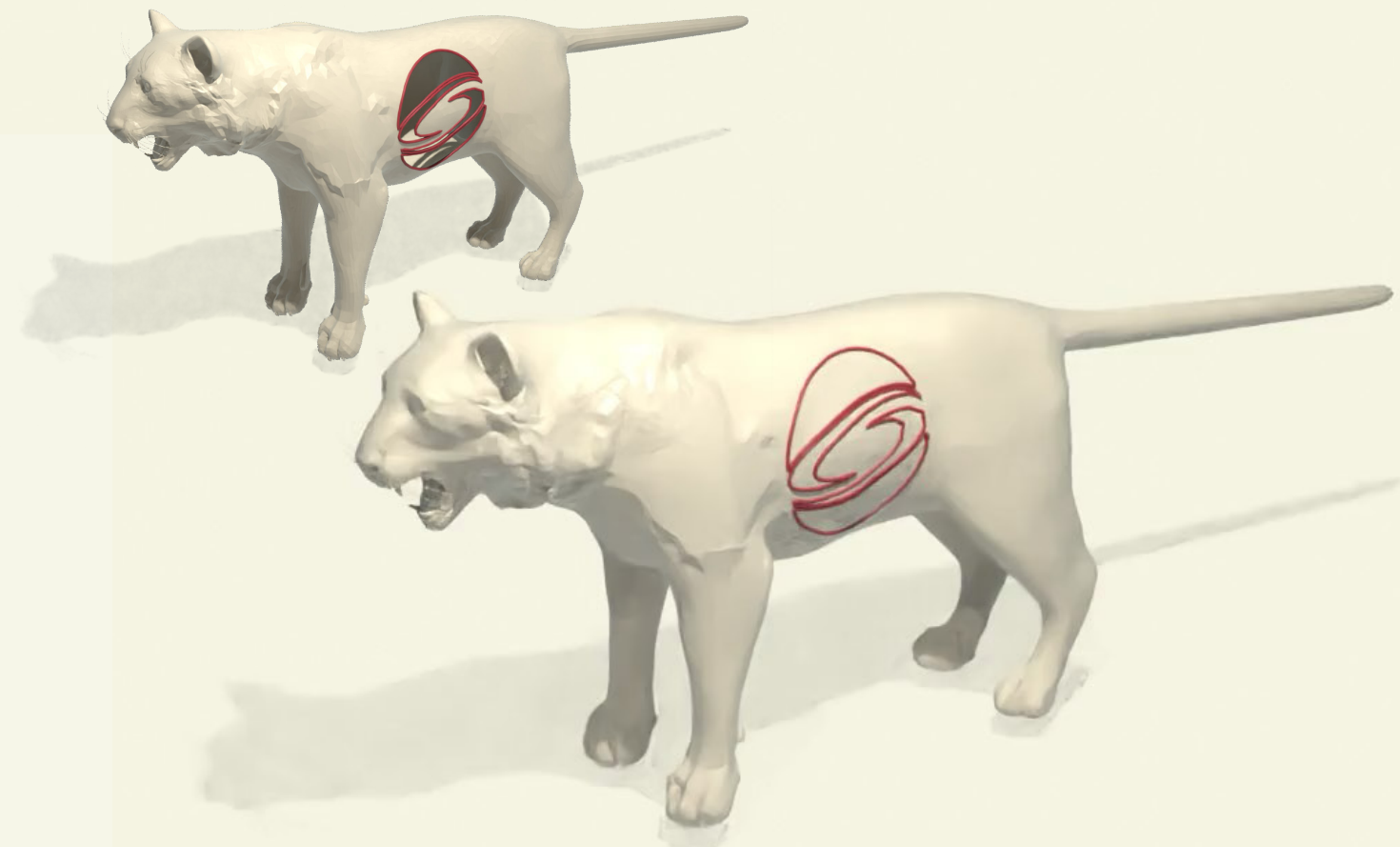
# Intersecting a ray with a level set



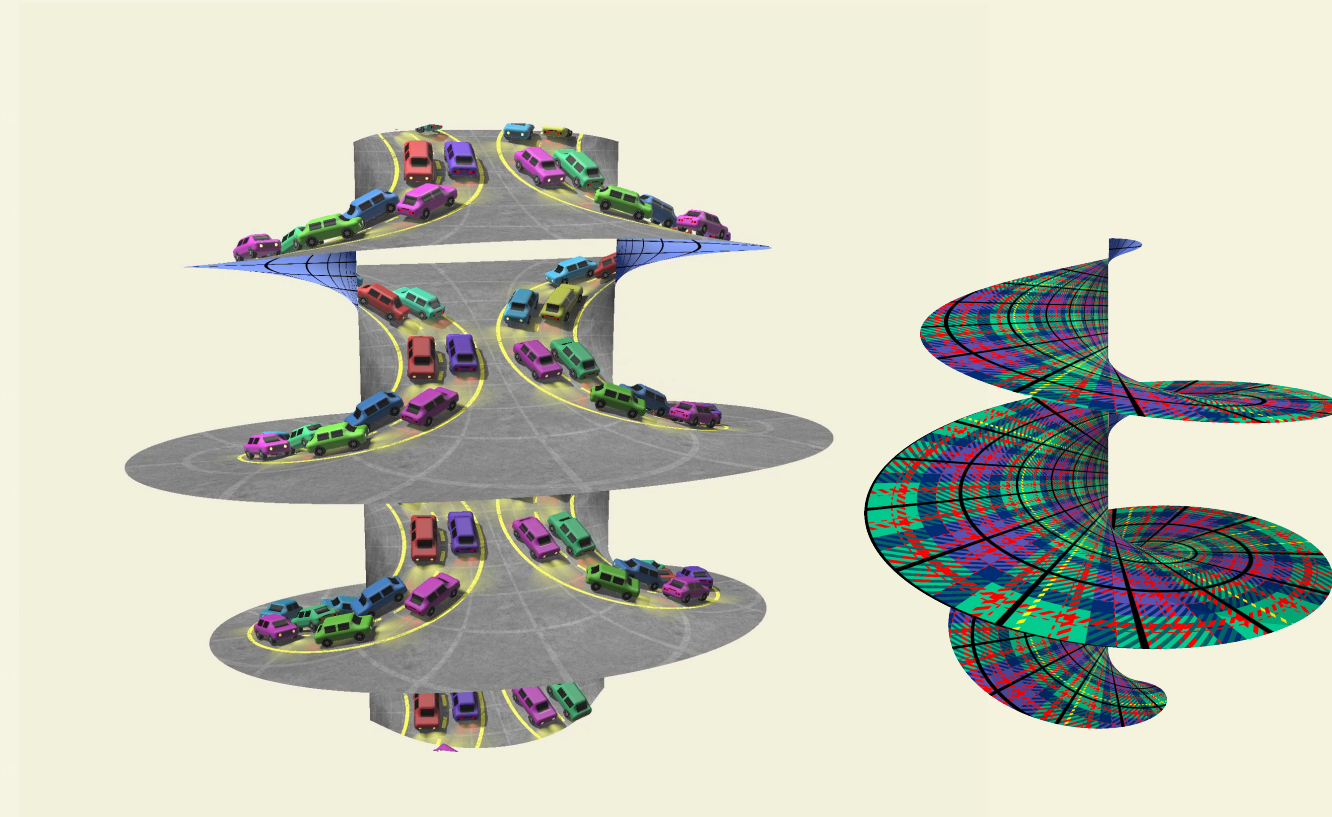
# Level sets of harmonic functions show up everywhere



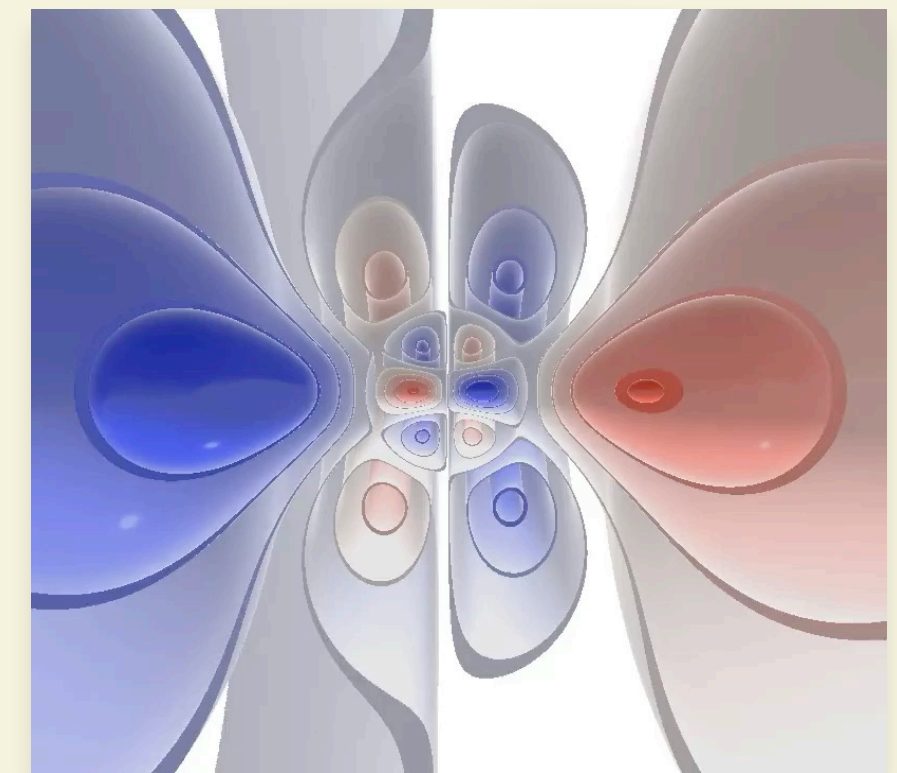
Poisson surface reconstruction  
[ Kazhdan *et al.* 2006 ]



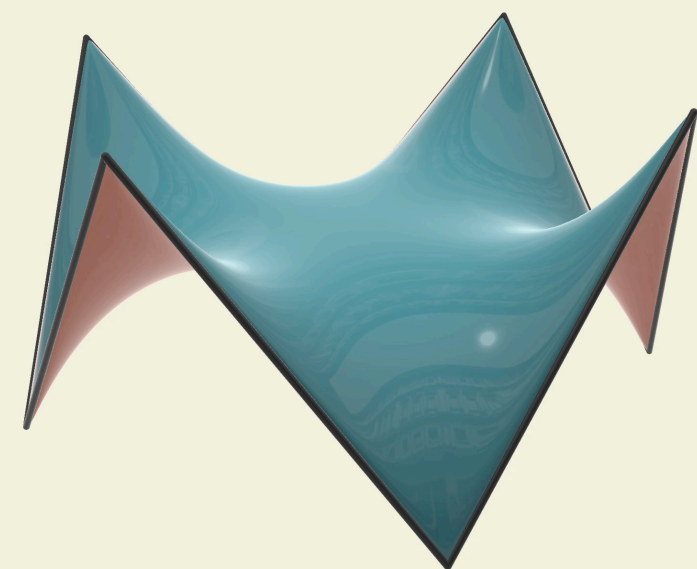
generalized winding numbers  
[ Jacobson *et al.* 2013 ]



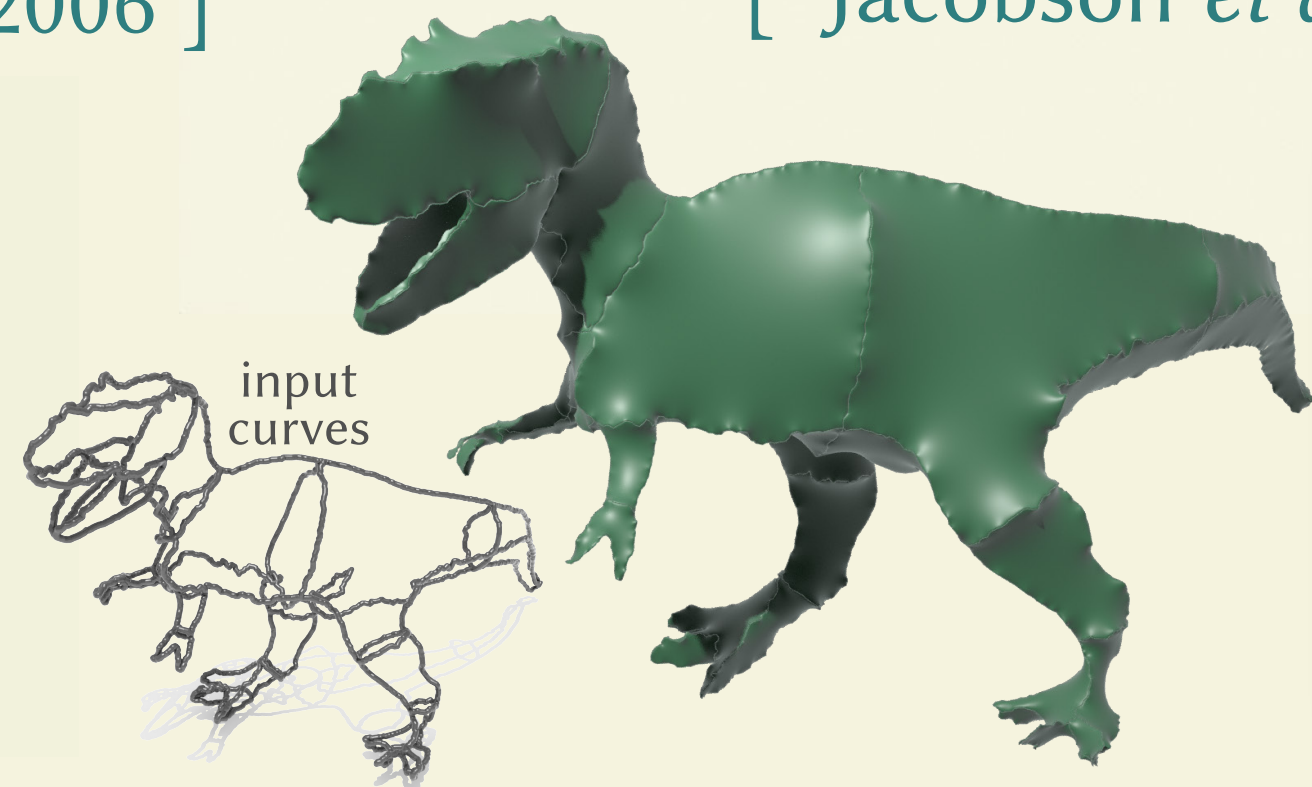
Riemann surfaces  
[ Riemann 1851 ]



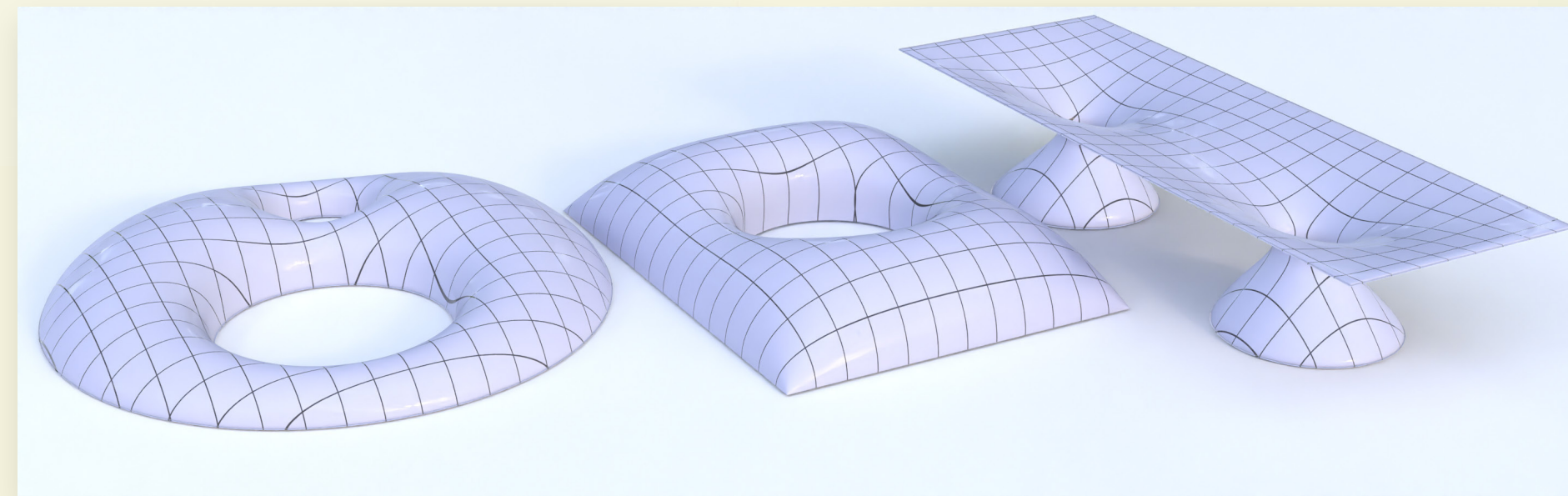
hyperspherical harmonics  
[ Fock 1935 ]



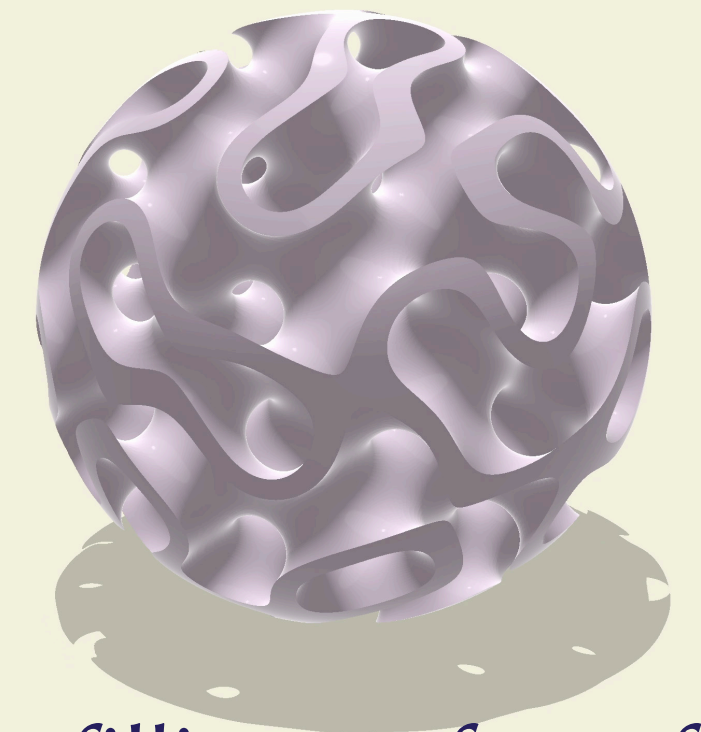
nonplanar polygons  
[ Maxwell 1873 ]



curve networks  
[ de Goes *et al.* 2011 ]



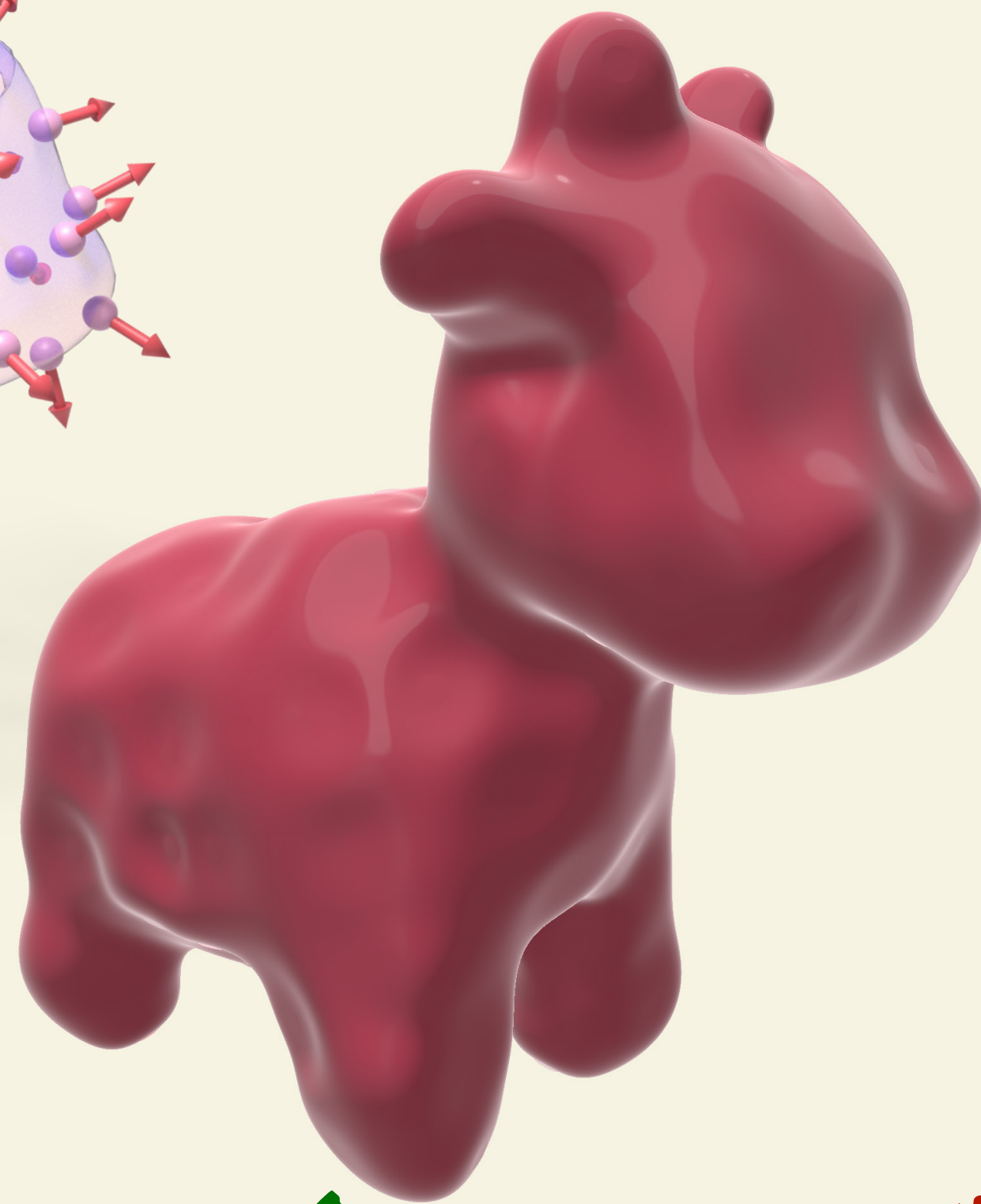
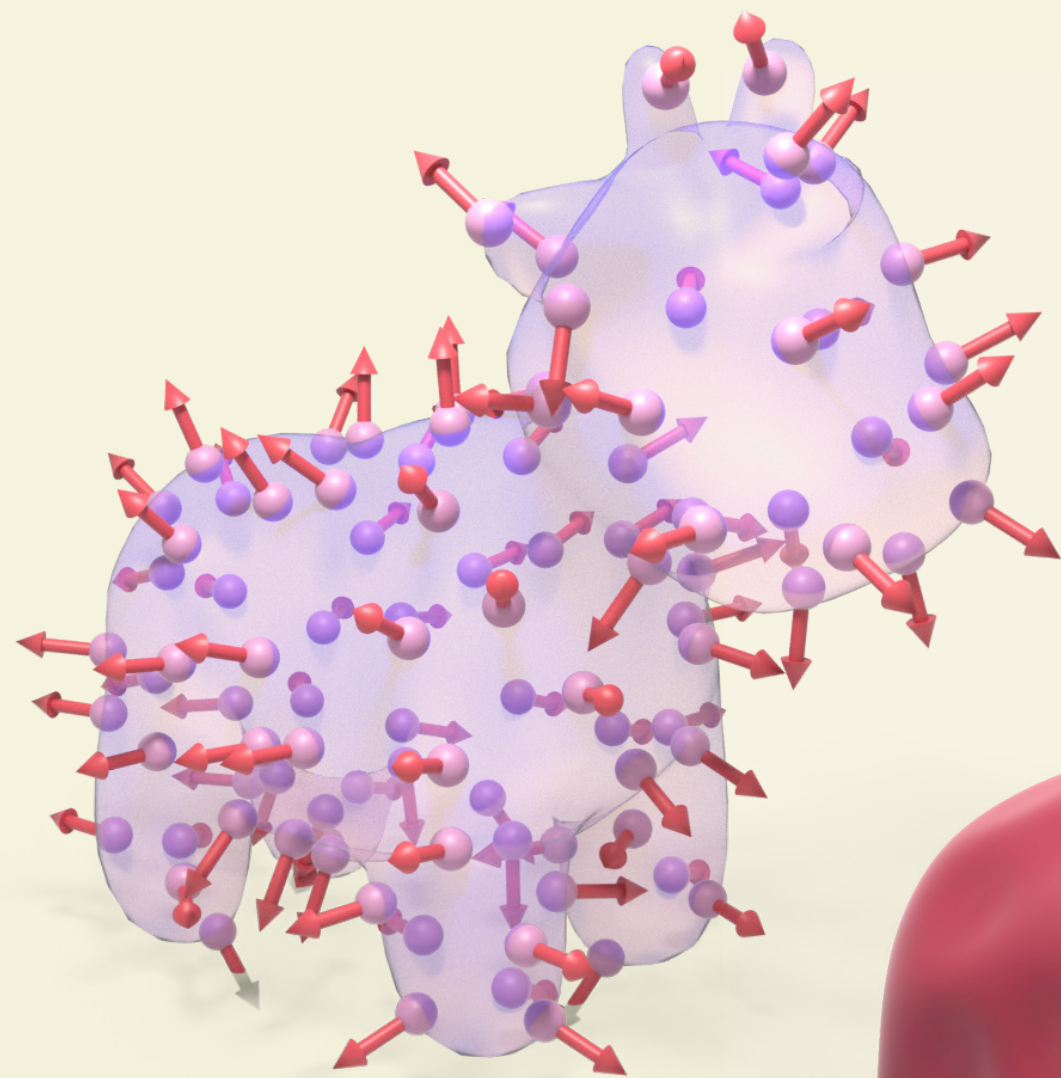
shell structures in architectural  
geometry [ Adiels *et al.* 2022 ]



space-filling surfaces for  
digital fabrication

... but, they're hard to render with existing techniques

may have singularities



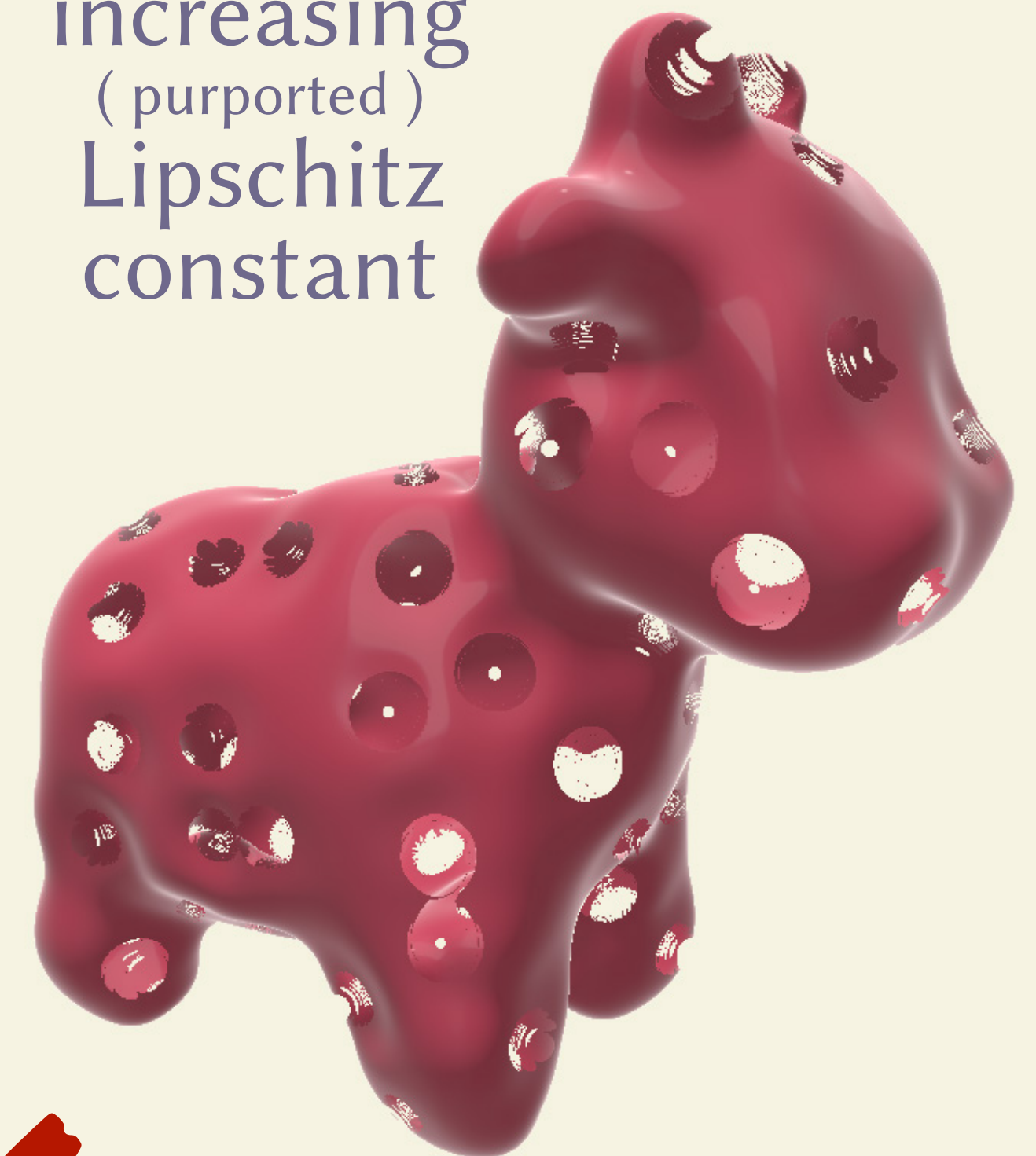
✓ ours

decreasing step size



✗ ray marching  
(with fixed step size)

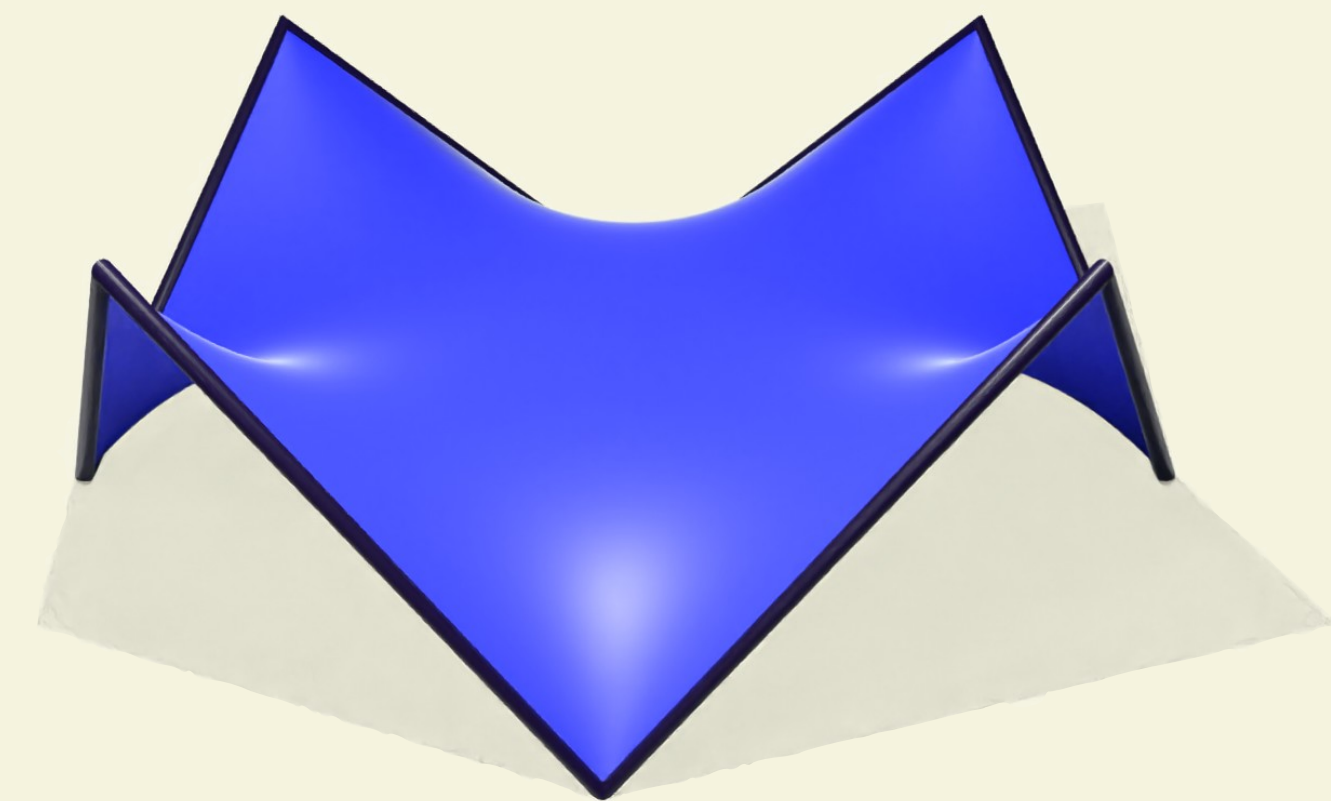
increasing  
(purported)  
Lipschitz  
constant



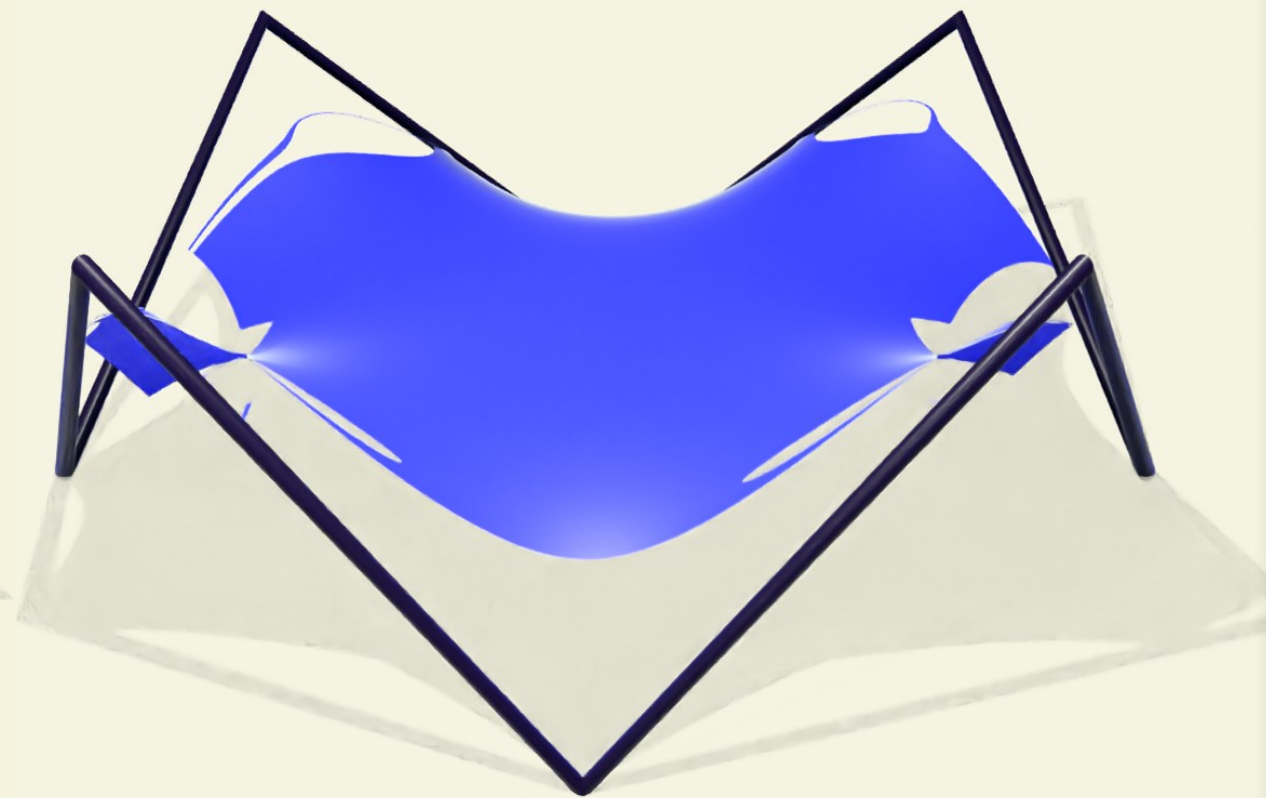
✗ sphere tracing  
(with purported Lipschitz constant)

... but, they're hard to render with existing techniques

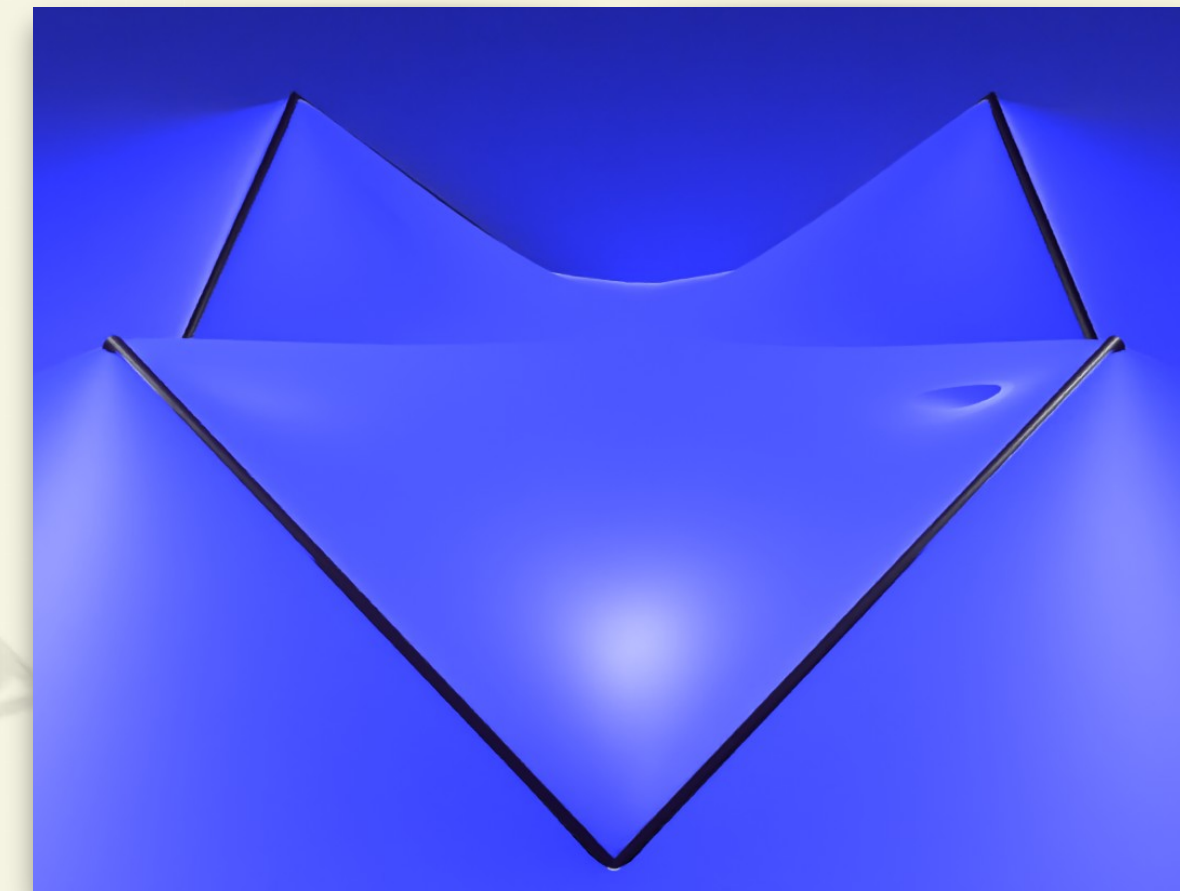
may have boundaries



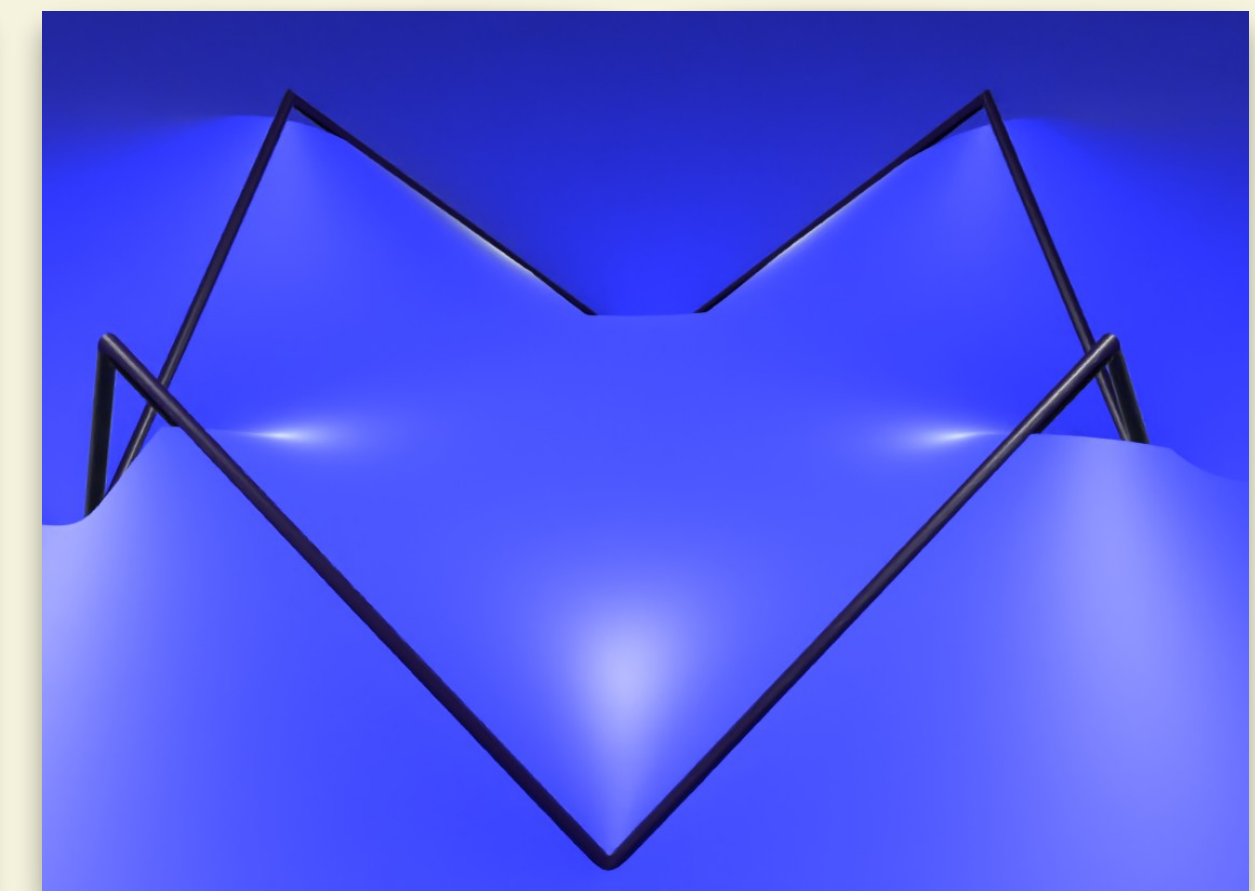
ours



Newton's method



interval arithmetic

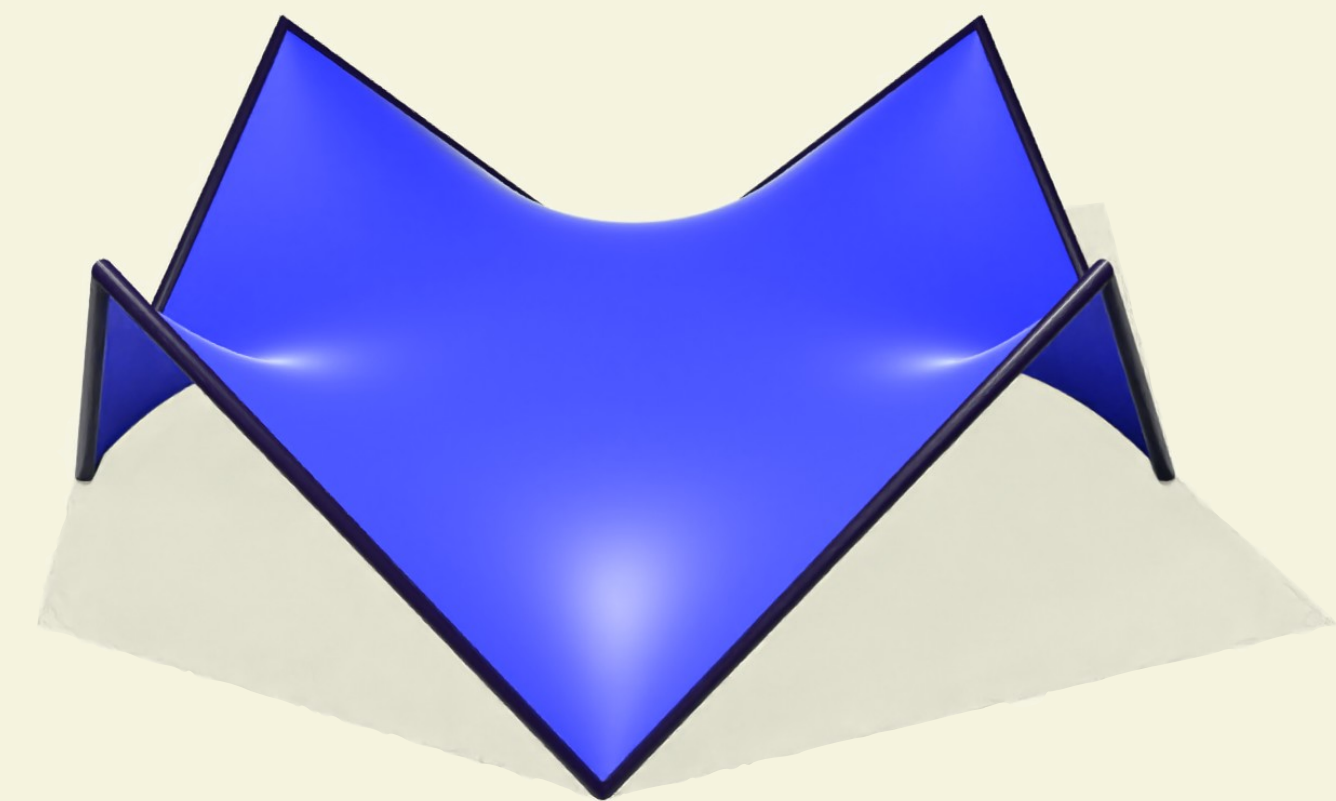


bisection search

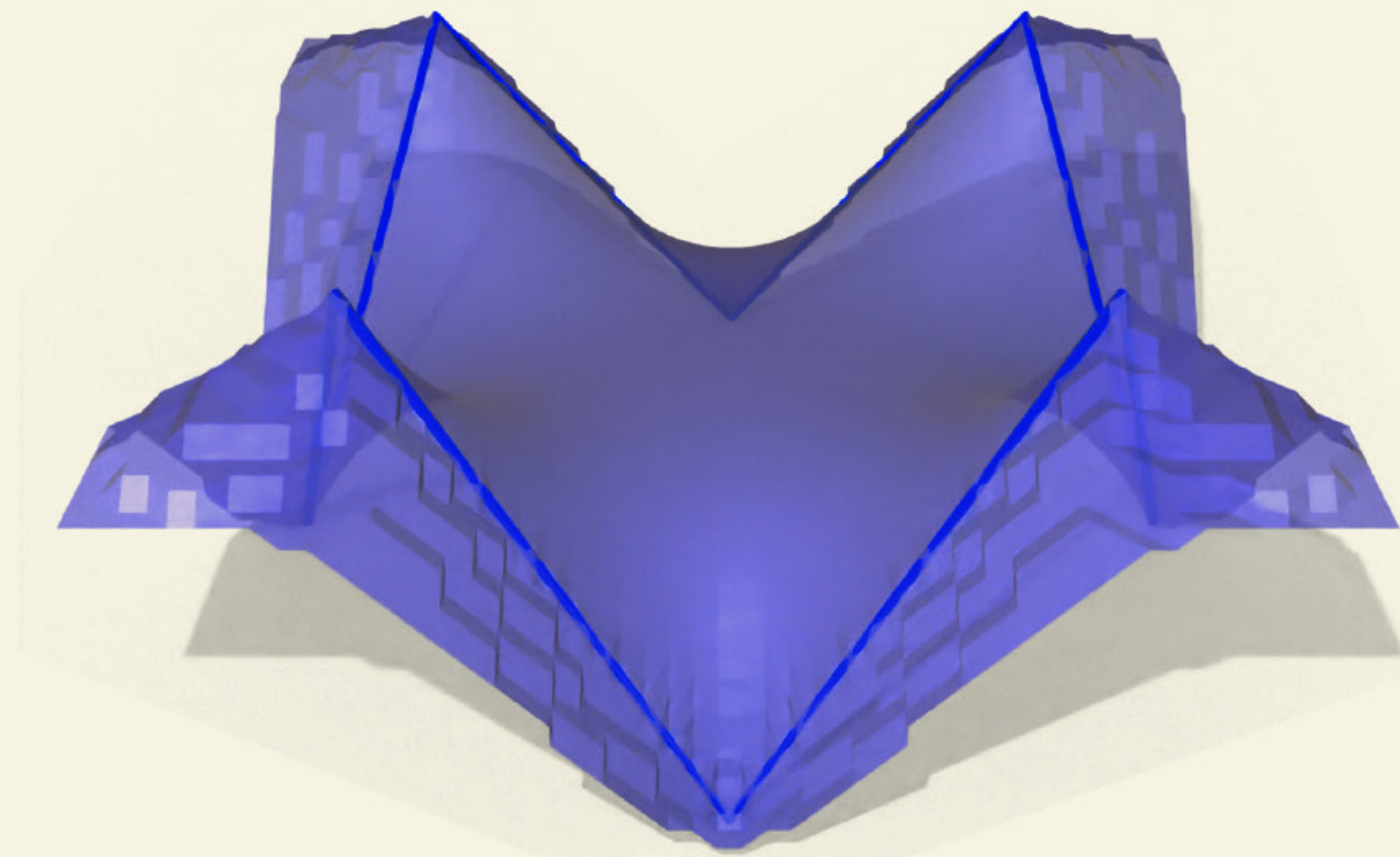


... but, they're hard to render with existing techniques

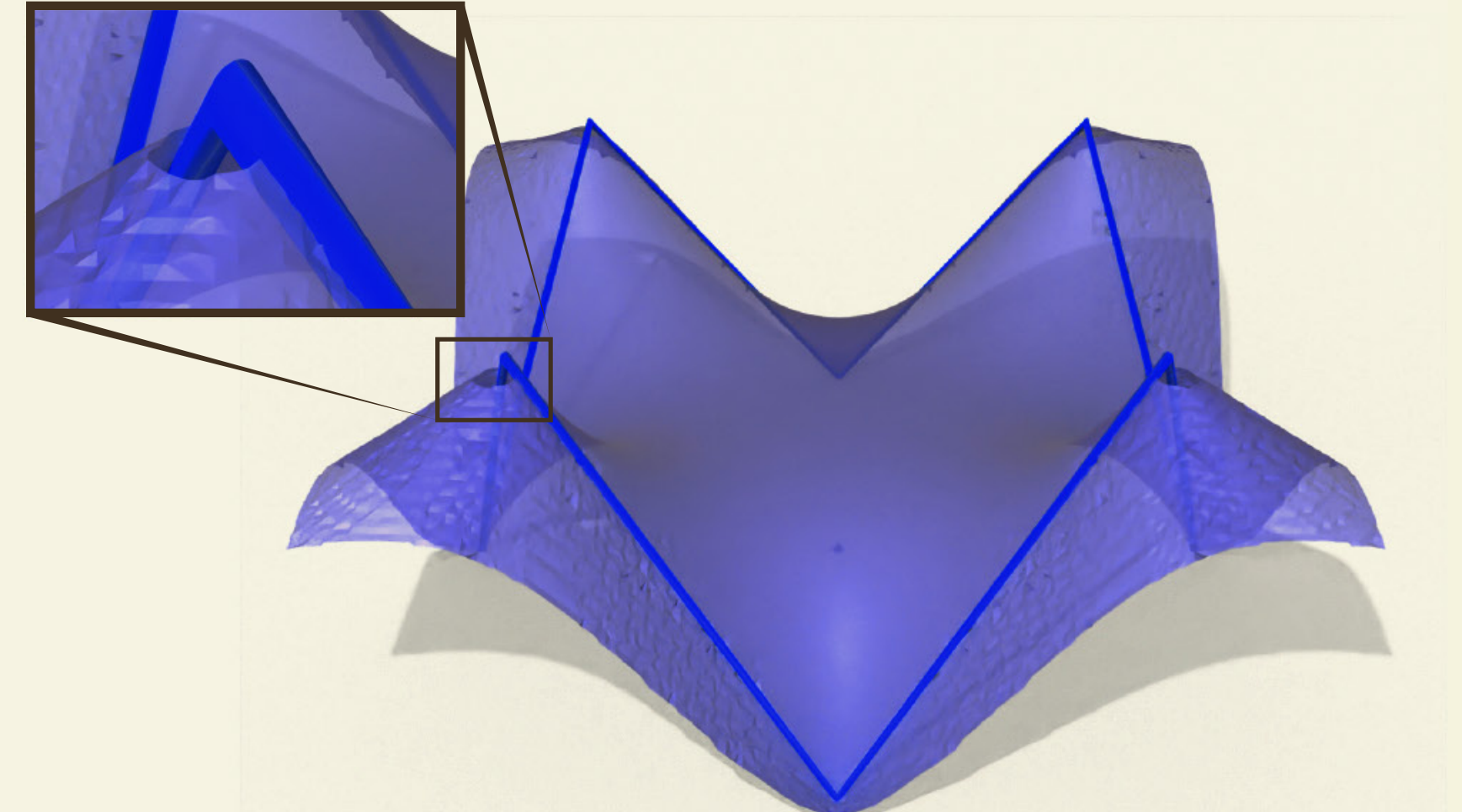
may have boundaries



✓ ours



✗ marching cubes

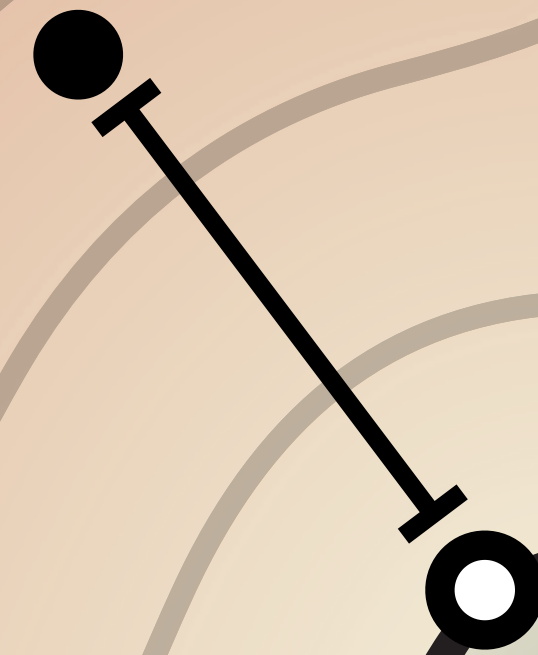


✗ Mathematica (ContourPlot3D)

# Sphere tracing

[ Hart 1996 ]

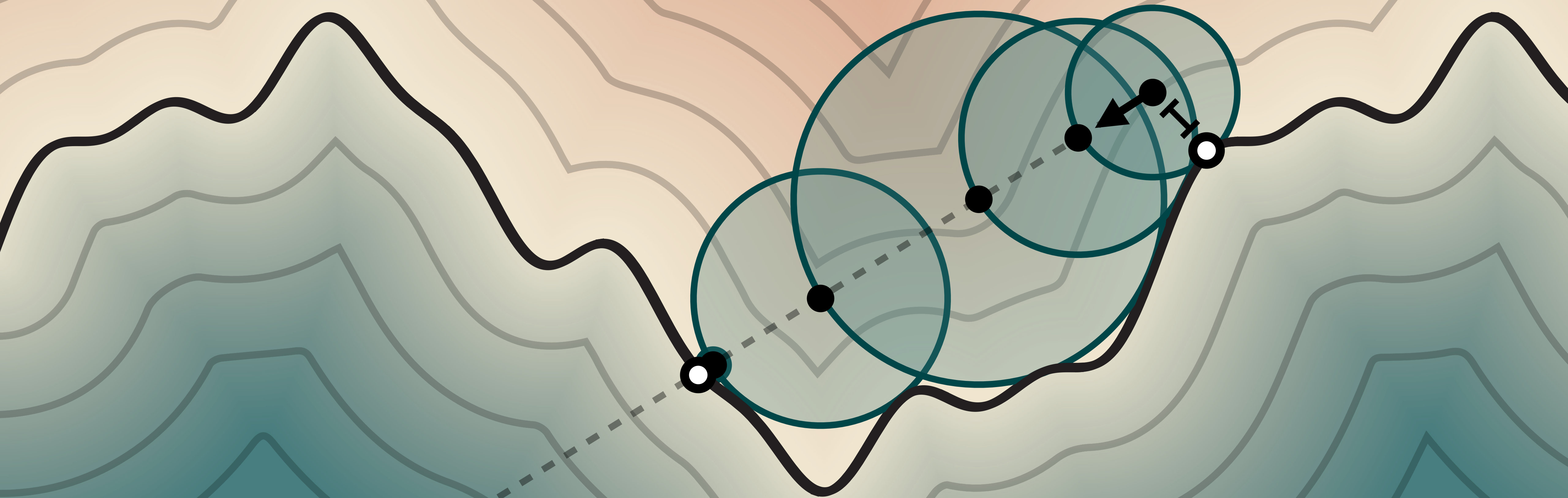
$f(x)$  = distance to curve



compute intersections for *signed distance functions (SDFs)*

# Sphere tracing

[ Hart 1996 ]



compute intersections for *signed distance functions (SDFs)*

# Sphere tracing: beyond SDFs

[ Hart 1996 ]

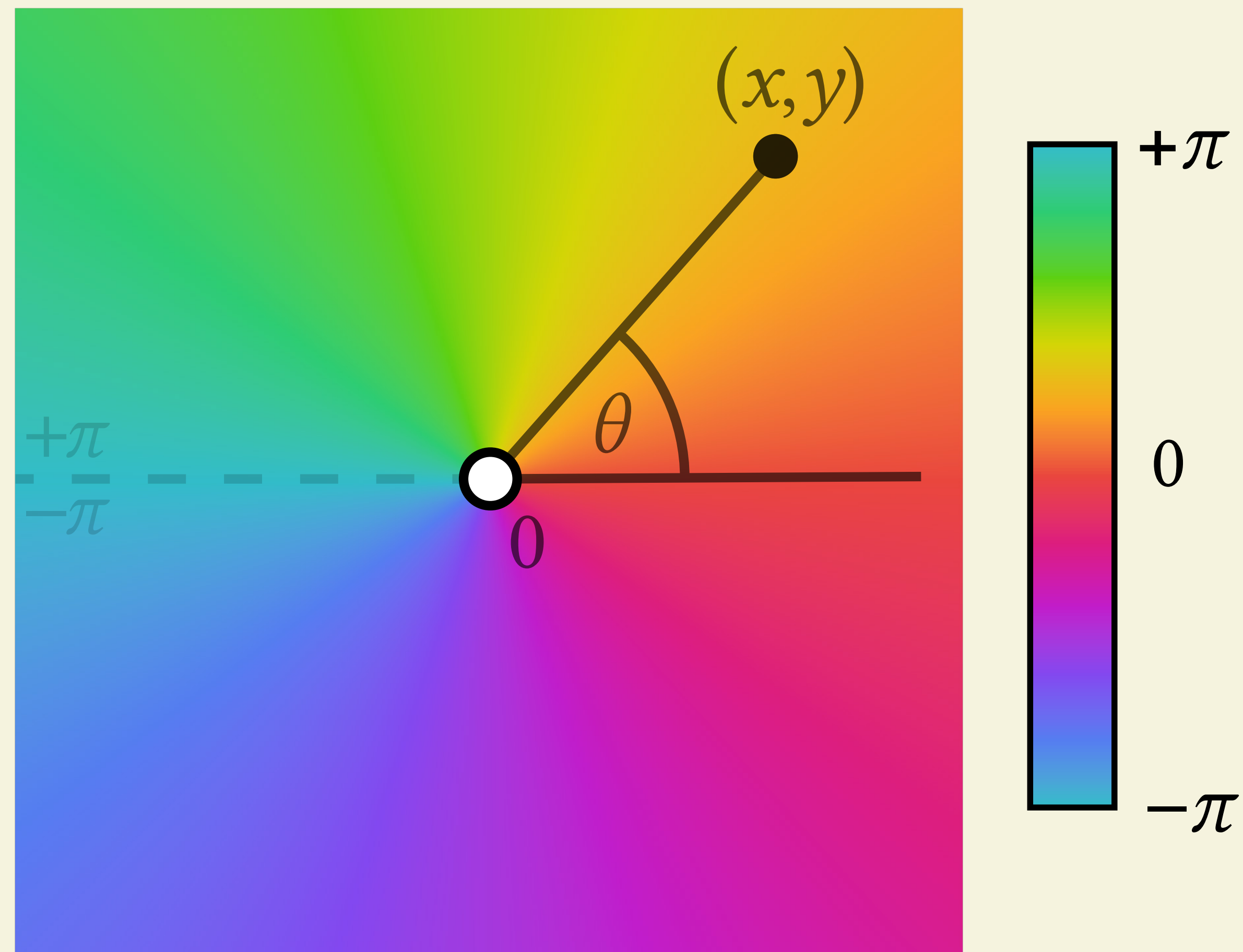
- Easy to generalize to *Lipschitz* functions:  
(essentially,  $|\nabla f| \leq L$ )
- Important fact:  
 $|f(x) - f(y)| \leq L|x - y|$
- provides a conservative bound on distance



[ Inigo Quilez 2015 ]

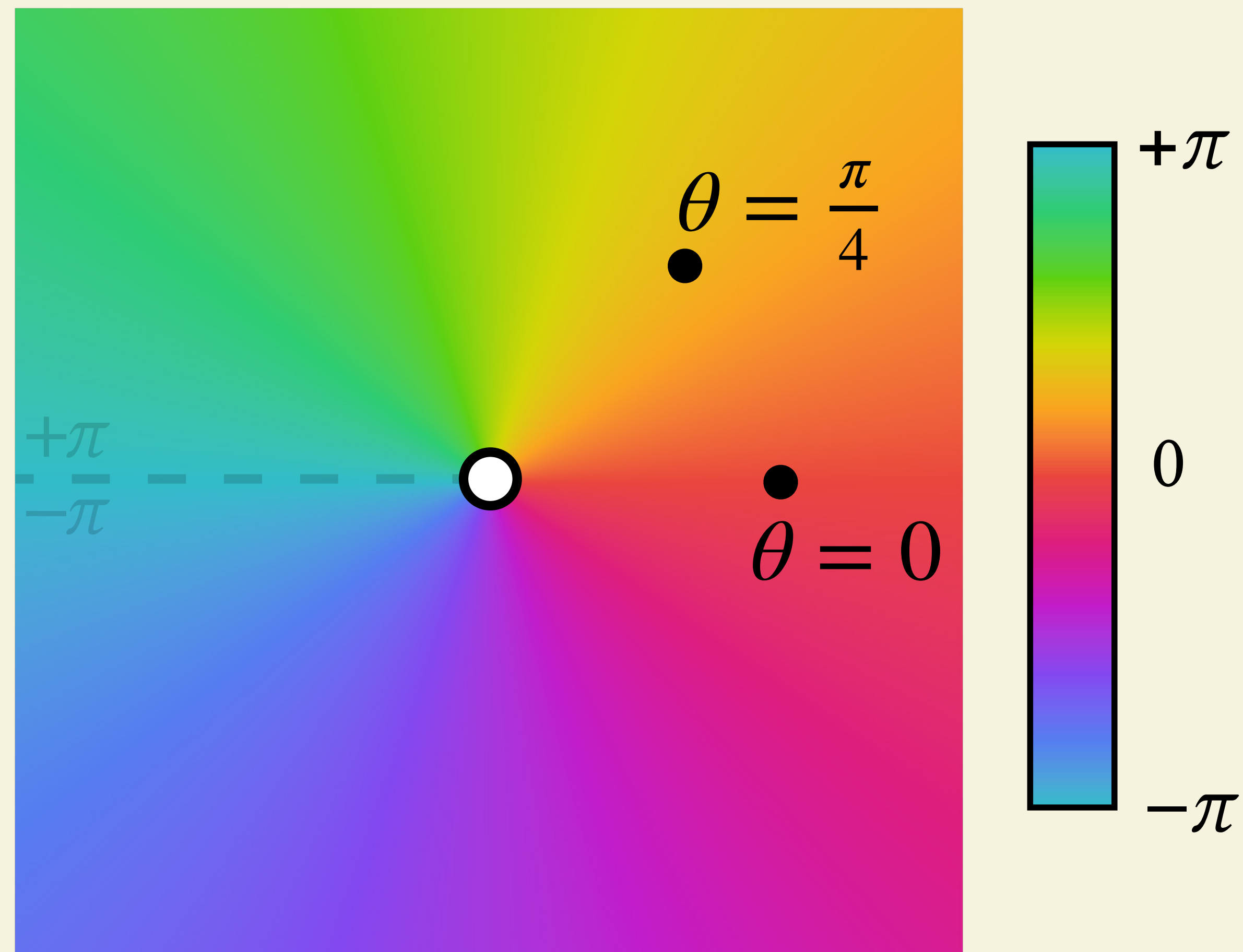
# Problem: many harmonic functions are not Lipschitz

$$\theta(x, y) = \text{atan2}(y, x)$$



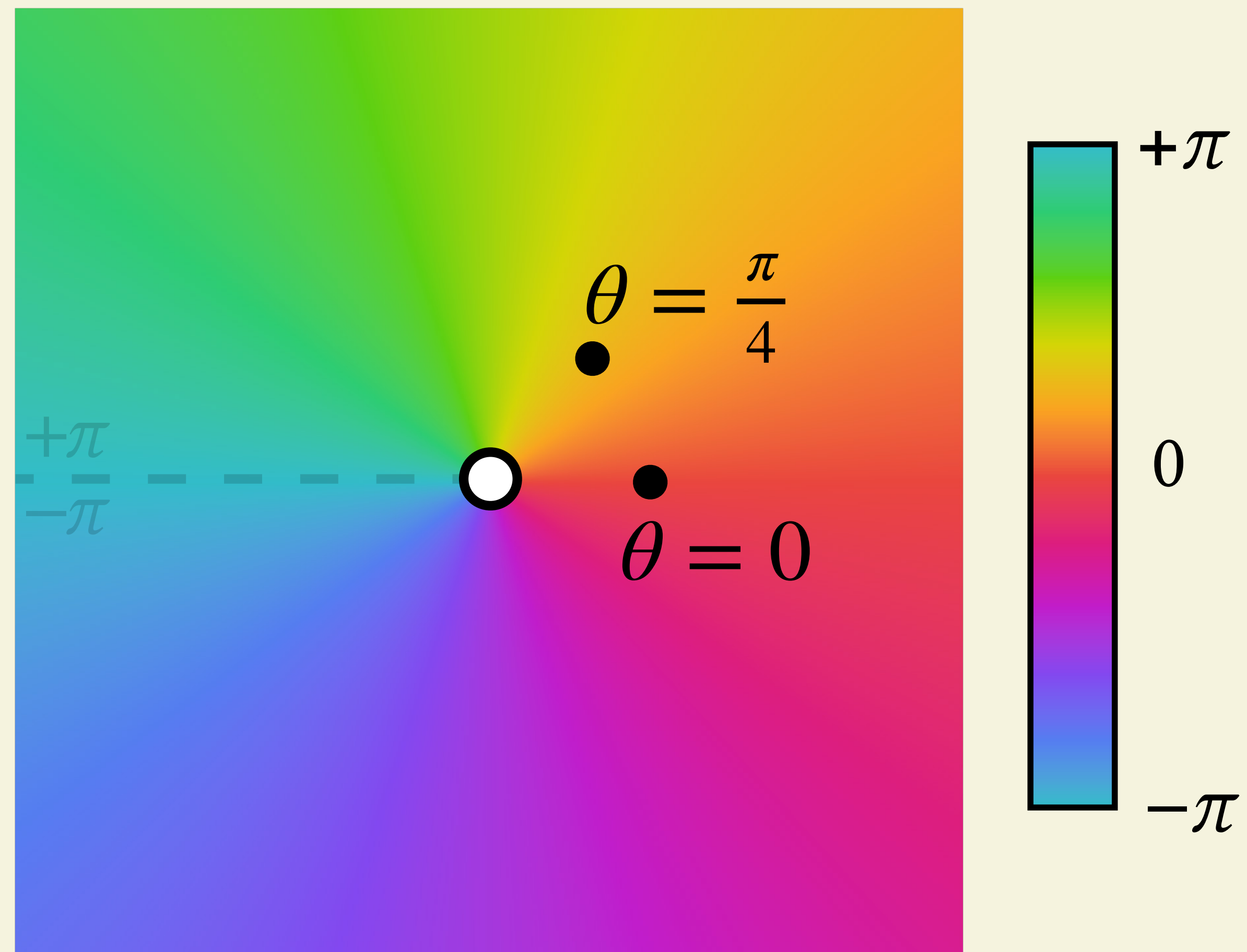
# Problem: many harmonic functions are not Lipschitz

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# Problem: many harmonic functions are not Lipschitz

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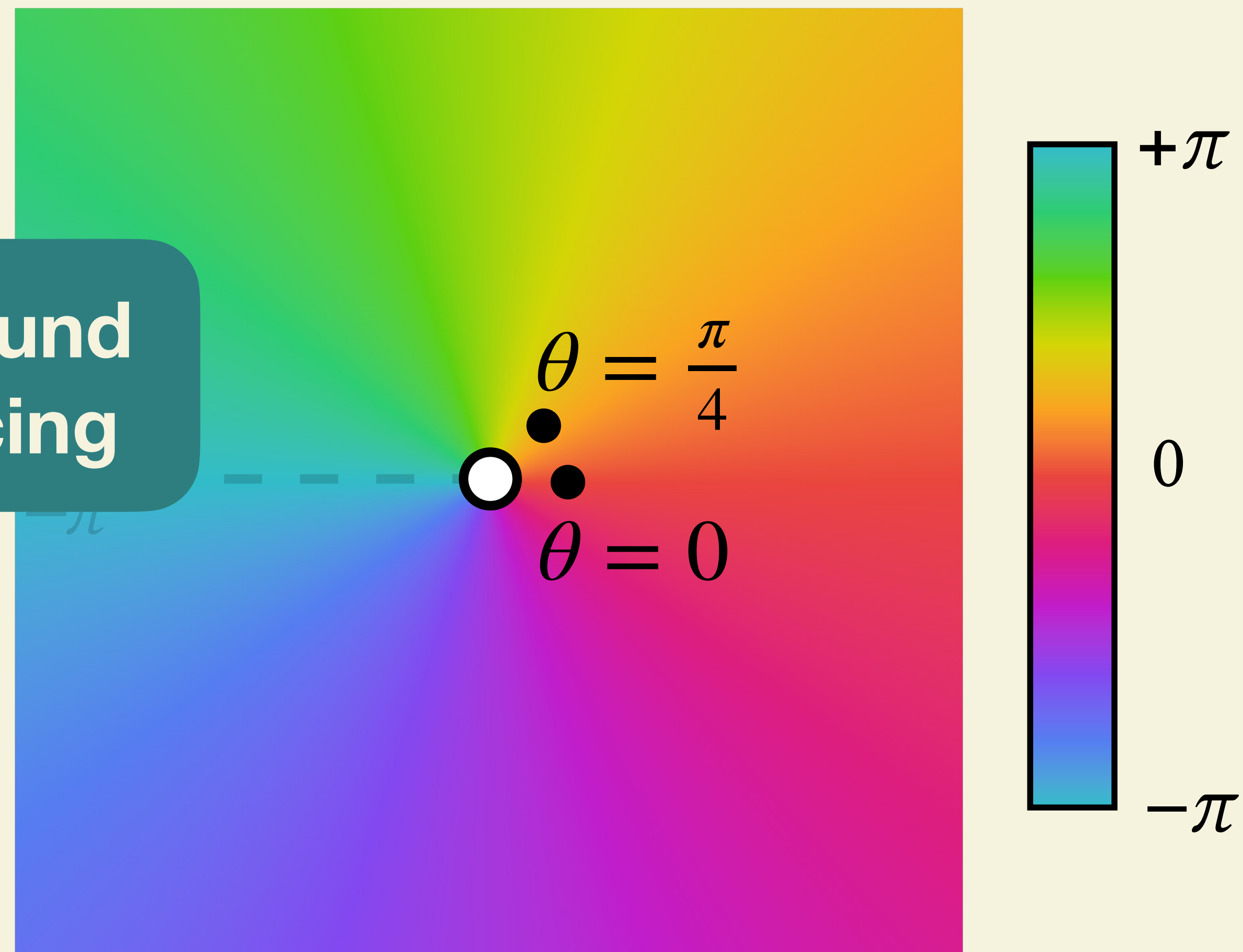


# Problem: many harmonic functions are not Lipschitz

No matter how close points get, function values never get closer

**no distance bound for sphere tracing**

$$\theta(x, y) = \text{atan2}(y, x)$$





# Main idea: get distance bounds from Harnack's inequality

Let  $f$  be a positive harmonic function on a ball:

$$\frac{1 - r/R}{(1 + r/R)^2} f(x) \leq f(y) \leq \frac{1 + r/R}{(1 - r/R)^2} f(x)$$

lower bound

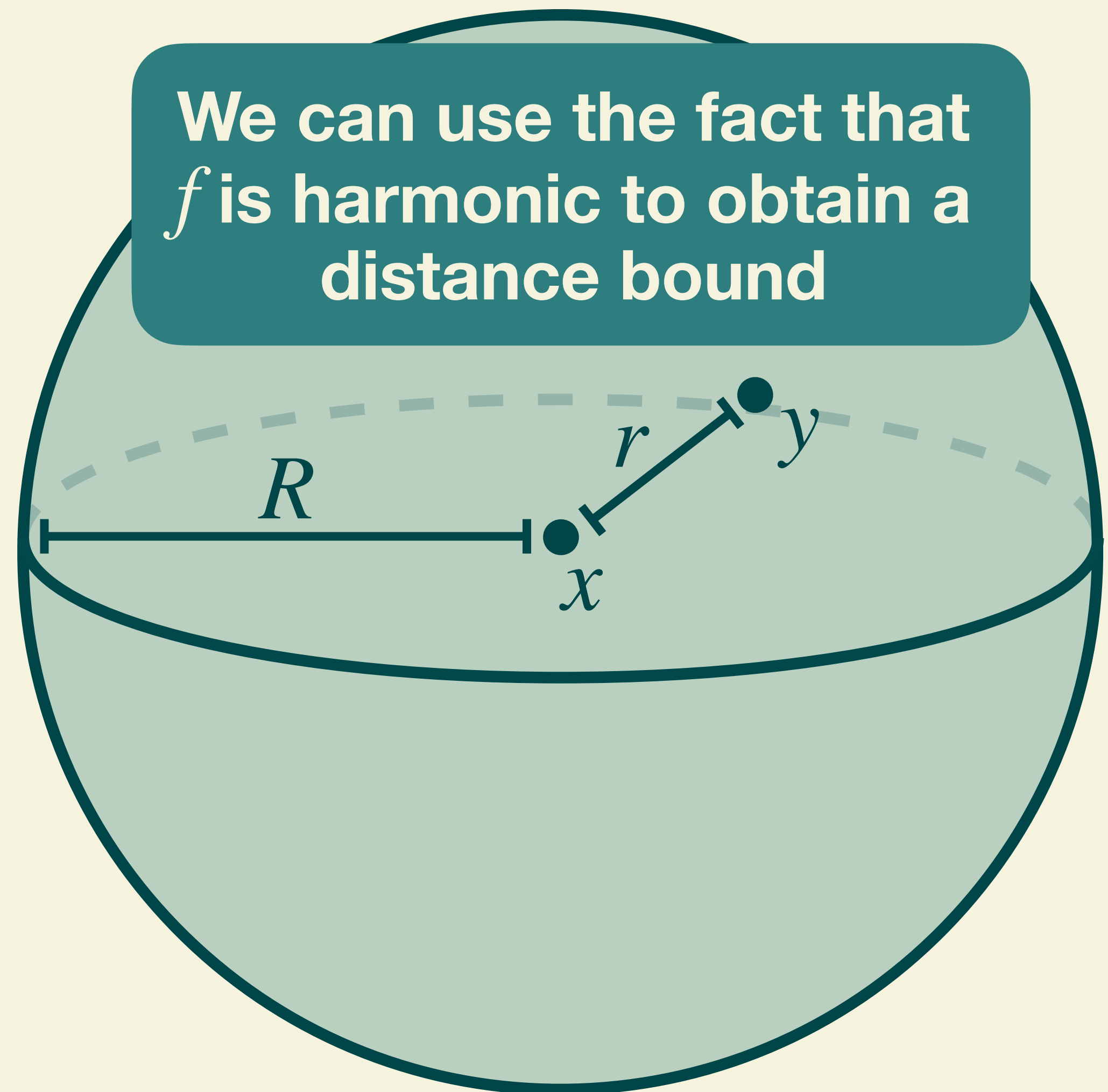
upper bound

always safe to take step of size

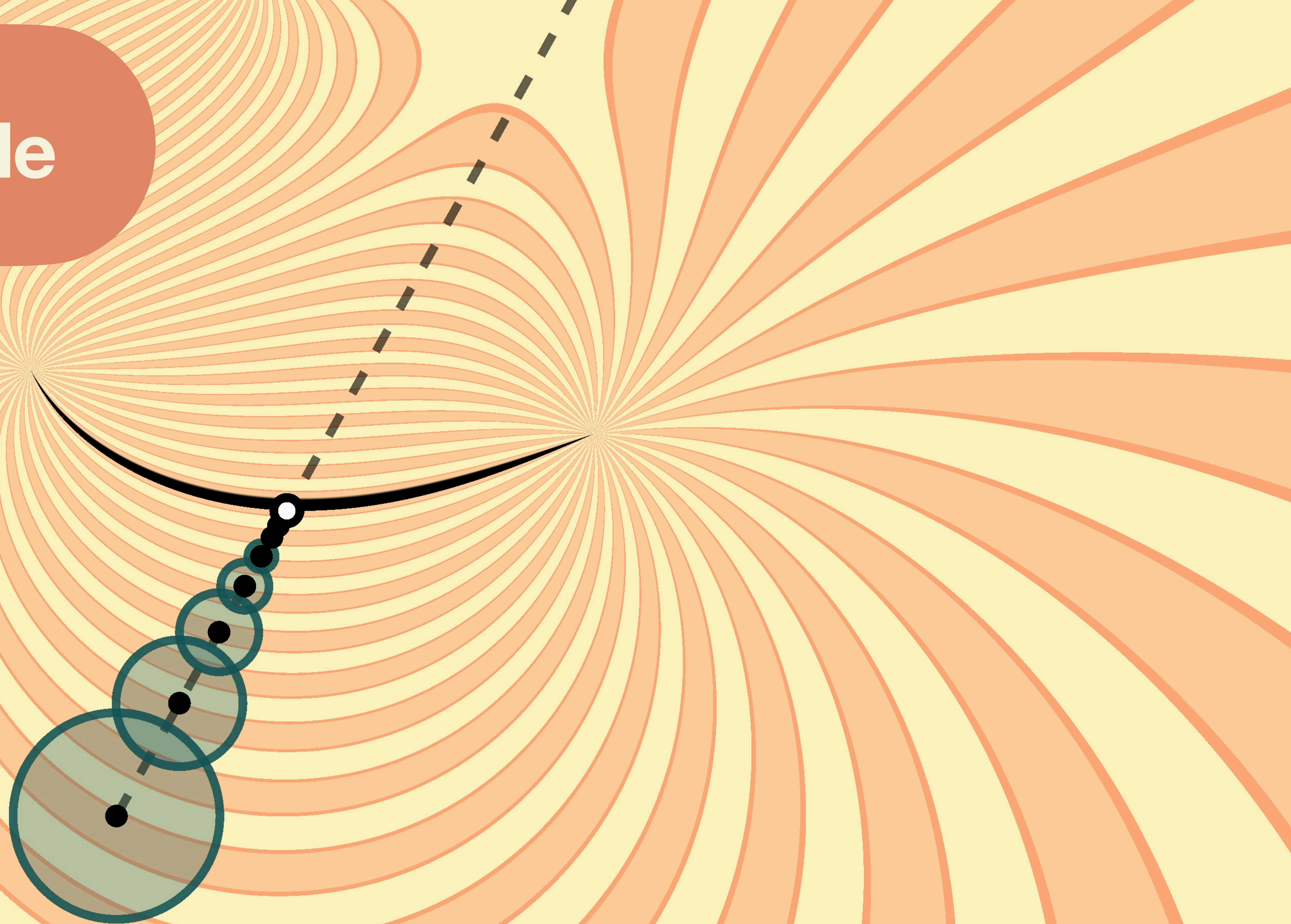
$$\frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$$

where  $a = \frac{f(x)}{f^*}$

We can use the fact that  $f$  is harmonic to obtain a distance bound

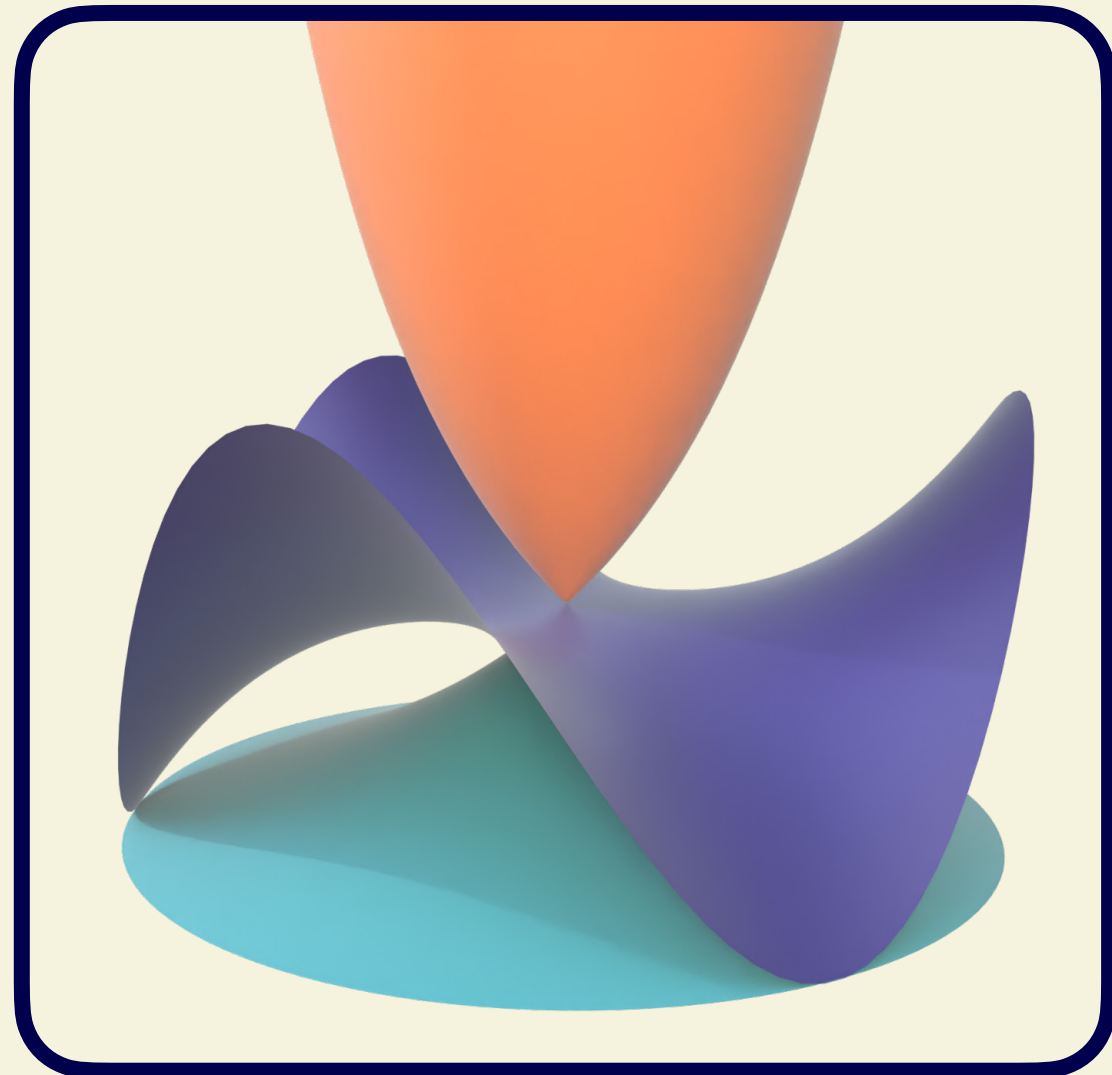


# 2D example

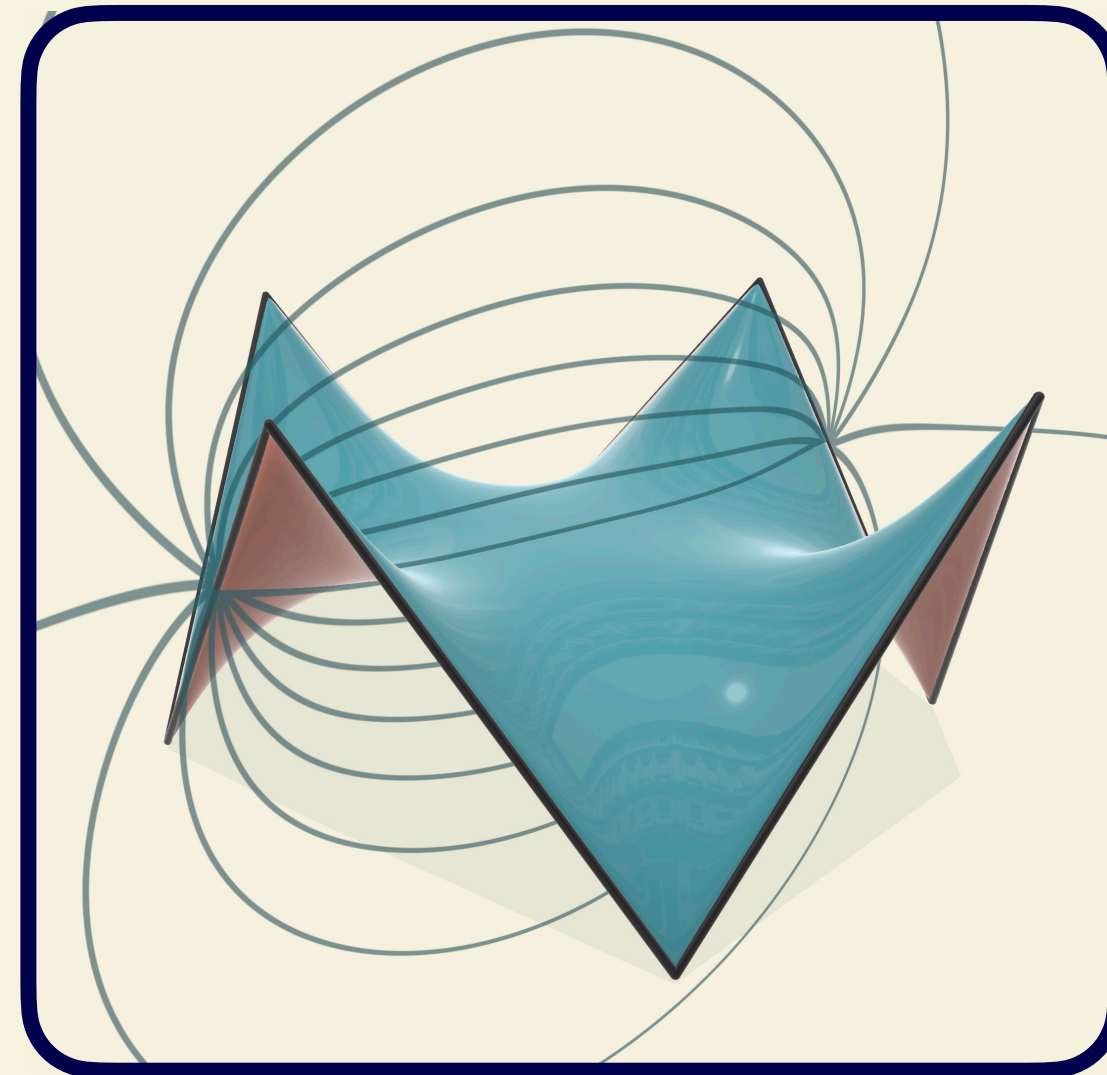


# Outline

## I. HARNACK'S INEQUALITY



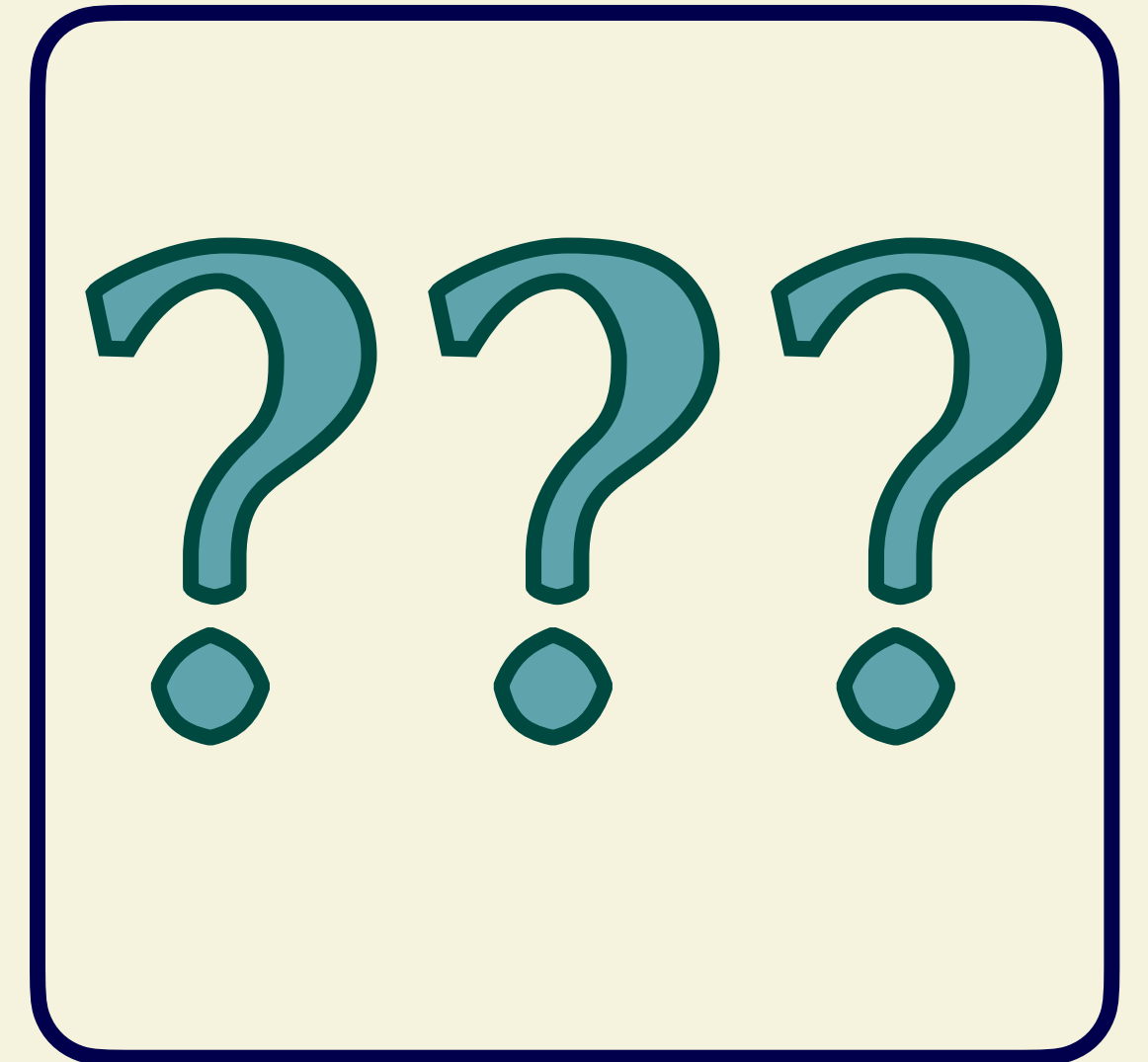
## II. HARNACK TRACING



## III. EXAMPLES



## IV. FUTURE WORK



# I. Harnack's Inequality



# Harnack's Inequality

(in  $\mathbb{R}^d$ )

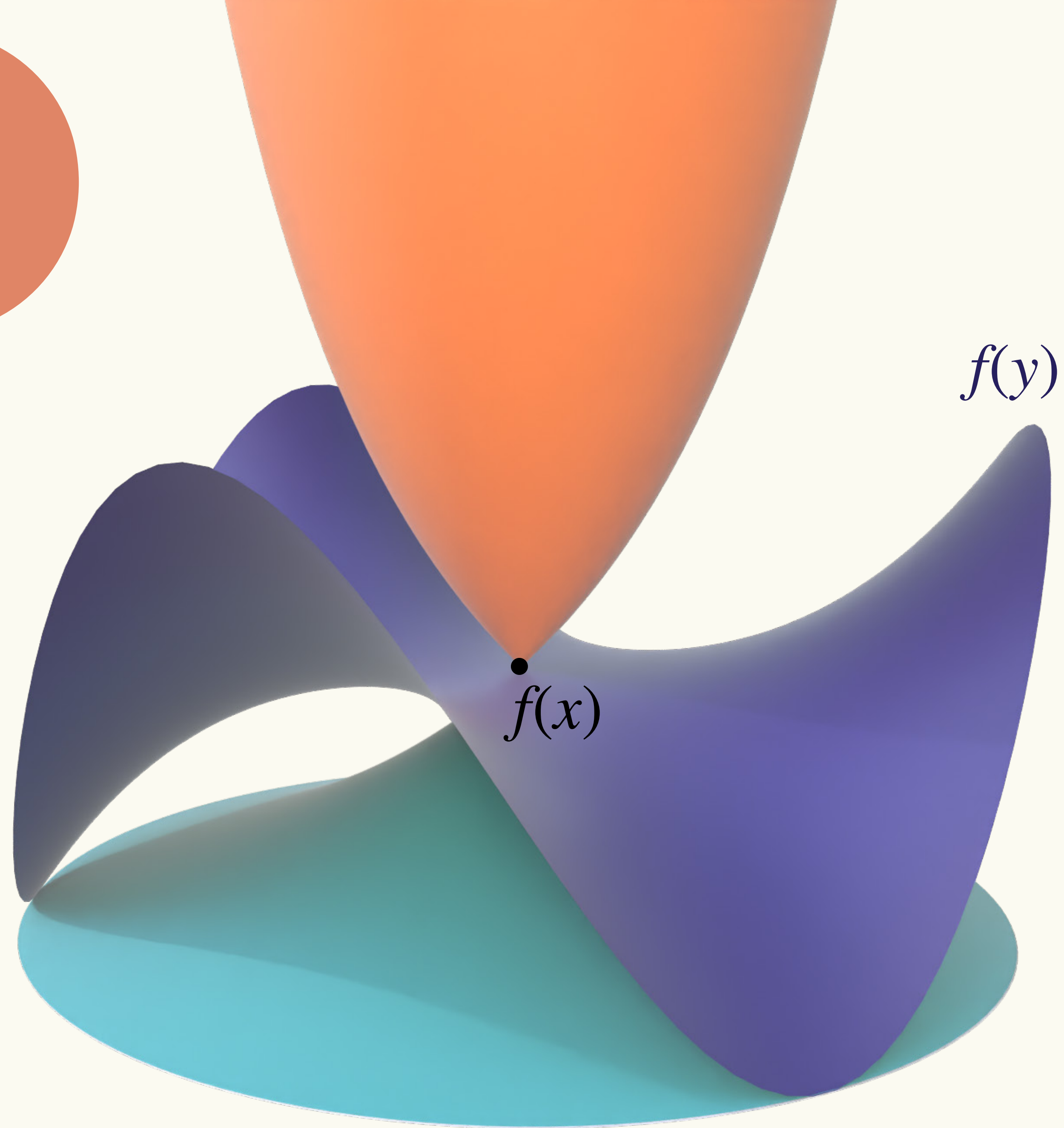
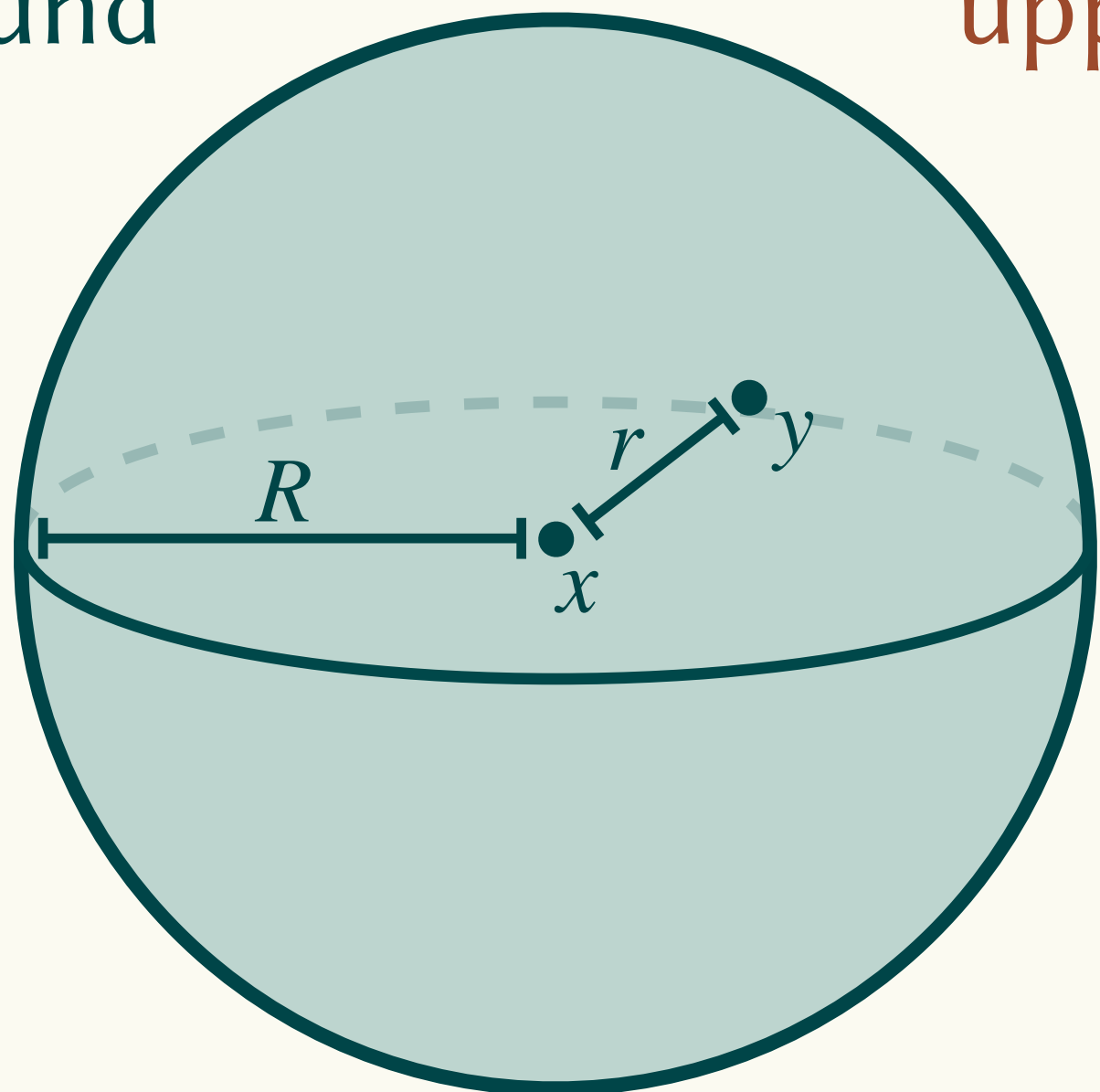
$$\frac{1 - r/R}{(1 + r/R)^{d-1}} f(x)$$

lower bound

$$\leq f(y) \leq$$

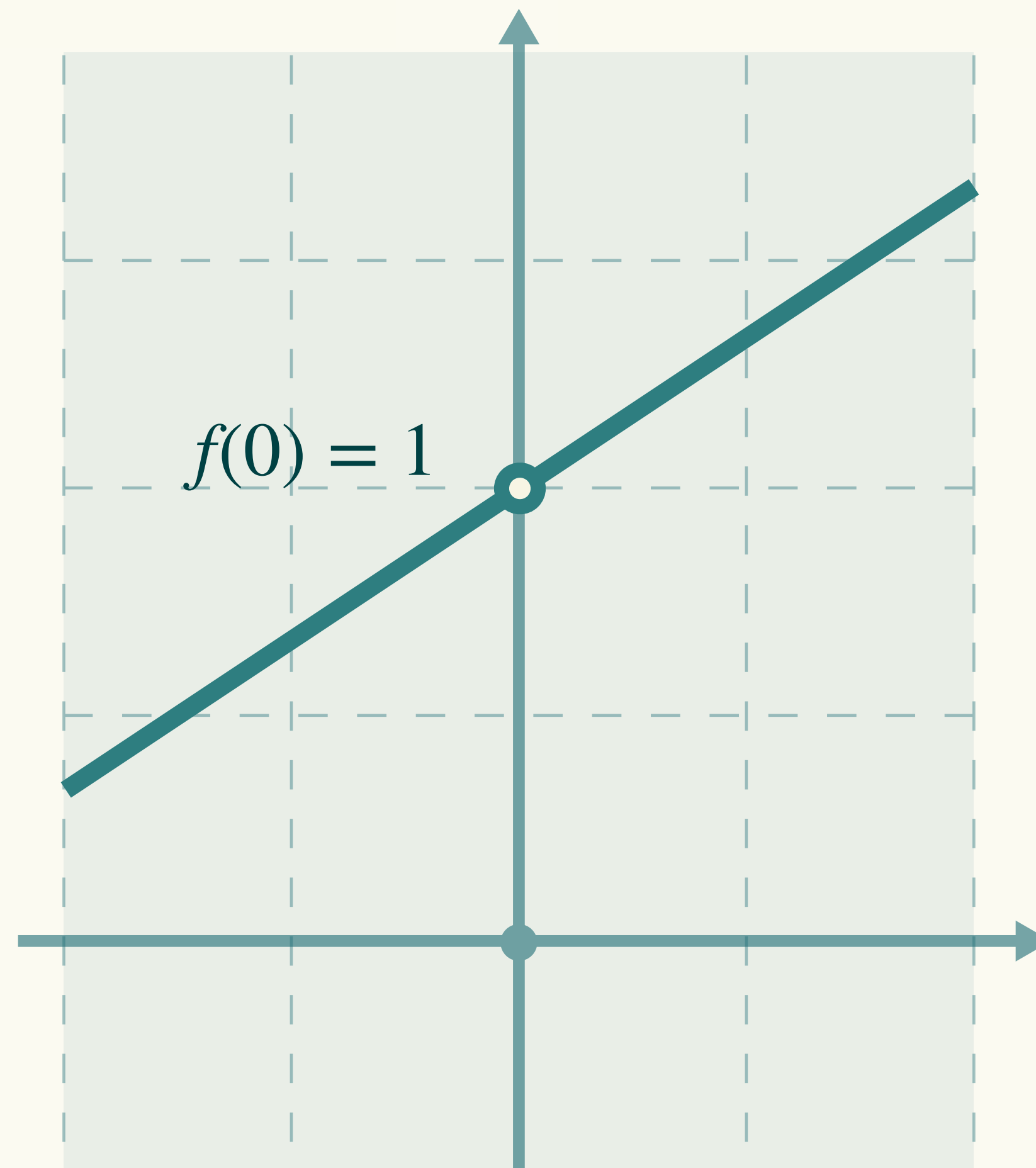
$$\frac{1 + r/R}{(1 - r/R)^{d-1}} f(x)$$

upper bound



# Prelude: Bounding Positive Linear Functions

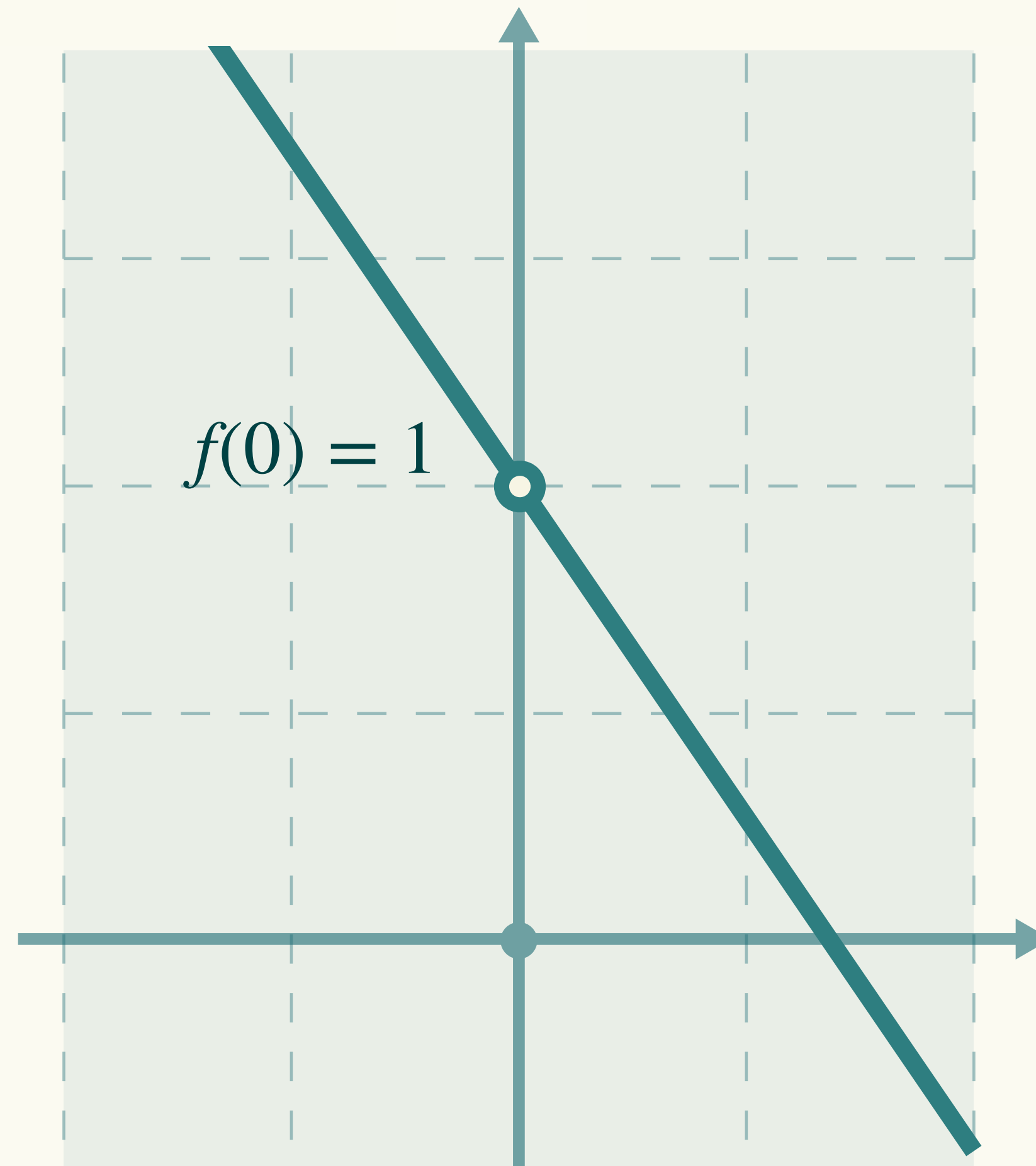
a linear function can  
change arbitrarily fast



\*technically speaking, positive affine functions

# Prelude: Bounding Positive Linear Functions

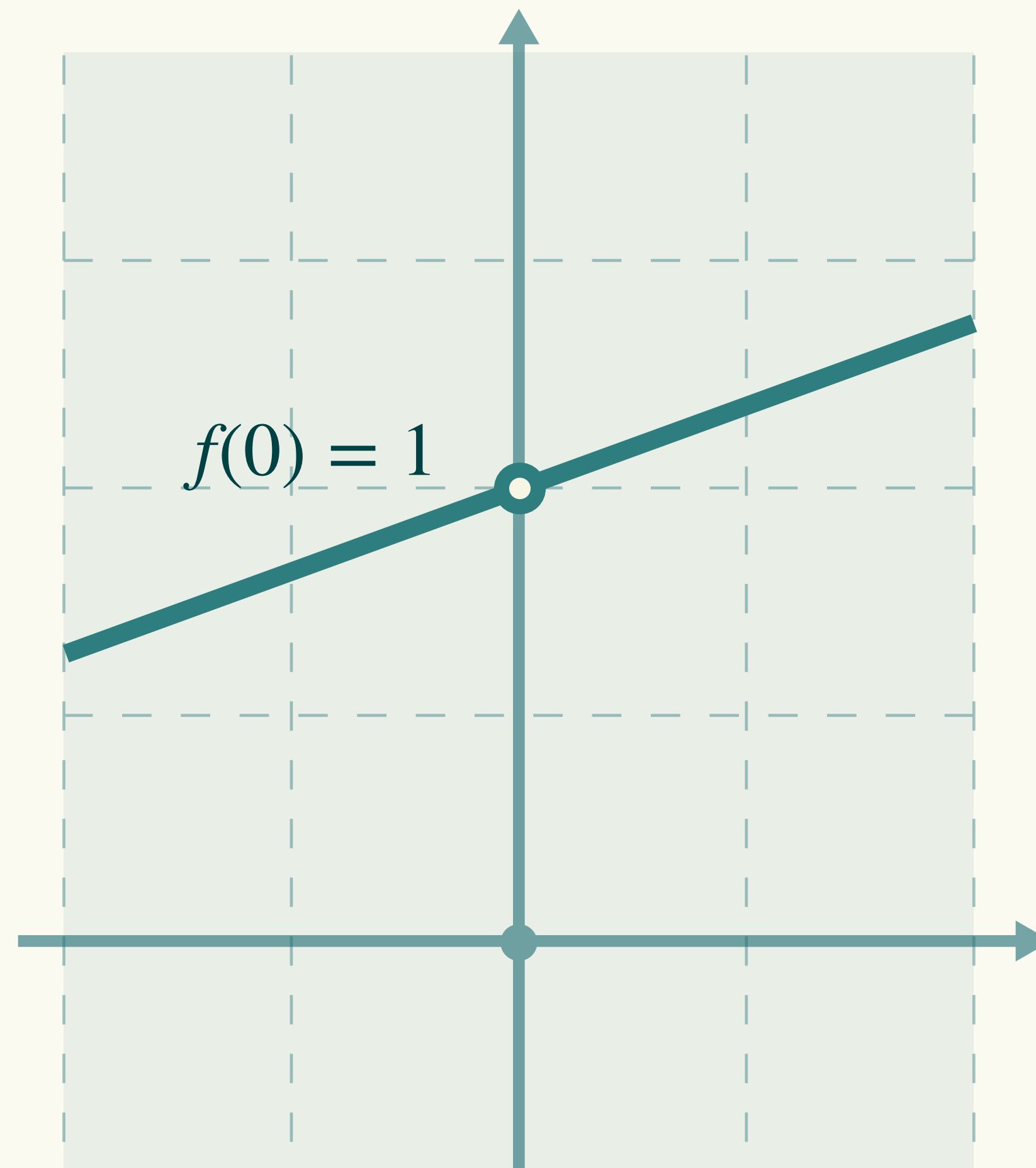
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# Prelude: Bounding Positive Linear Functions

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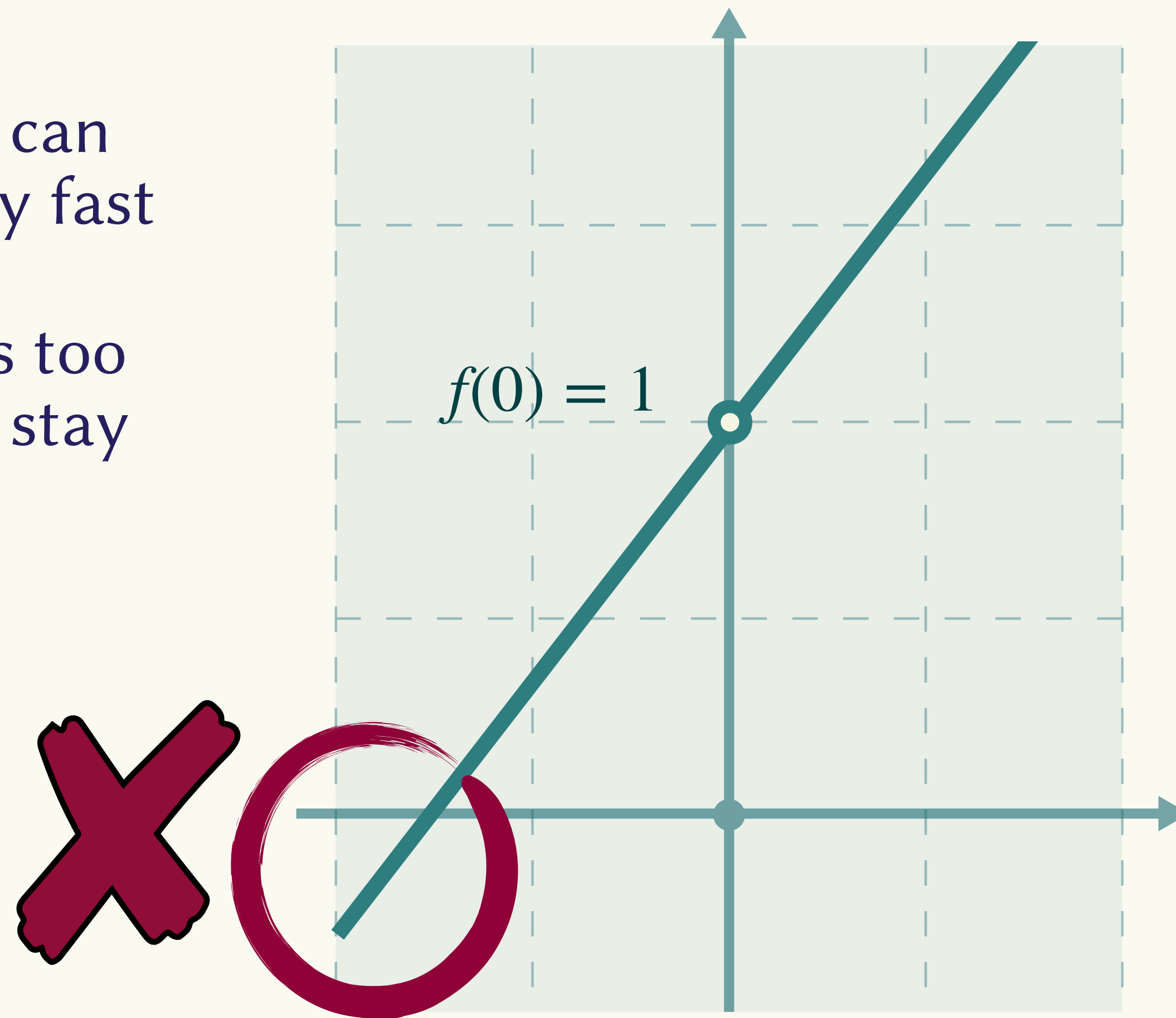
\*technically speaking, positive affine functions



# Prelude: Bounding Positive Linear Functions

a linear function can  
change arbitrarily fast

but if it changes too  
fast, it does not stay  
*positive*

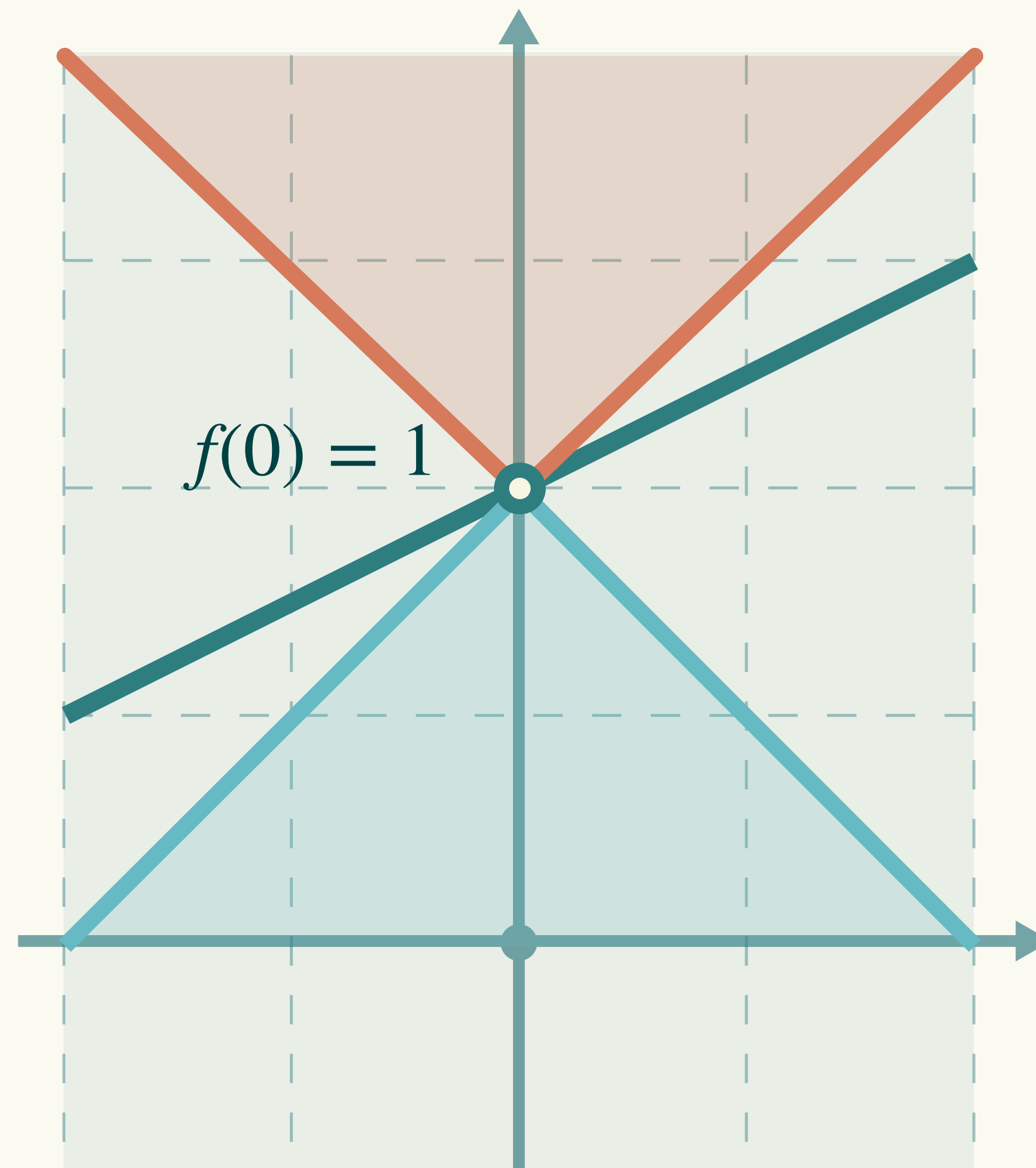


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# Prelude: Bounding Positive Linear Functions

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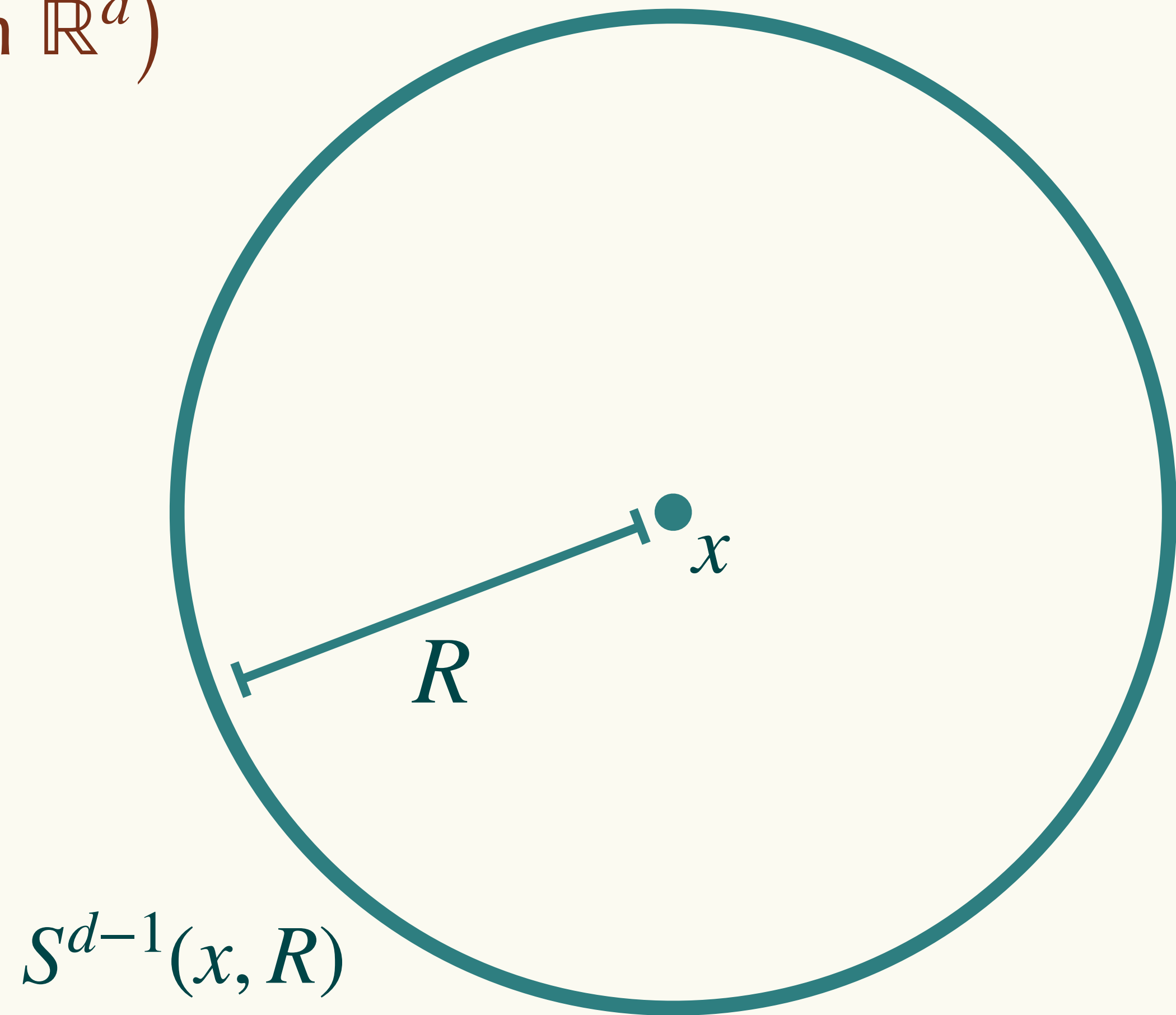
but if it changes too  
fast, it does not stay  
*positive*



positive linear functions  
must stay between the  
upper and lower bounds

# The Mean Value Property

(in  $\mathbb{R}^d$ )

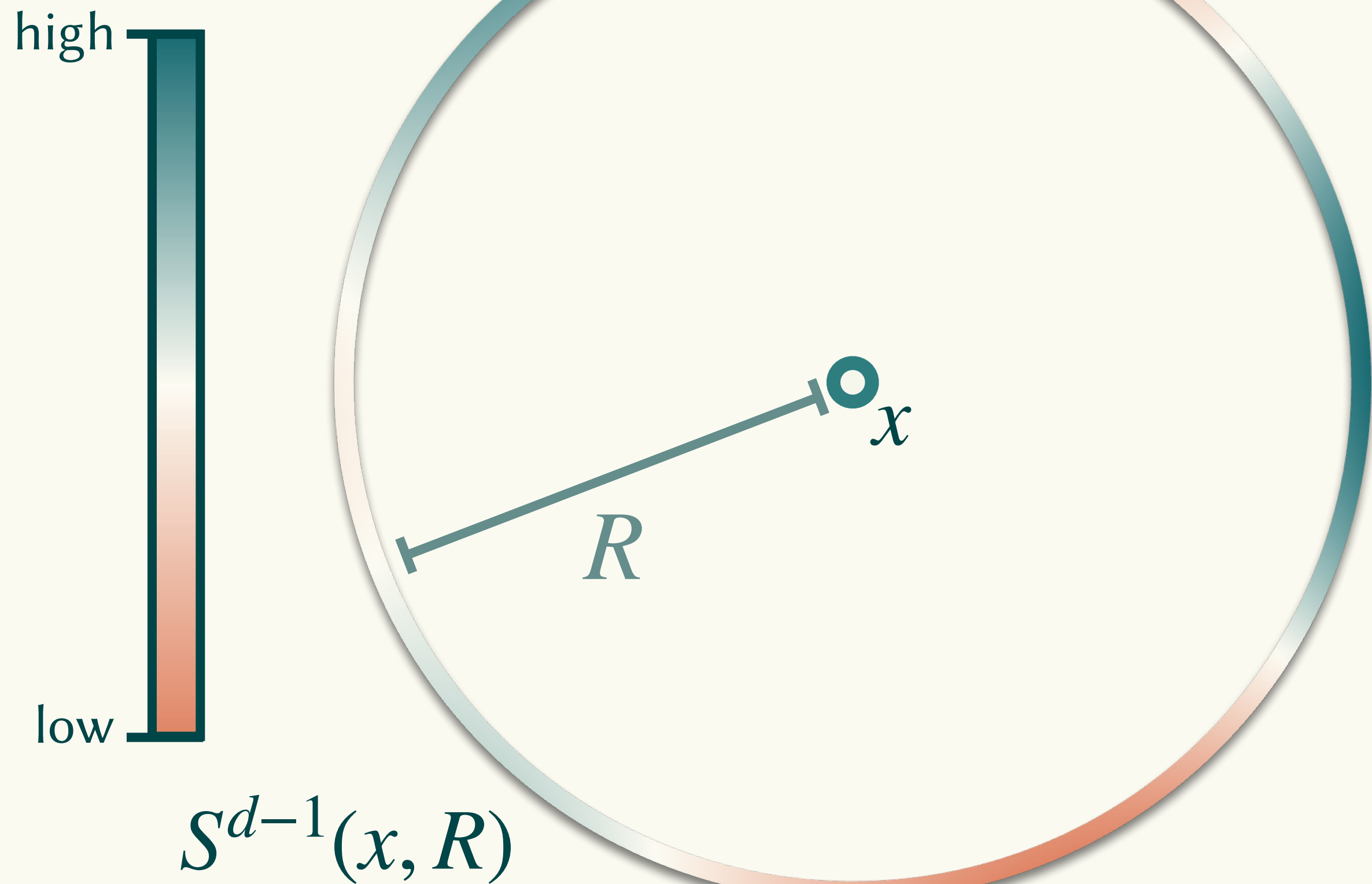


# The Mean Value Property

mean value property

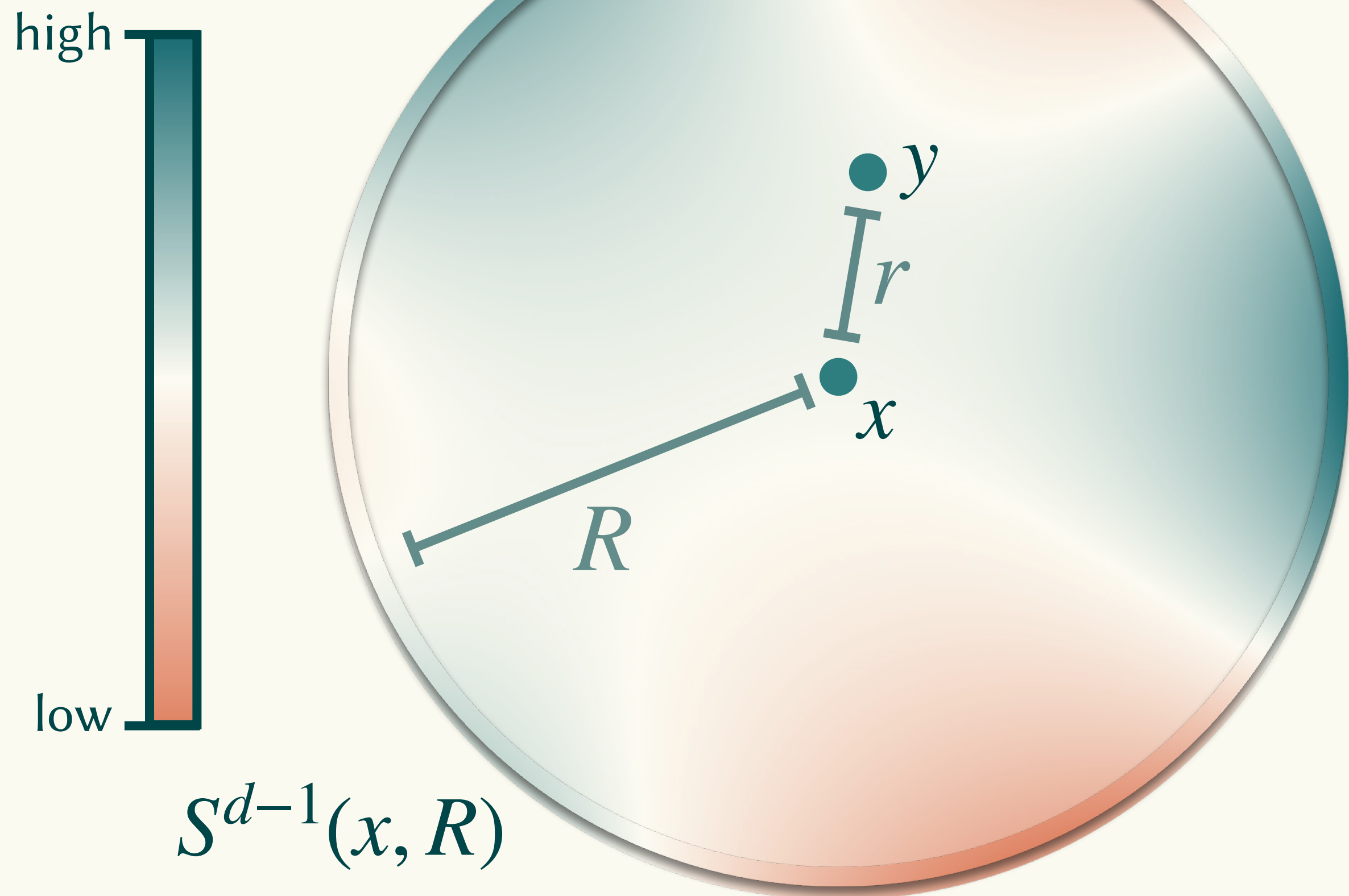
$$f(x) = \frac{1}{\text{vol}(S)} \int_S f(z) dz$$

(in  $\mathbb{R}^d$ )



# The Poisson Kernel

(in  $\mathbb{R}^d$ )



mean value property

$$f(x) = \frac{1}{\text{vol}(S)} \int_S f(z) dz$$

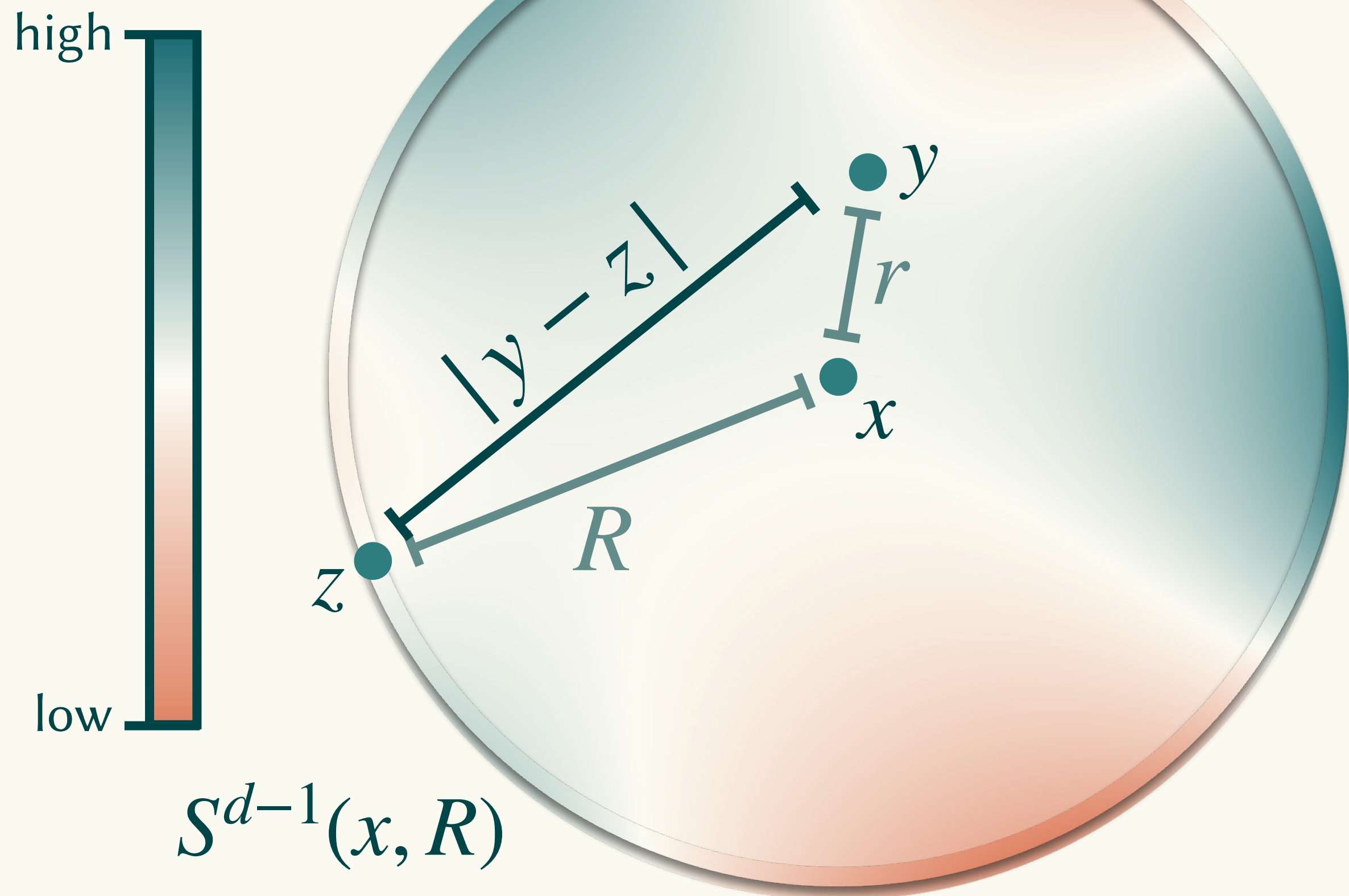
Poisson kernel

$$f(y) = \frac{1}{\text{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d} |y - z|^d} f(z) dz$$

weighted average

# The Poisson Kernel

(in  $\mathbb{R}^d$ )



mean value property

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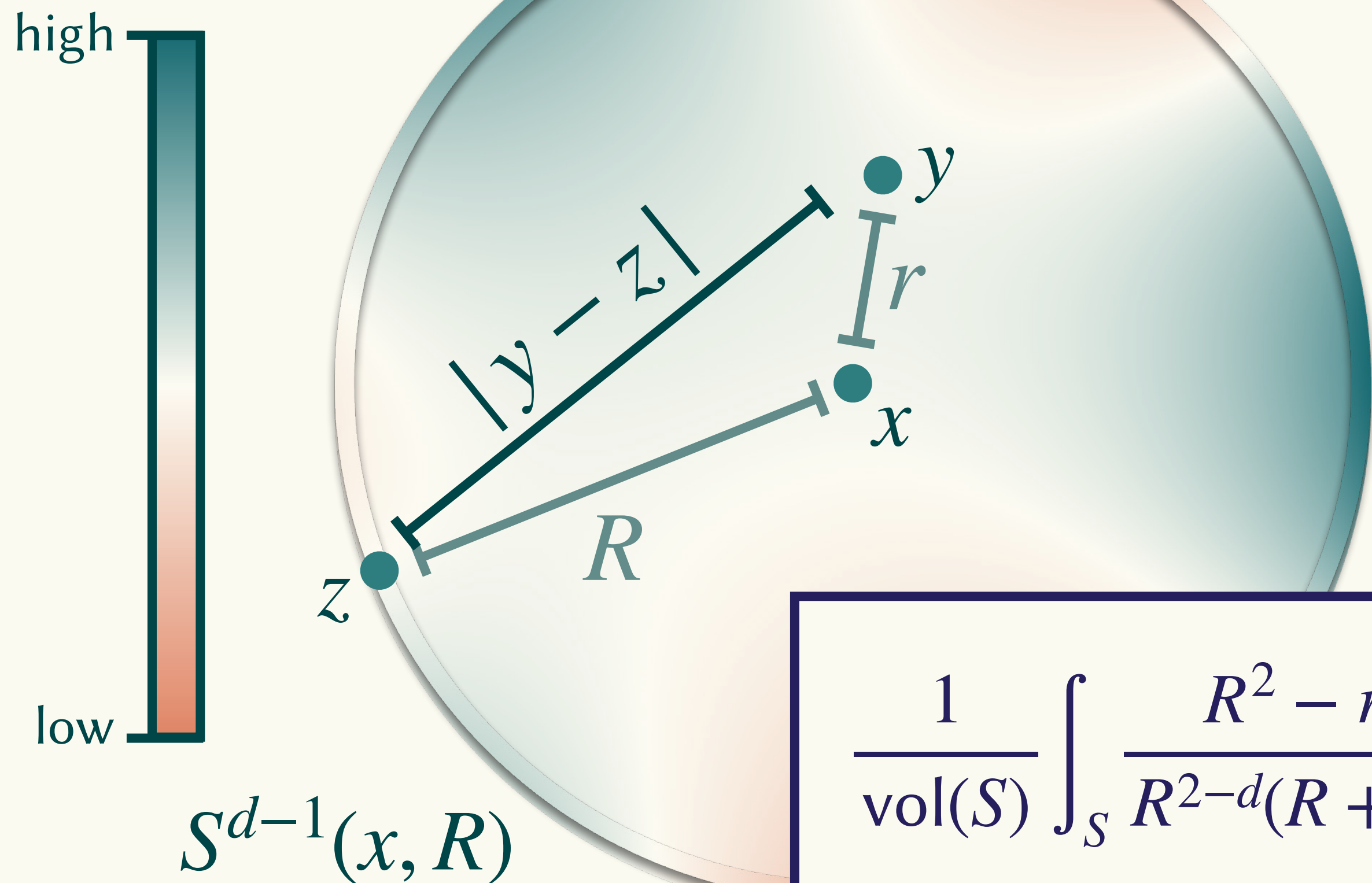
Poisson kernel

$$f(y) = \frac{1}{\text{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d} |y-z|^d} f(z) dz$$

$$R - r \leq |y-z| \leq R + r$$

# The Poisson Kernel

(in  $\mathbb{R}^d$ )



mean value property

$$f(x) = \frac{1}{\text{vol}(S)} \int_S f(z) dz$$

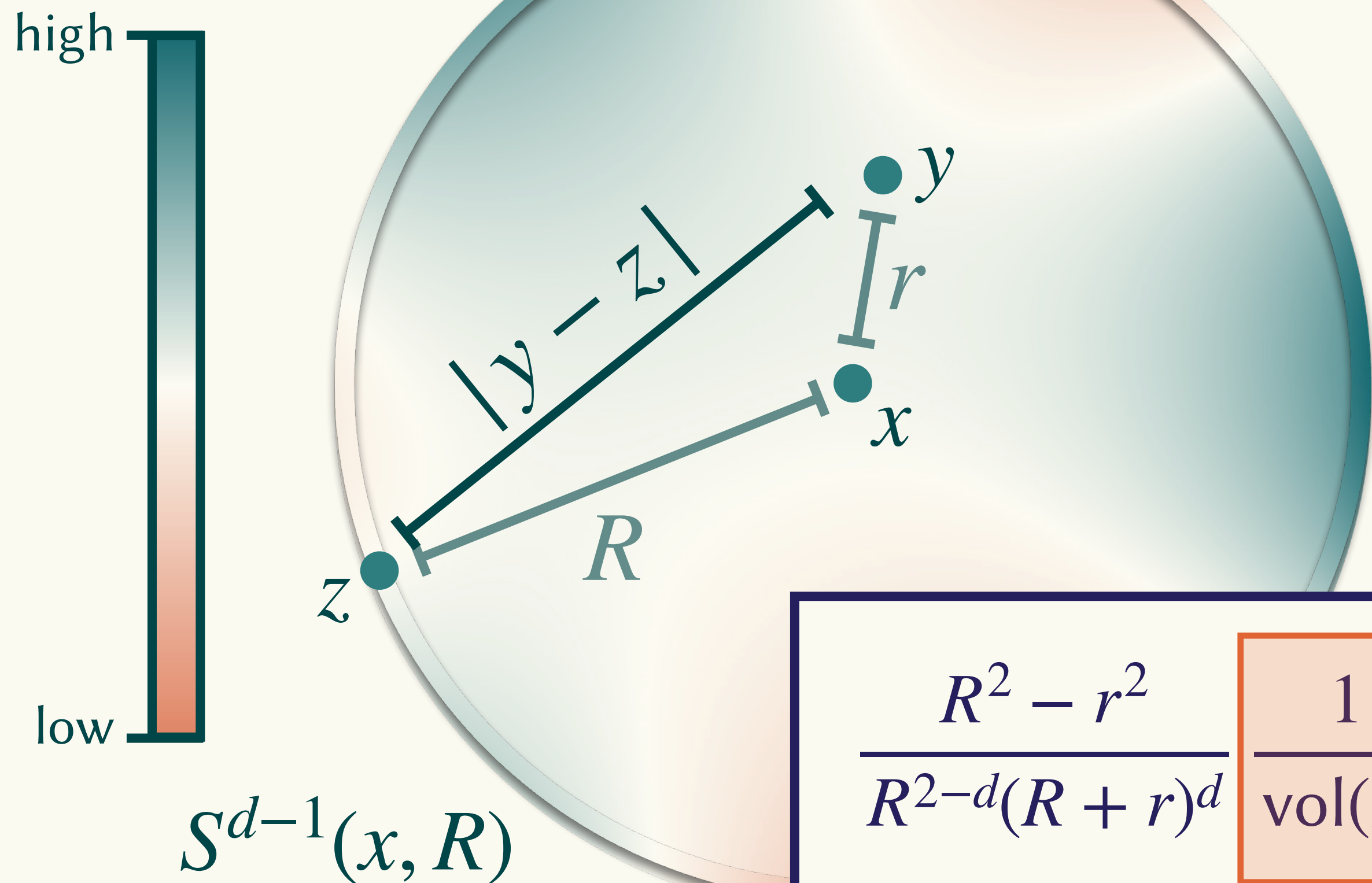
Poisson kernel

$$f(y) = \frac{1}{\text{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d} |y-z|^d} f(z) dz$$

$$\frac{1}{\text{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d}(R+r)^d} f(z) dz \leq f(y) \leq \frac{1}{\text{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d}(R-r)^d} f(z) dz$$

# The Poisson Kernel

(in  $\mathbb{R}^d$ )



mean value property

$$f(x) = \frac{1}{\text{vol}(S)} \int_S f(z) dz$$

Poisson kernel

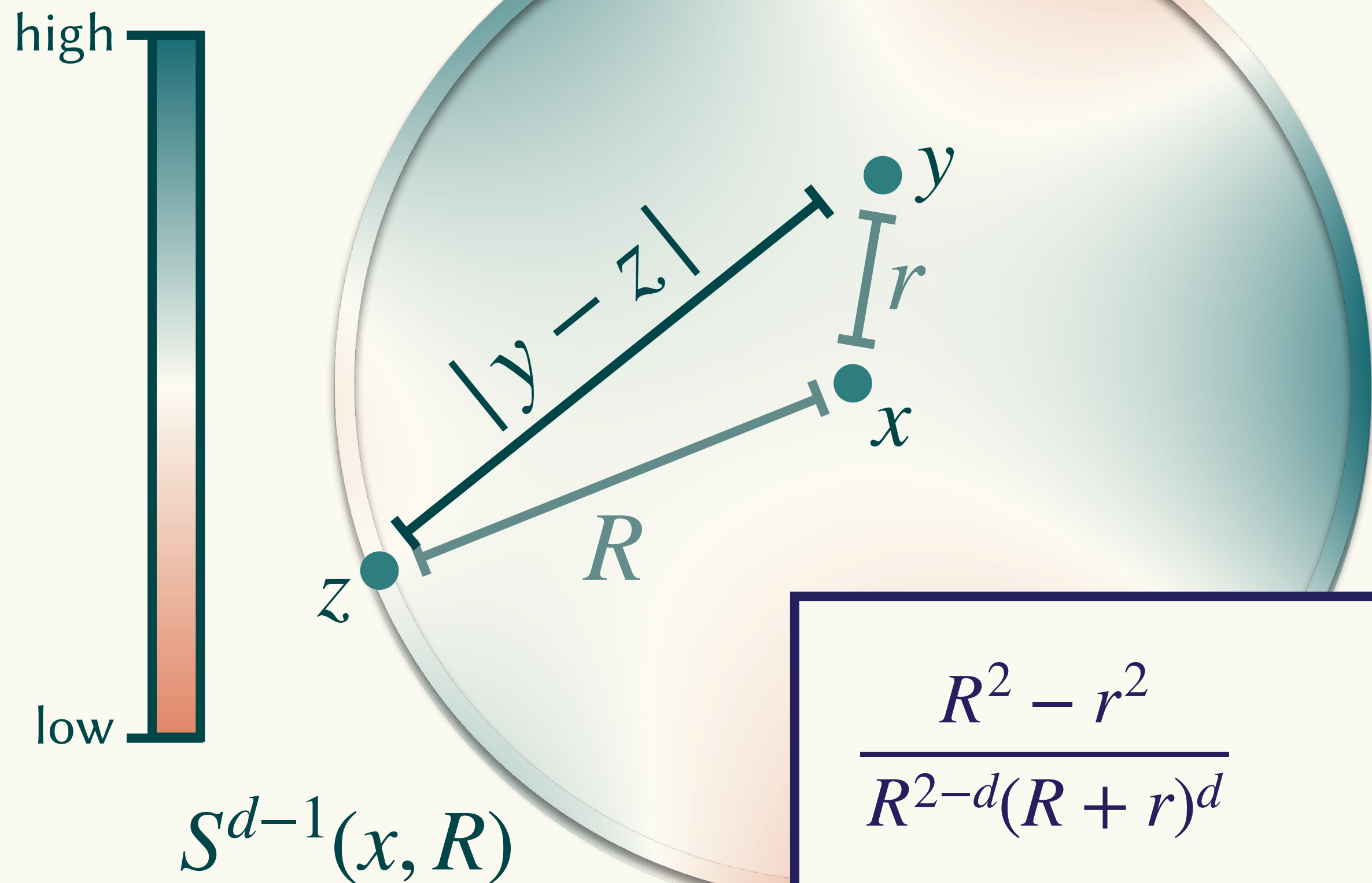
$$f(y) = \frac{1}{\text{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d} |y-z|^d} f(z) dz$$

$$\frac{R^2 - r^2}{R^{2-d}(R+r)^d} \frac{1}{\text{vol}(S)} \int_S f(z) dz \leq f(y) \leq \frac{R^2 - r^2}{R^{2-d}(R-r)^d} \frac{1}{\text{vol}(S)} \int_S f(z) dz$$



# The Poisson Kernel

(in  $\mathbb{R}^d$ )



mean value property

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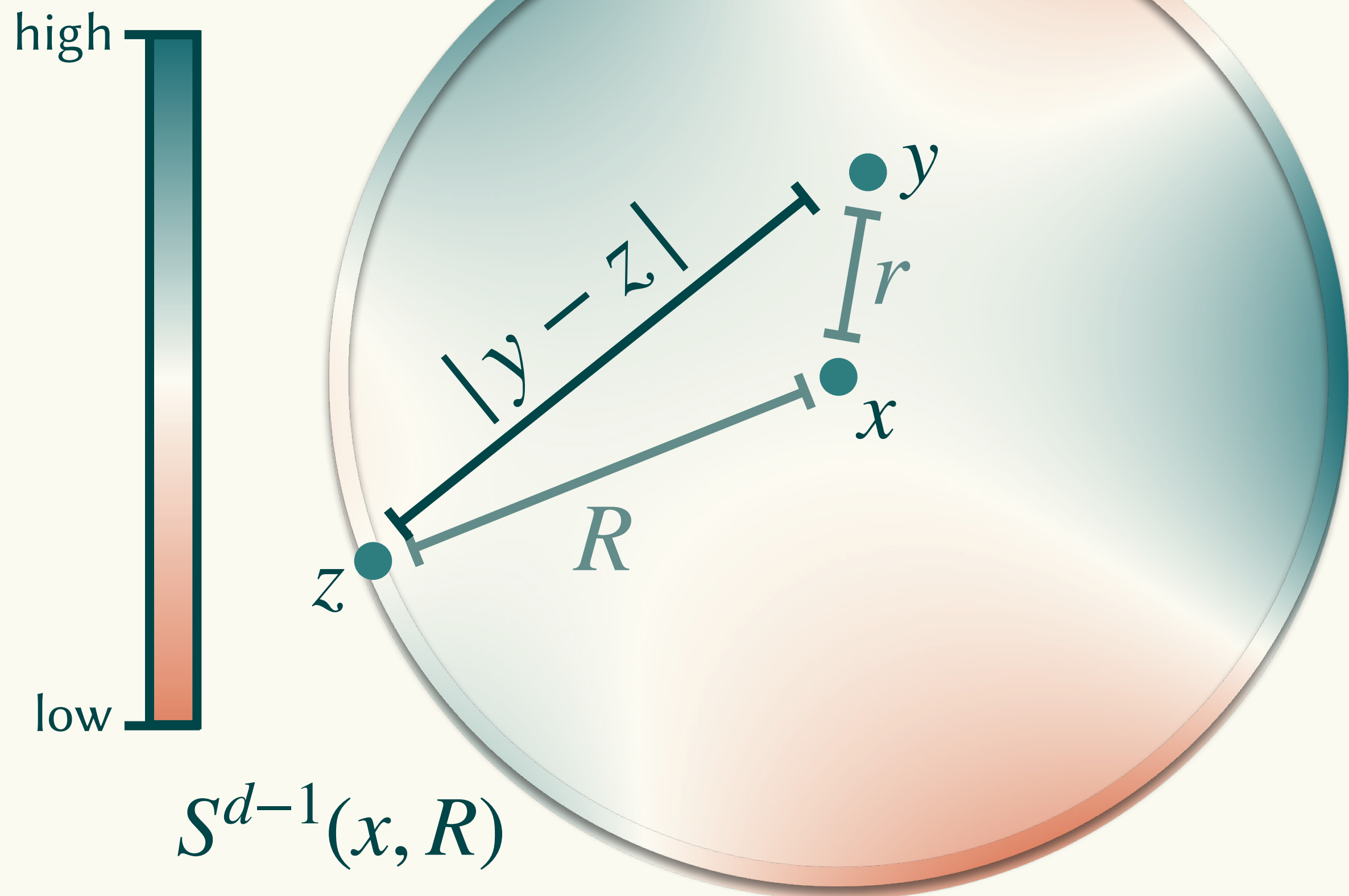
Poisson kernel

$$f(y) = \frac{1}{\text{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d} |y-z|^d} f(z) dz$$

$$\frac{R^2 - r^2}{R^{2-d}(R+r)^d} f(x) \leq f(y) \leq \frac{R^2 - r^2}{R^{2-d}(R-r)^d} f(x)$$

# Harnack's Inequality

(in  $\mathbb{R}^d$ )



mean value property

$$f(x) = \frac{1}{\text{vol}(S)} \int_S f(z) dz$$

Poisson kernel

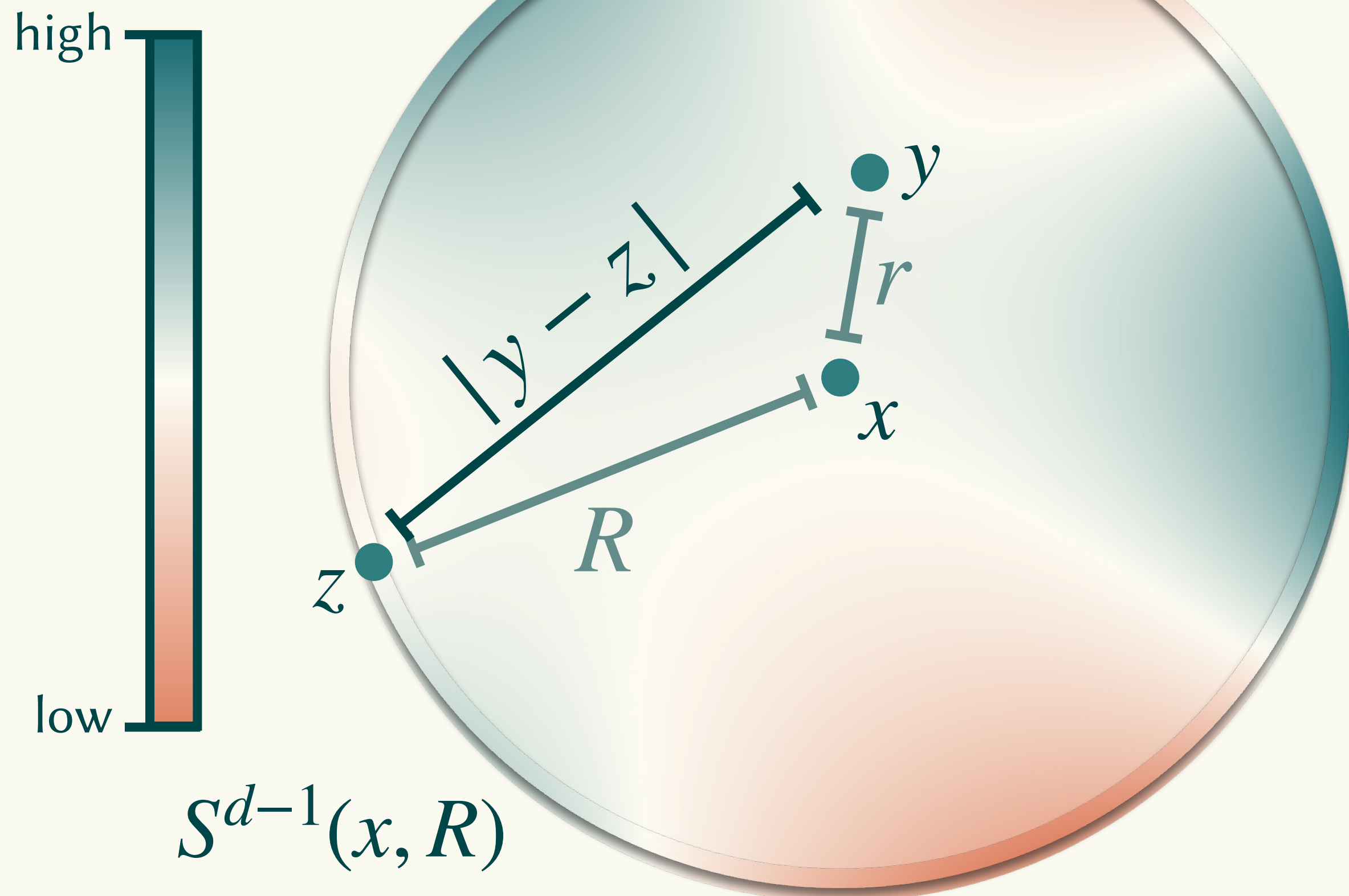
$$f(y) = \frac{1}{\text{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d} |y-z|^d} f(z) dz$$

Harnack's inequality

$$\frac{R^2 - r^2}{R^{2-d}(R+r)^d} f(x) \leq f(y) \leq \frac{R^2 - r^2}{R^{2-d}(R-r)^d} f(x)$$

# Harnack's Inequality

(in  $\mathbb{R}^d$ )



mean value property

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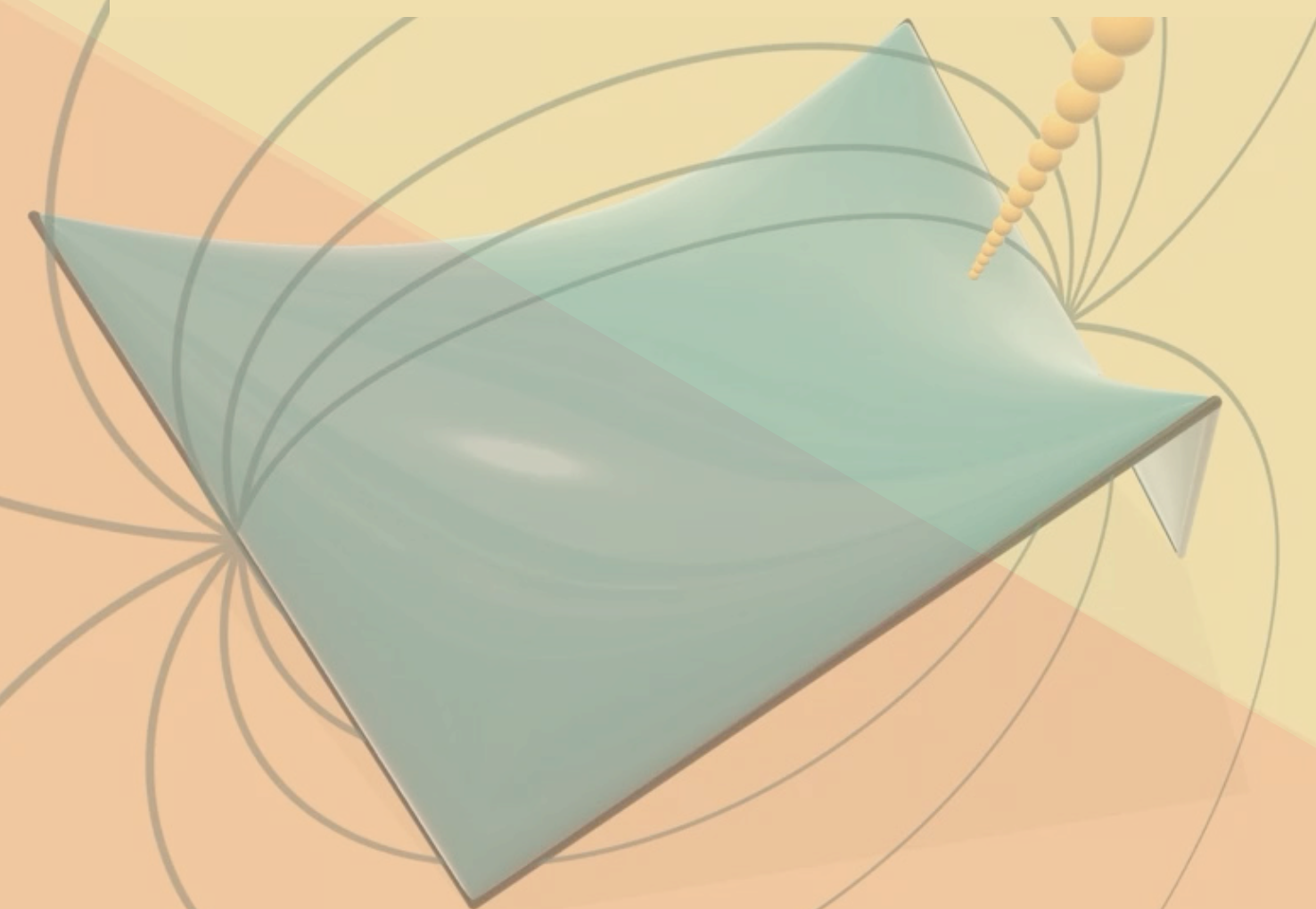
Poisson kernel

$$f(y) = \frac{1}{\text{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d} |y-z|^d} f(z) dz$$

Harnack's inequality

$$\frac{1 - r/R}{(1 + r/R)^{d-1}} f(x) \leq f(y) \leq \frac{1 + R/r}{(1 - r/R)^{d-1}} f(x)$$

# II. Harnack Tracing



# Distance bounds from Harnack's inequality

Let  $f$  be a positive harmonic function on a ball:

$$\frac{1 - r/R}{(1 + r/R)^2} f(x) \leq f(y) \leq \frac{1 + r/R}{(1 - r/R)^2} f(x)$$

lower bound

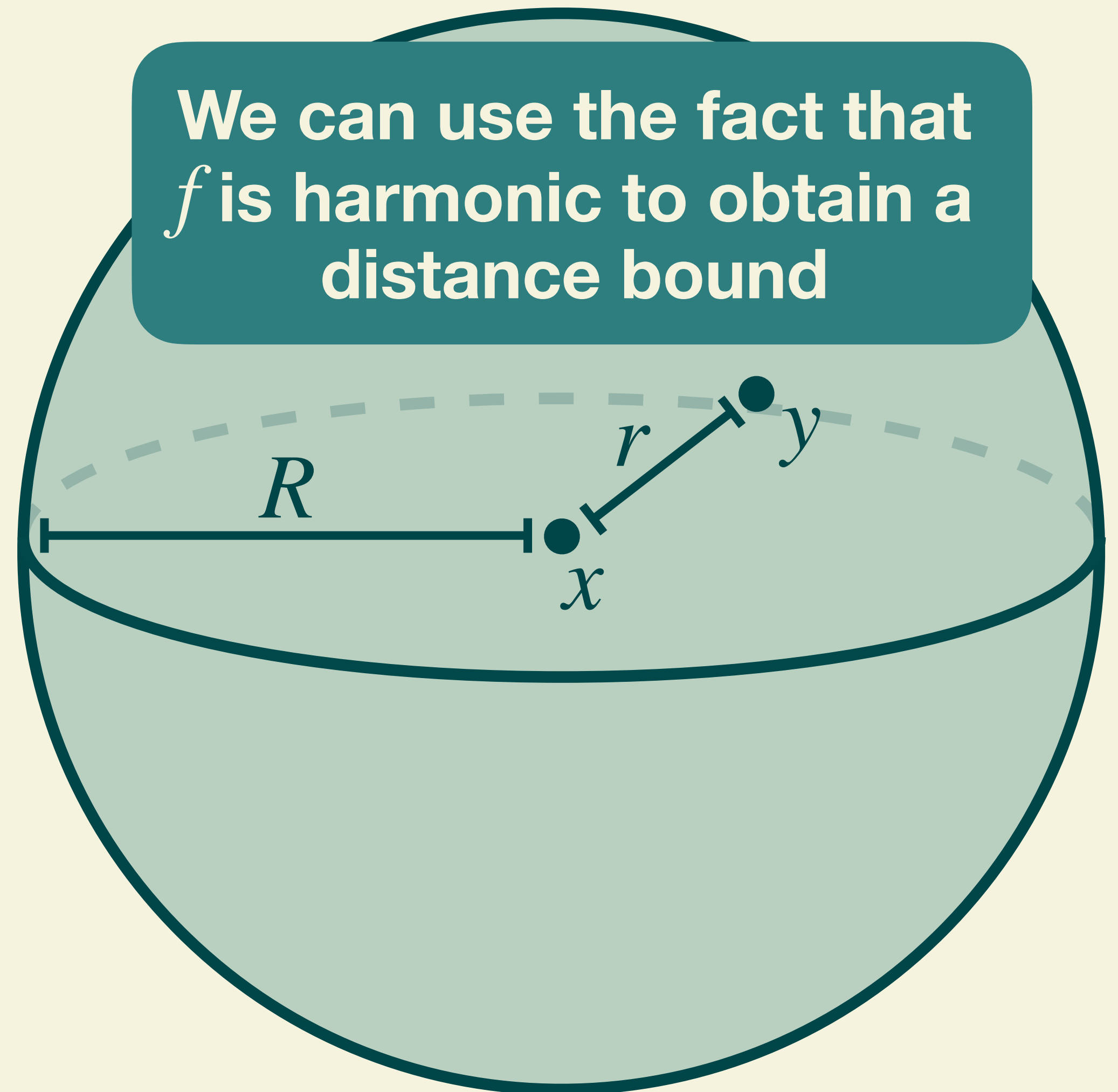
upper bound

always safe to take step of size

$$\frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$$

where  $a = \frac{f(x)}{f^*}$

We can use the fact that  $f$  is harmonic to obtain a distance bound



# Distance bounds from Harnack's inequality

Let  $f$  be a positive harmonic function on a ball:

What if  $f$  is not positive?  
Just add a constant to make it positive on the ball

$$\frac{1 + r/R}{1 - r/R} f(x)$$

upper bound

lower bound

always safe to take step of size

$$r = \frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$$

$$\text{where } a = \frac{f(x)}{f^*}$$

All you need is a valid ball radius and a lower bound on  $f$

# Algorithm sketch

## Harnack Tracing

Starting from point  $x_0$  in direction  $d$ :

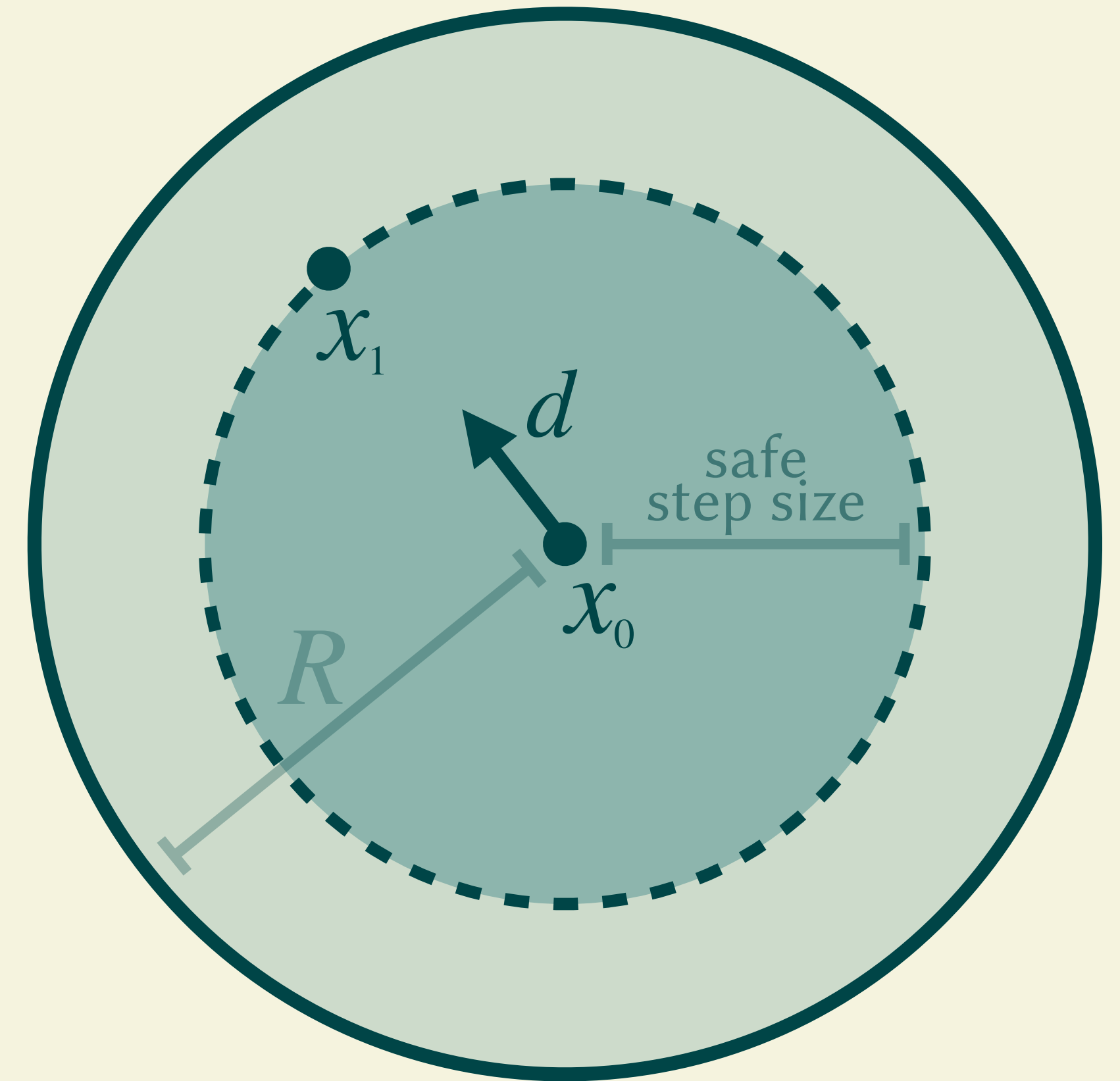
Pick ball radius

Shift  $f$  to be positive on ball

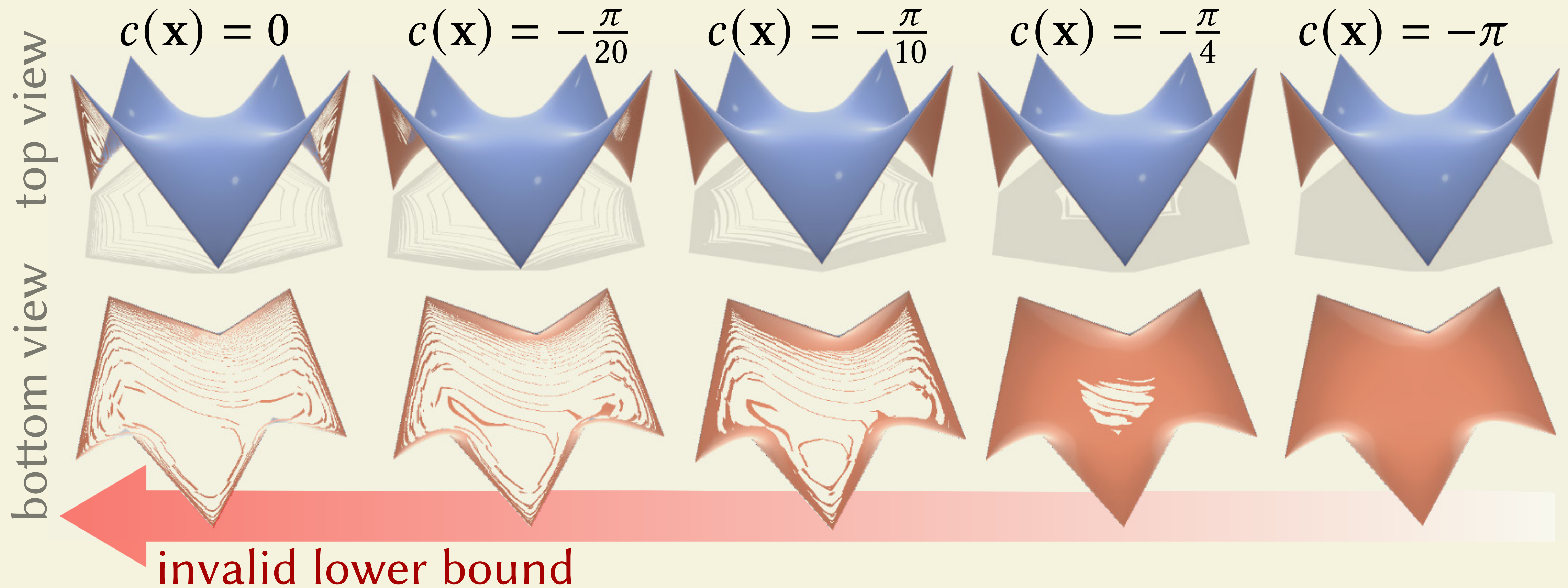
Calculate safe step size

Take safe step in ray direction

Repeat until  $f$  is sufficiently close to  $f^*$

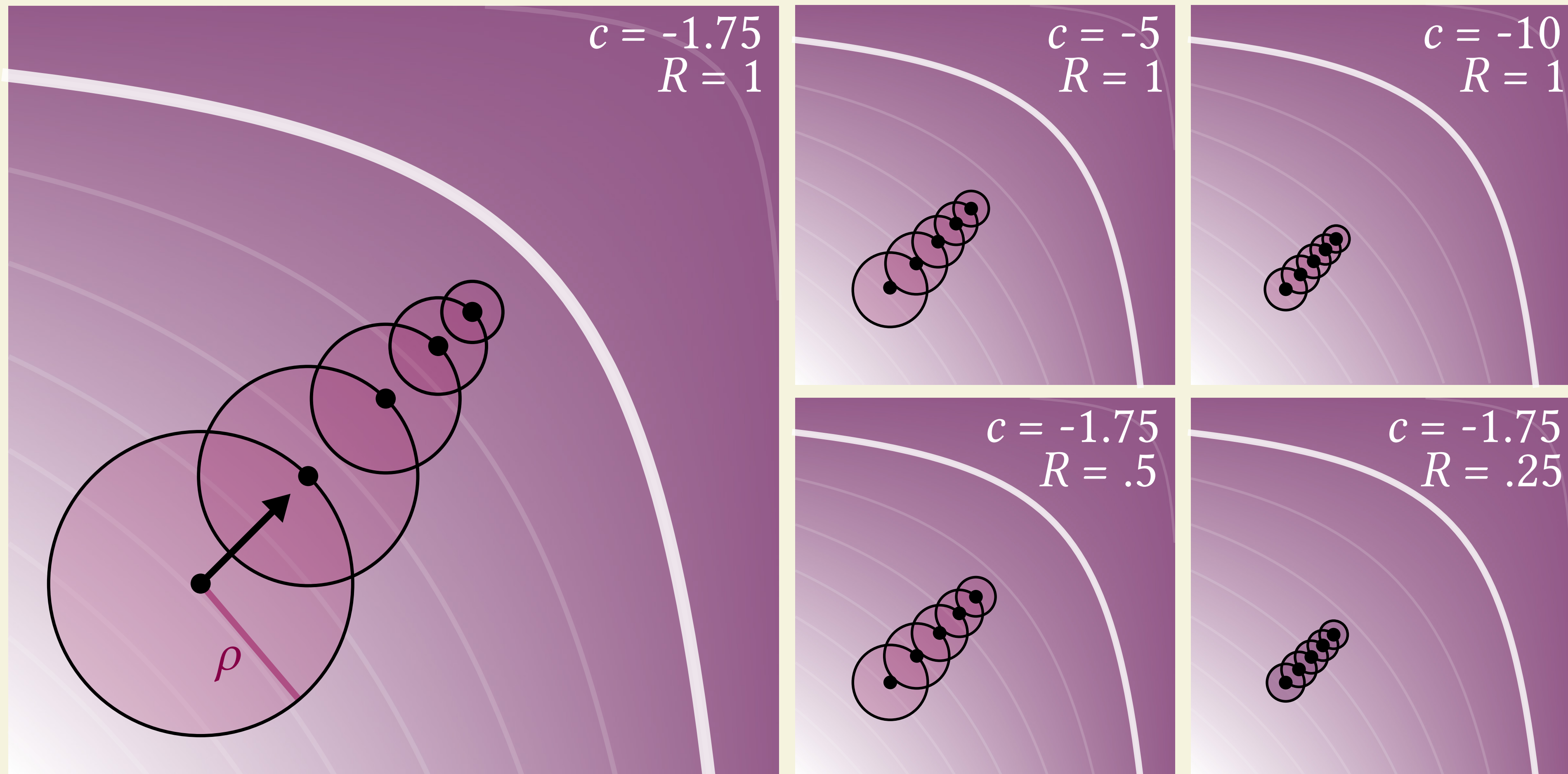


# Invalid lower bounds

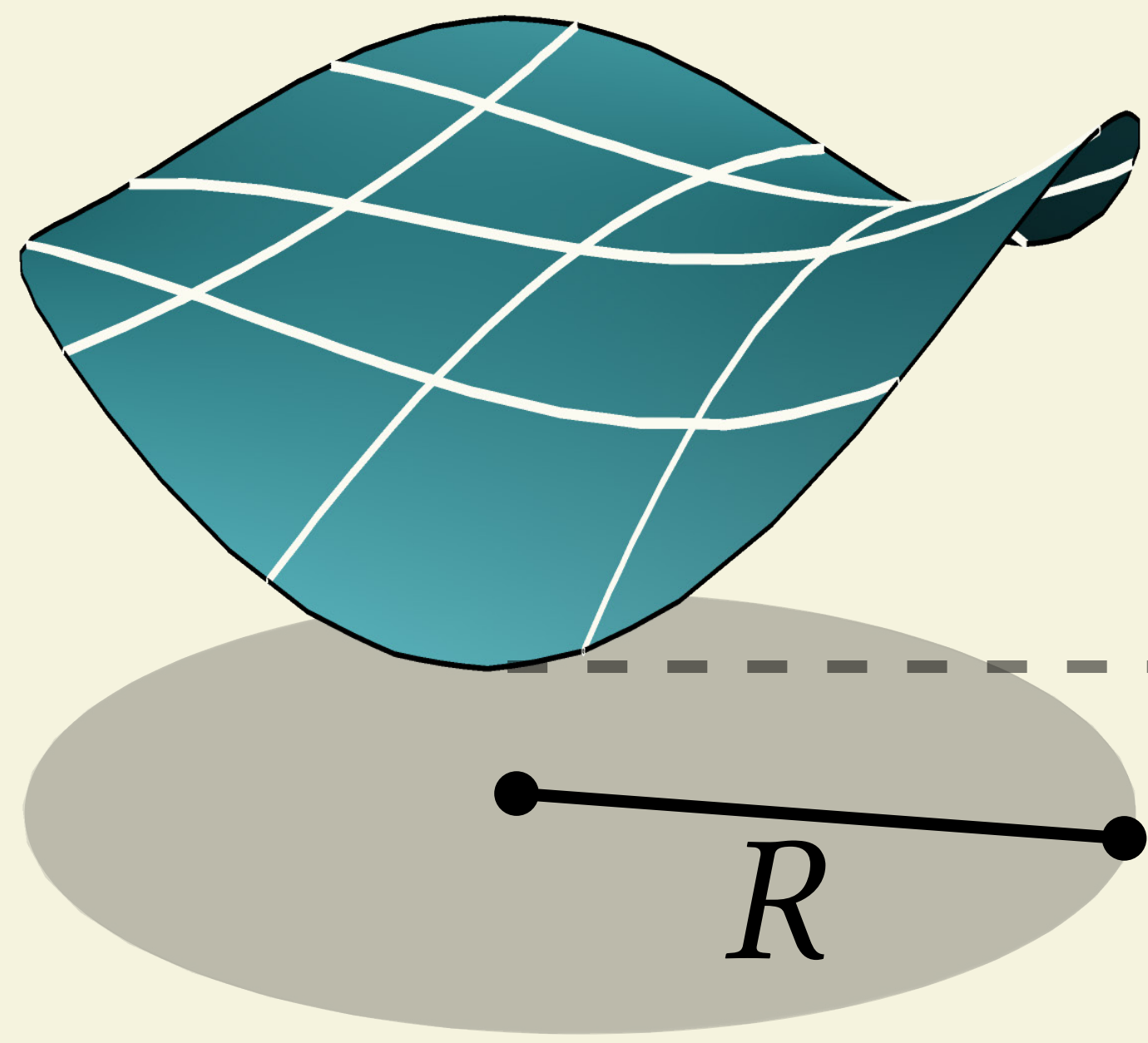




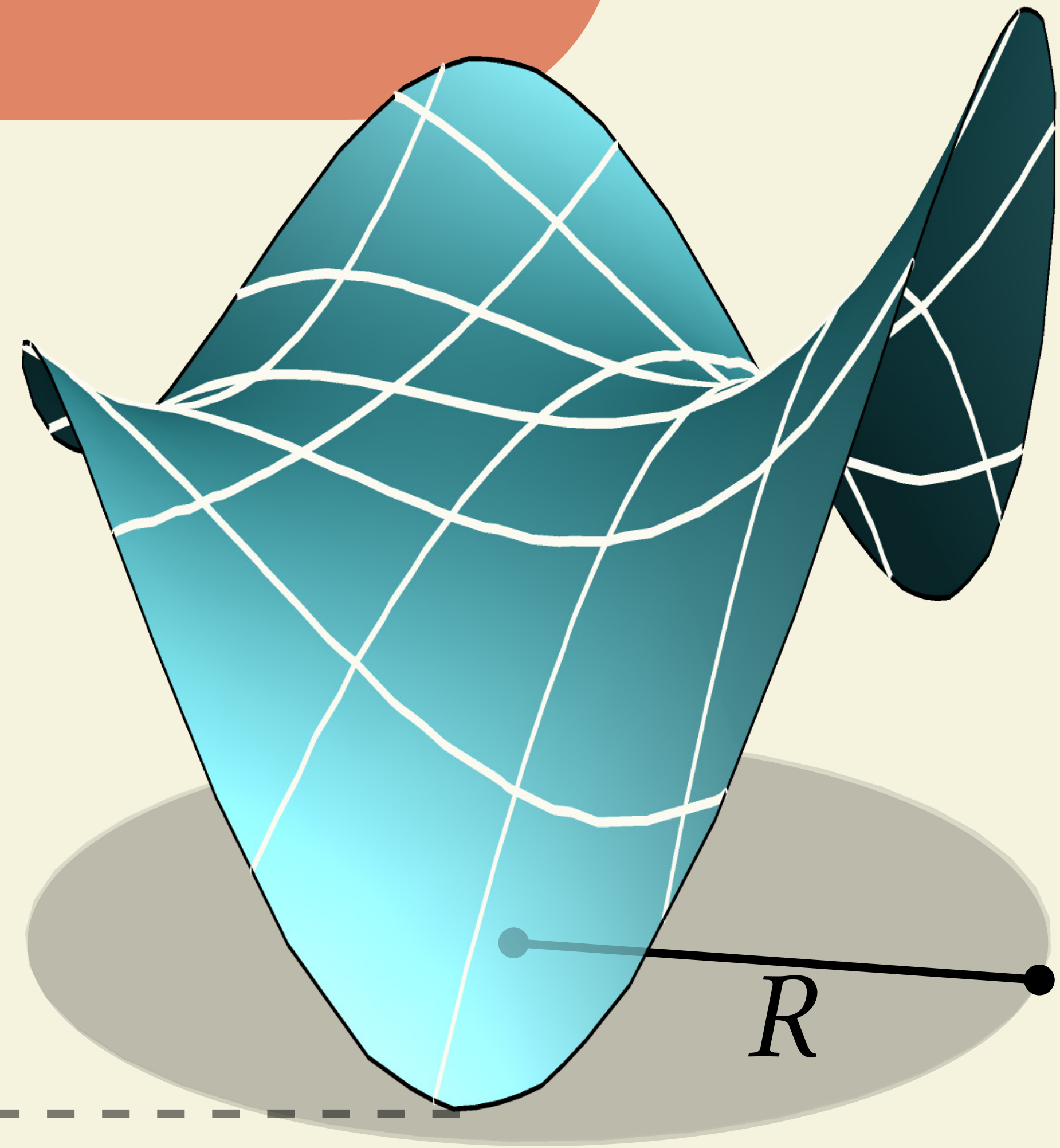
# Balancing the radius and shift



# Balancing the radius and shift



smaller radius, larger shift

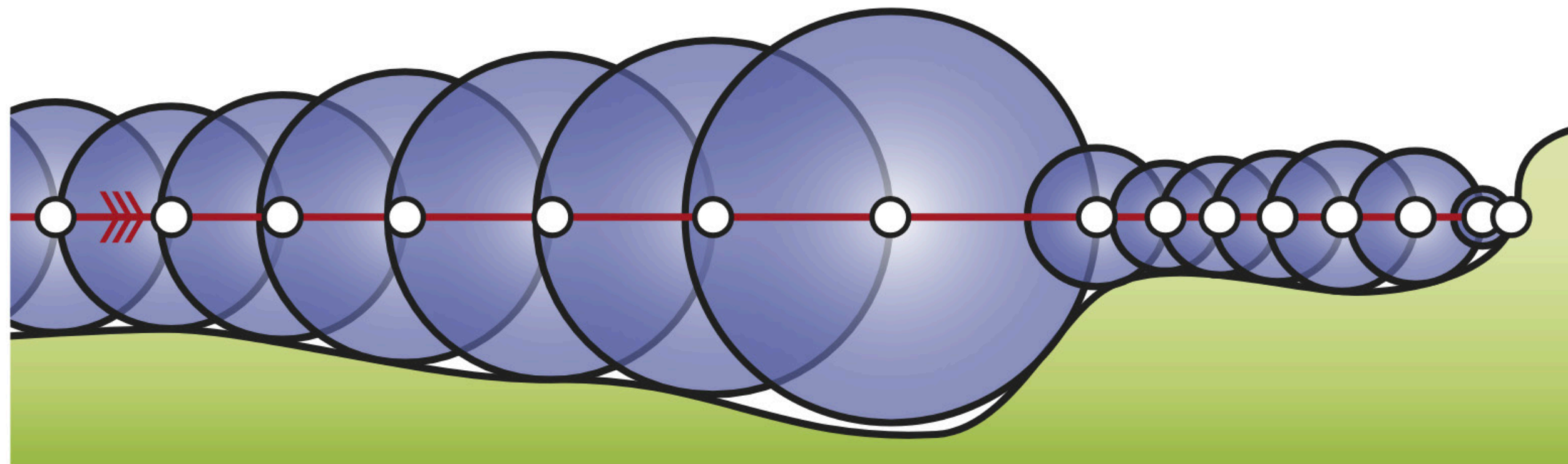


larger radius, smaller shift

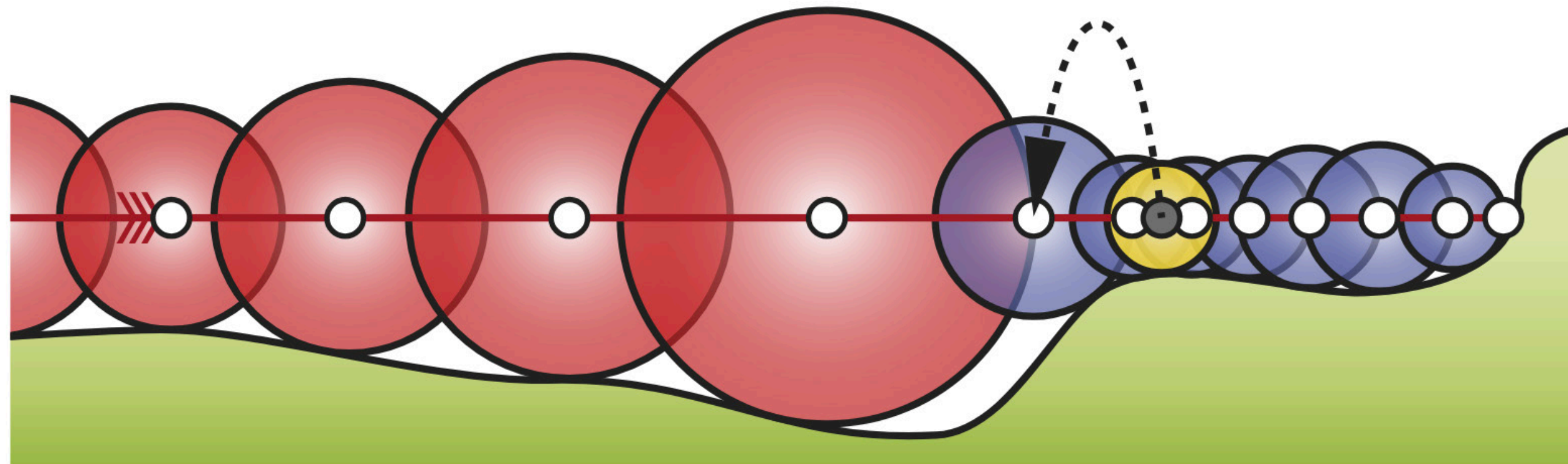
# Sphere tracing acceleration

[ Keinert *et al.* 2014 ]: “over-stepping”

conservative steps



valid oversteps



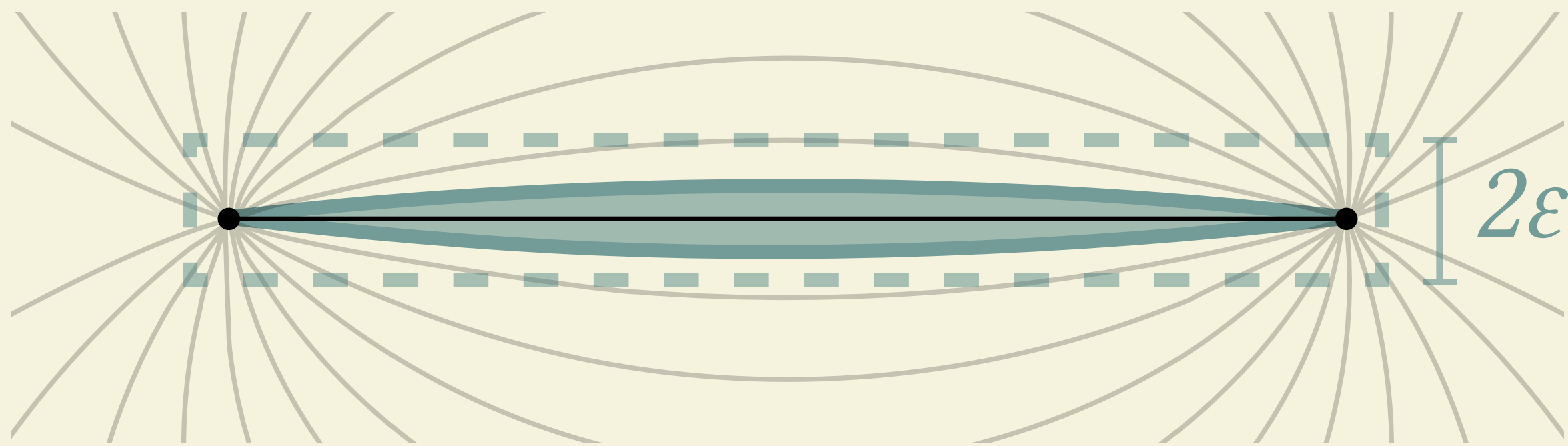
# Acceleration: gradient termination

How do you decide when you have “hit” the surface?

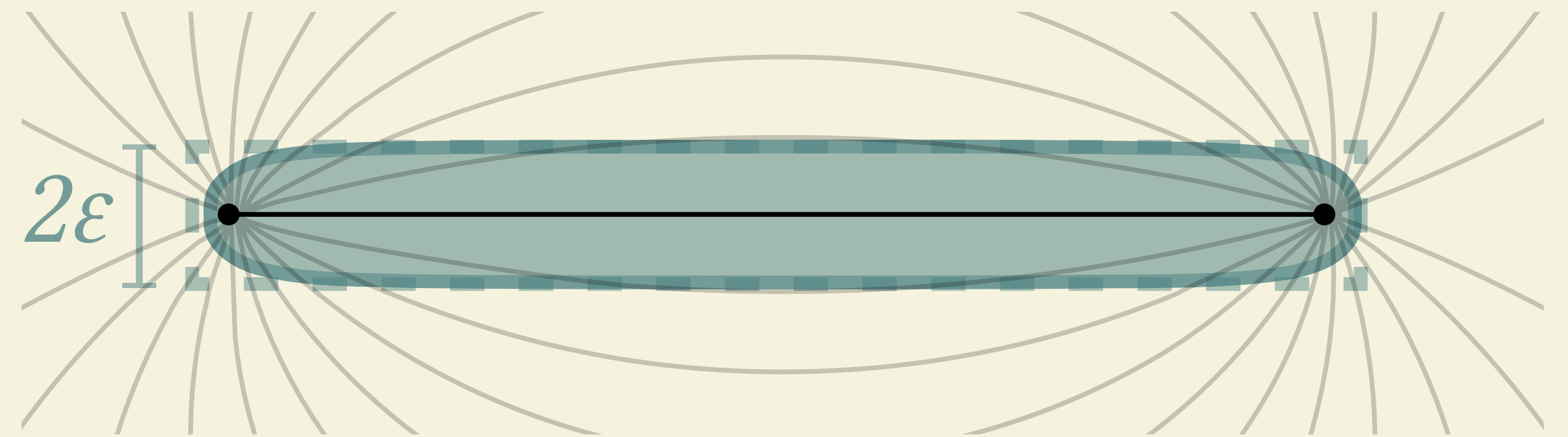
$$|f(\mathbf{x}) - f^*| < \varepsilon$$

# Acceleration: gradient termination

How do you decide when you have “hit” the surface?



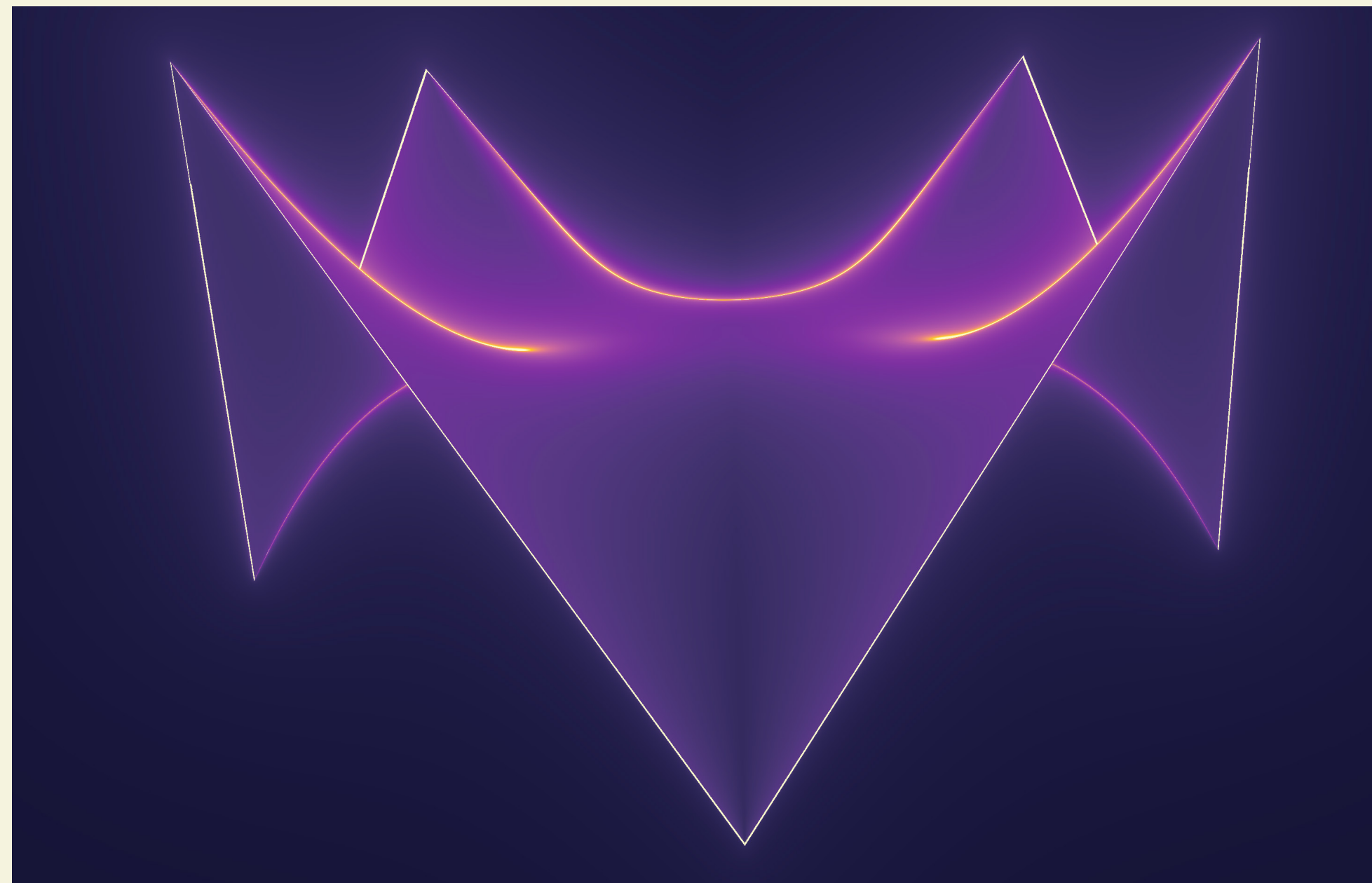
$$|f(\mathbf{x}) - f^*| < \epsilon$$



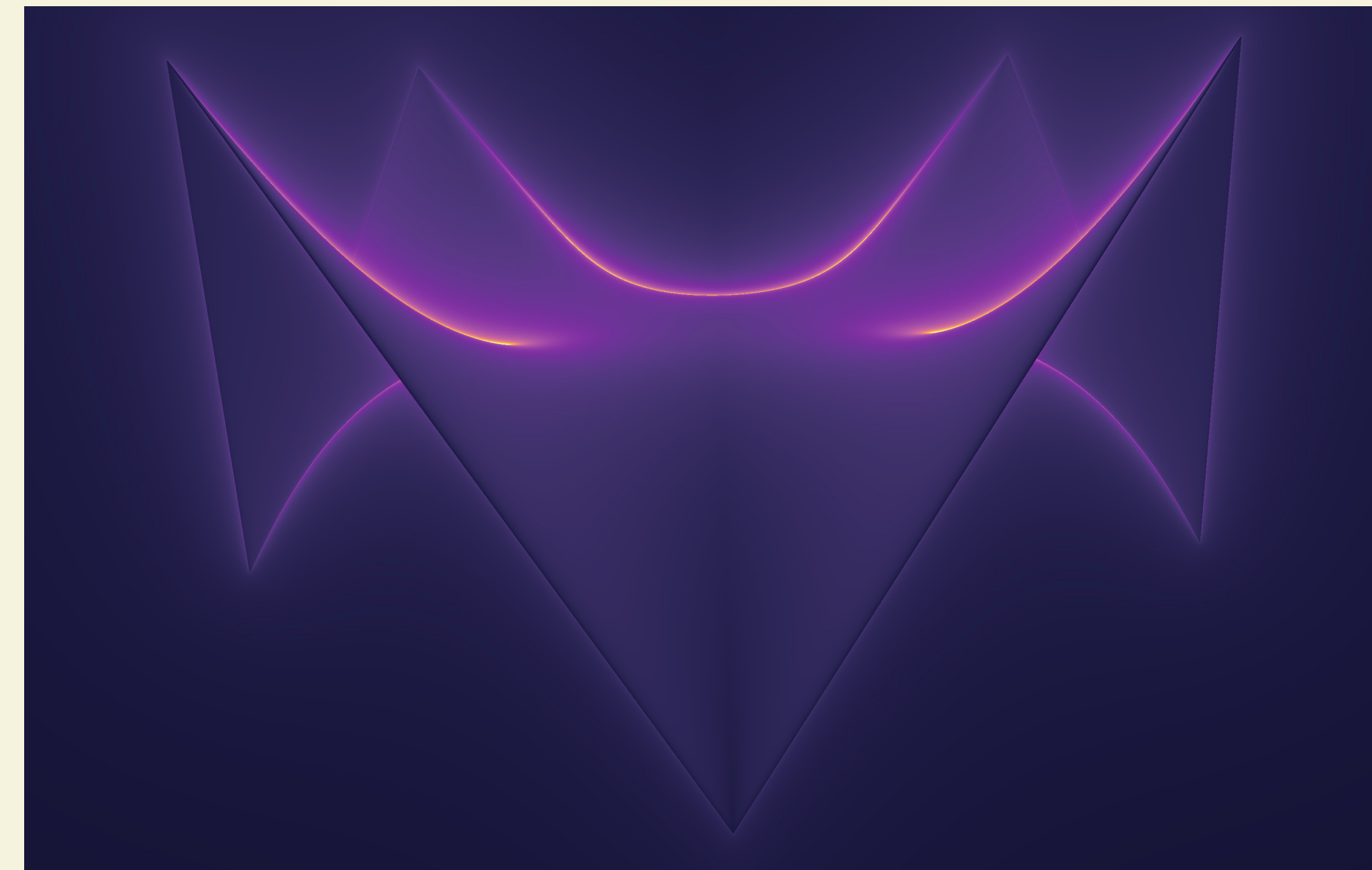
$$\frac{|f(\mathbf{x}) - f^*|}{|\nabla f(\mathbf{x})|} < \epsilon$$

# Acceleration: gradient termination

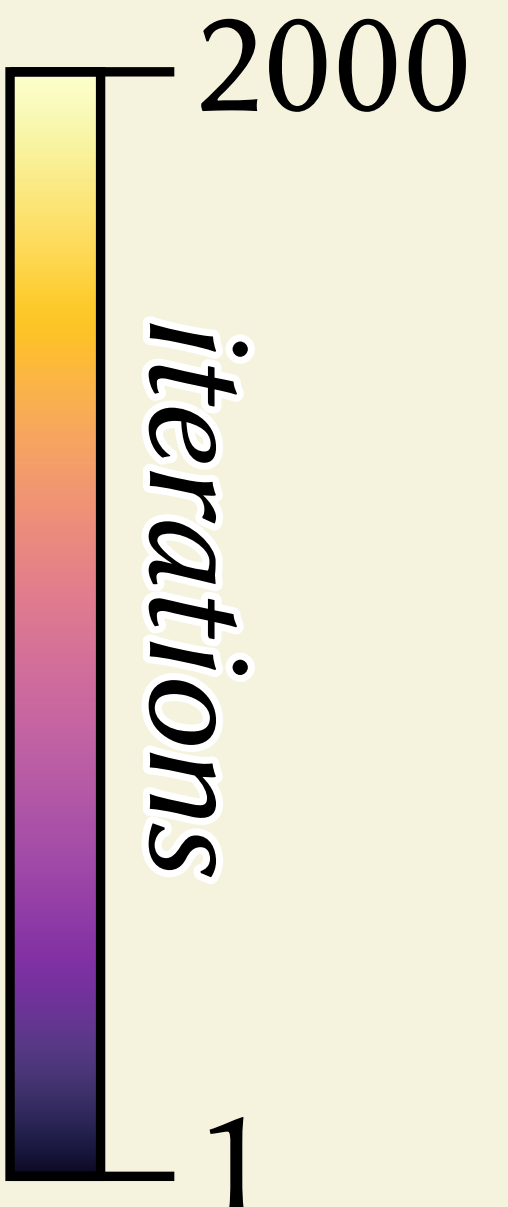
How do you decide when you have “hit” the surface?



$$|f(\mathbf{x}) - f^*| < \varepsilon$$

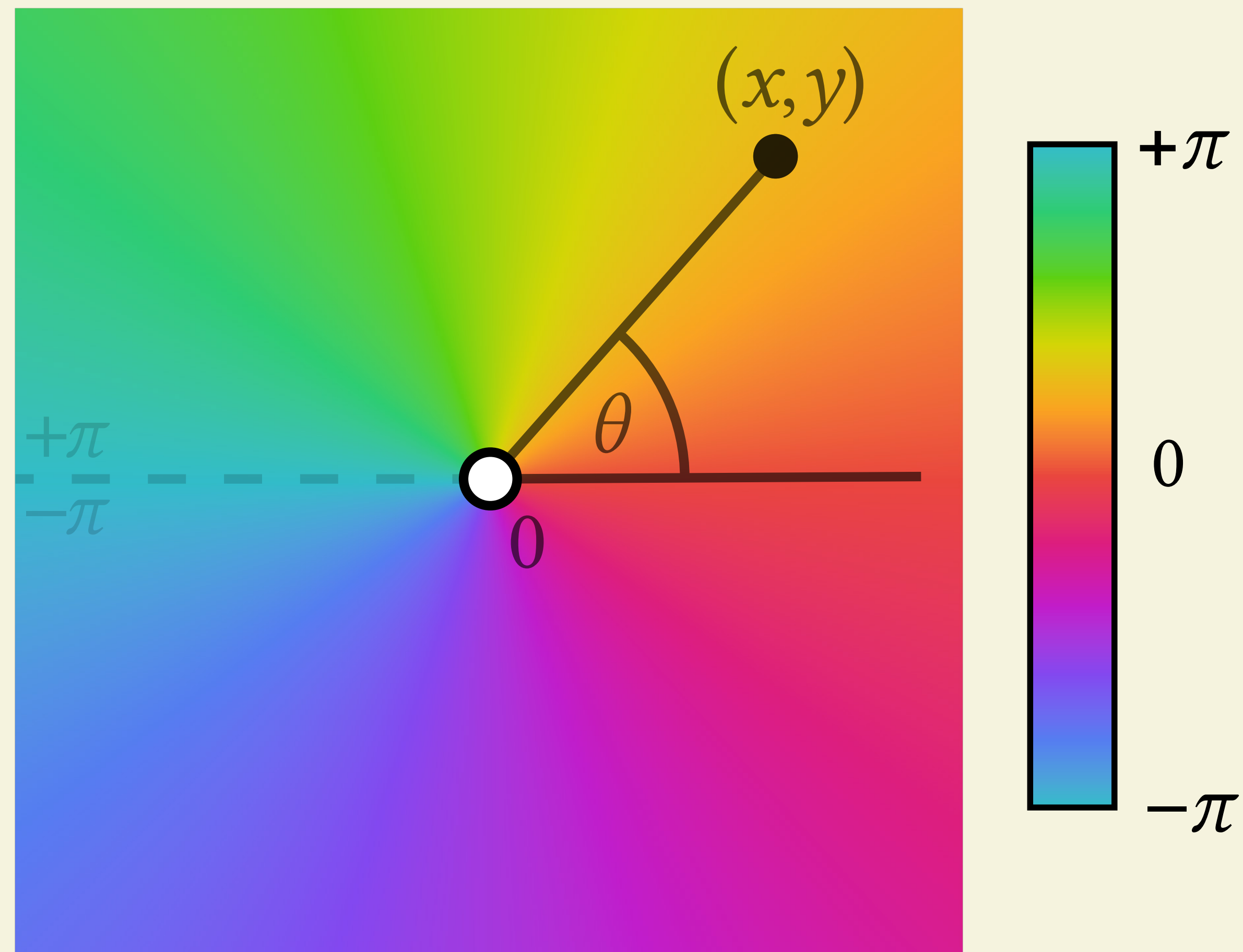


$$\frac{|f(\mathbf{x}) - f^*|}{|\nabla f(\mathbf{x})|} < \varepsilon$$



# Angle-valued functions

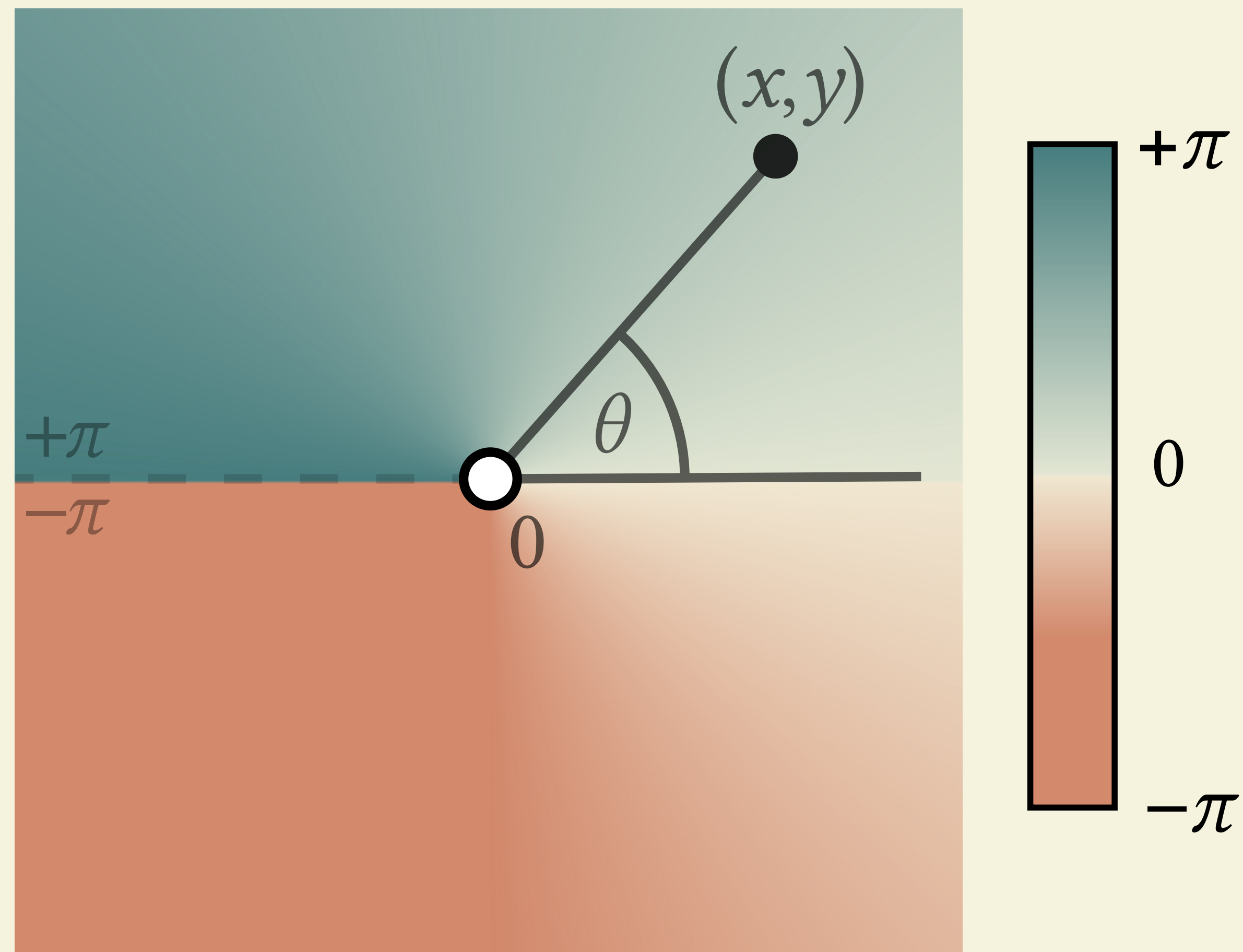
$$\theta(x, y) = \text{atan2}(y, x)$$



# Angle-valued functions

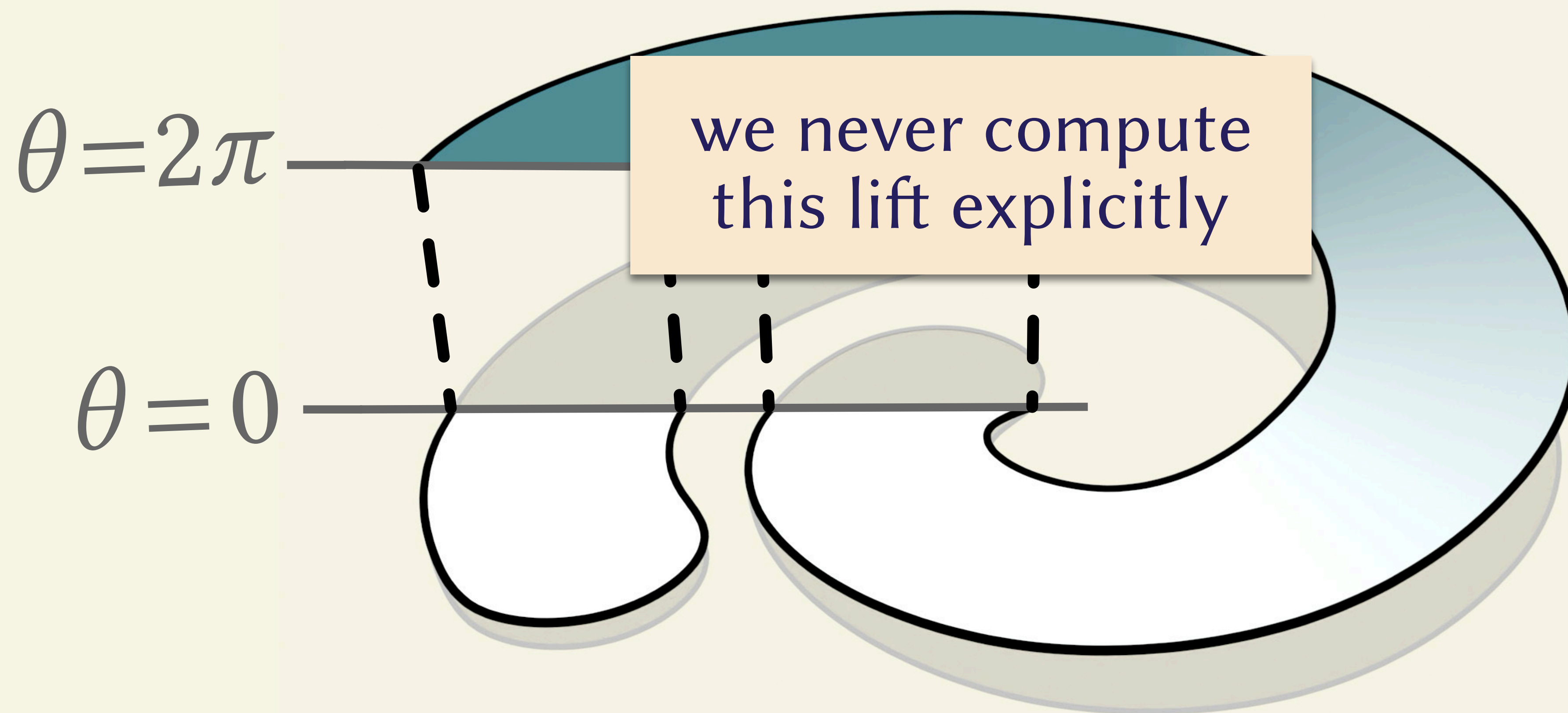
continuous when  
viewed modulo  $2\pi$

$$\theta(x, y) = \text{atan2}(y, x)$$



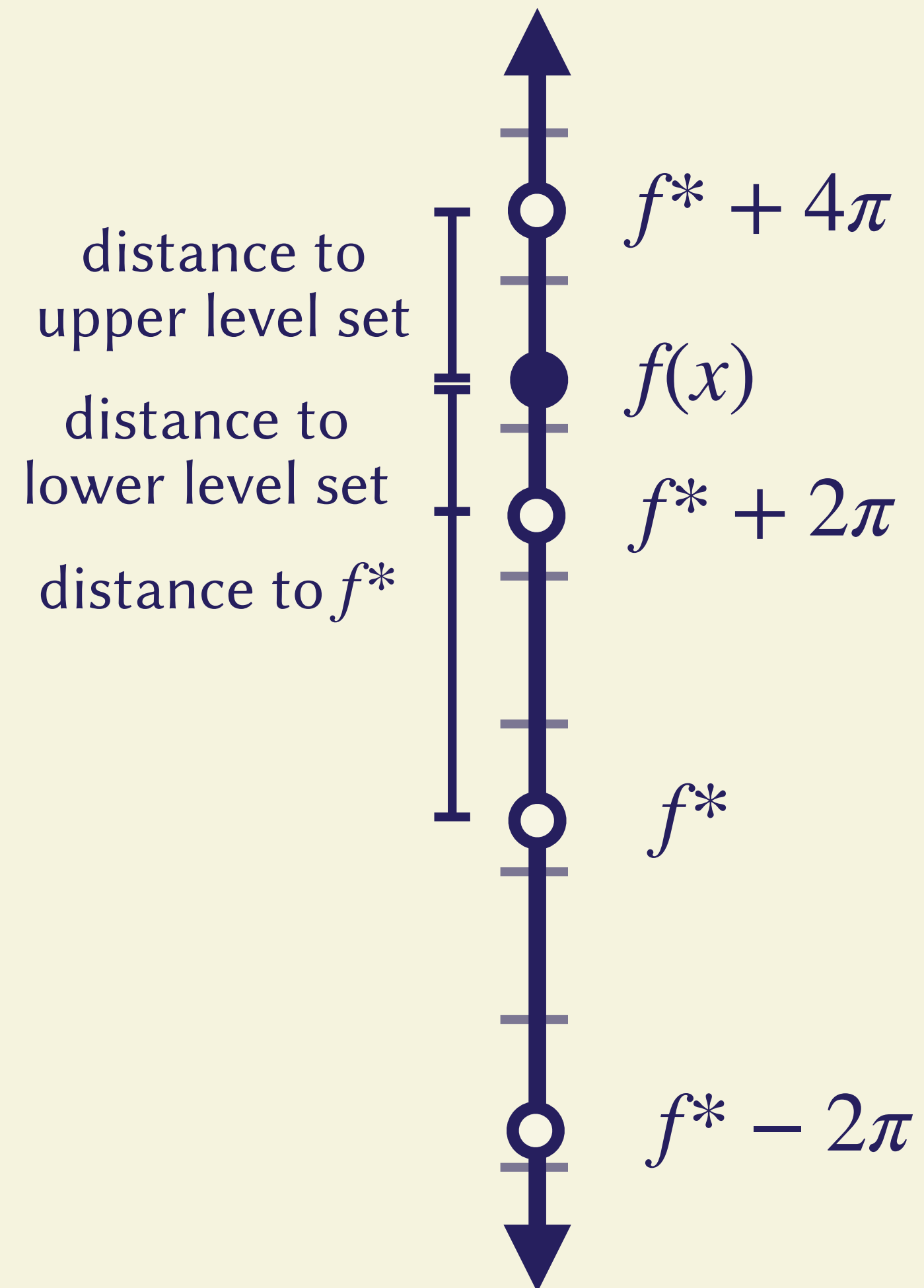


# Angle-valued functions $\rightarrow$ continuous functions



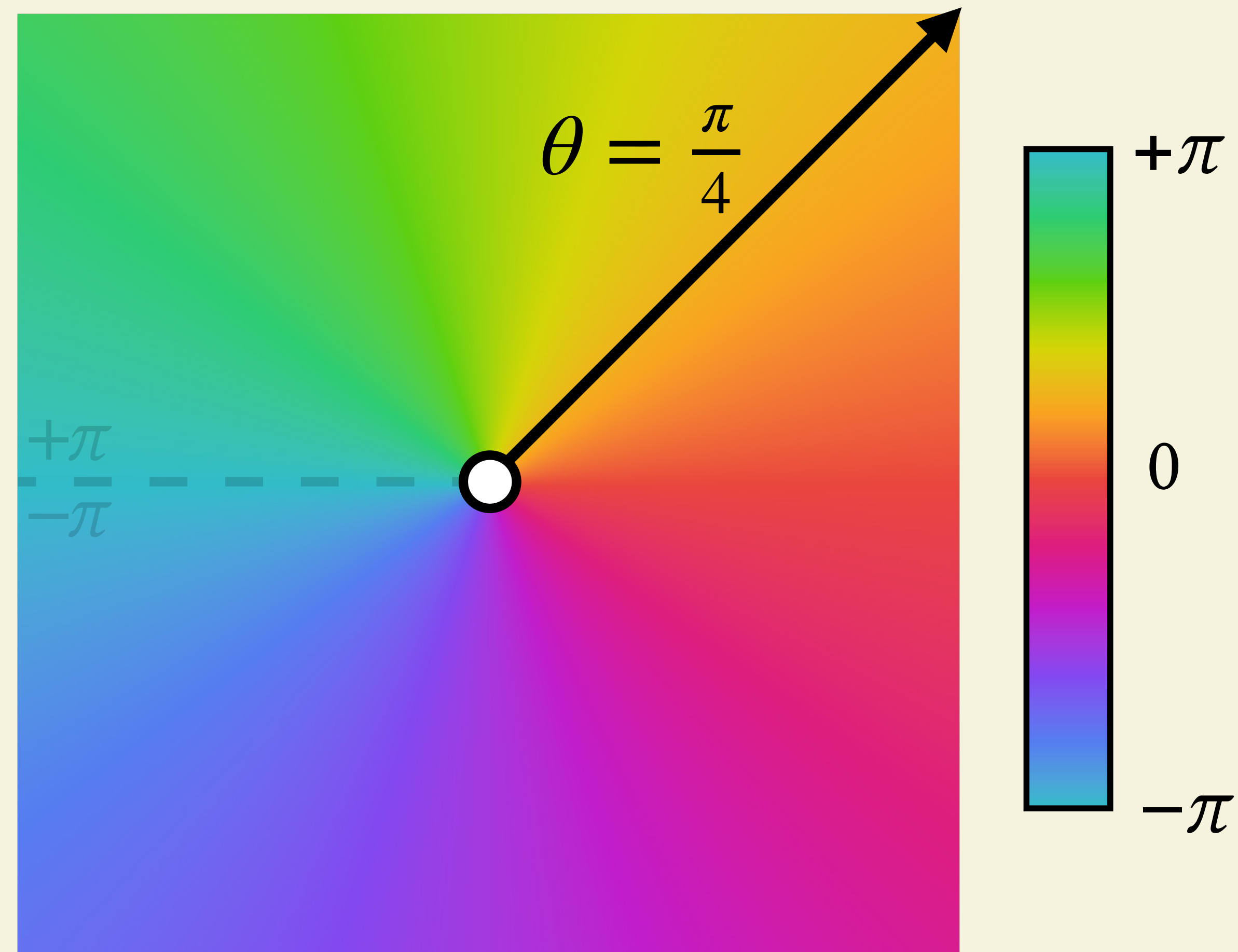
DISCONTINUOUS FUNCTION

# In practice: look for level sets above and below

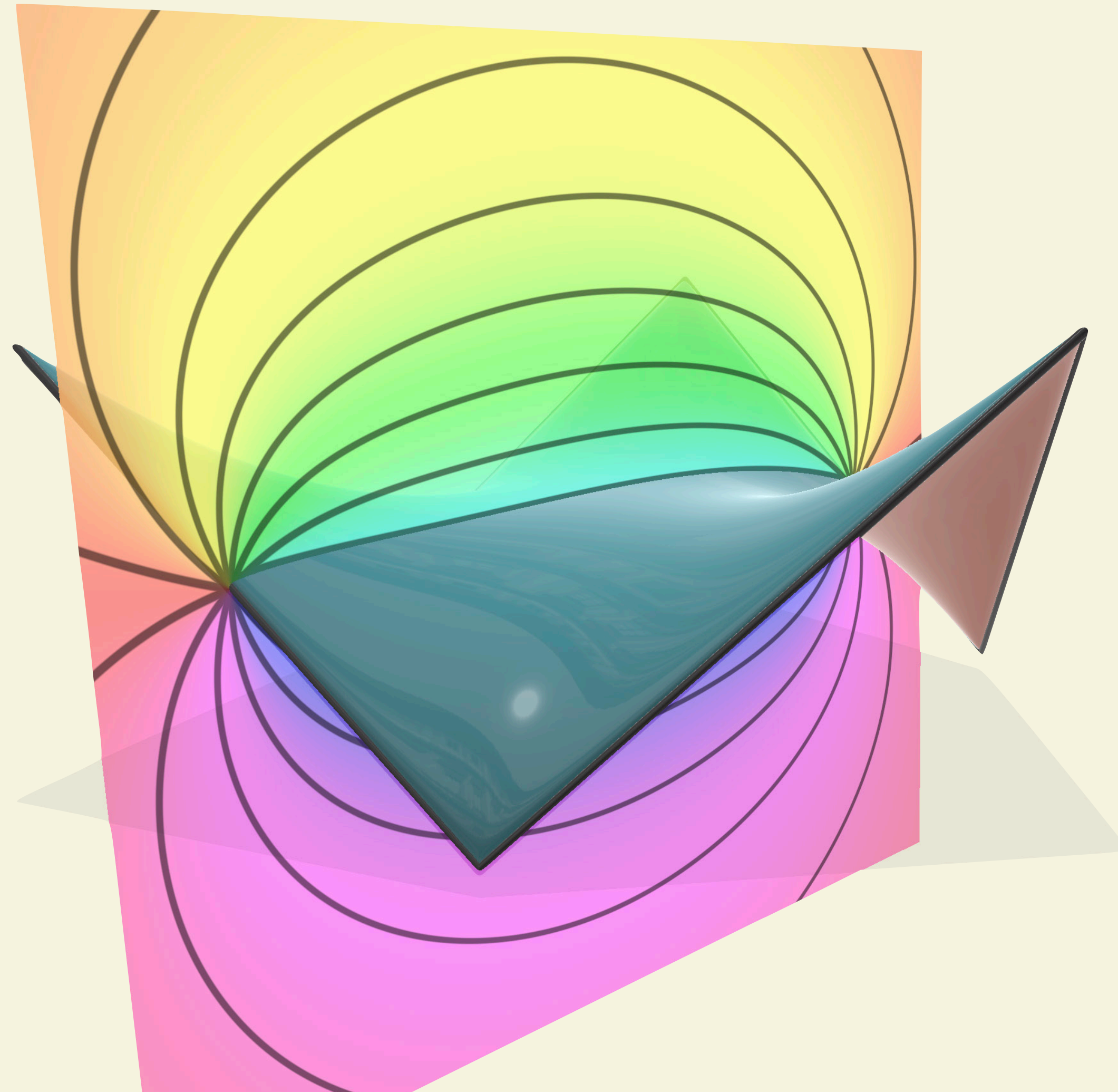


# Angle-valued functions allow for boundaries

$$\theta(x, y) = \text{atan2}(y, x)$$



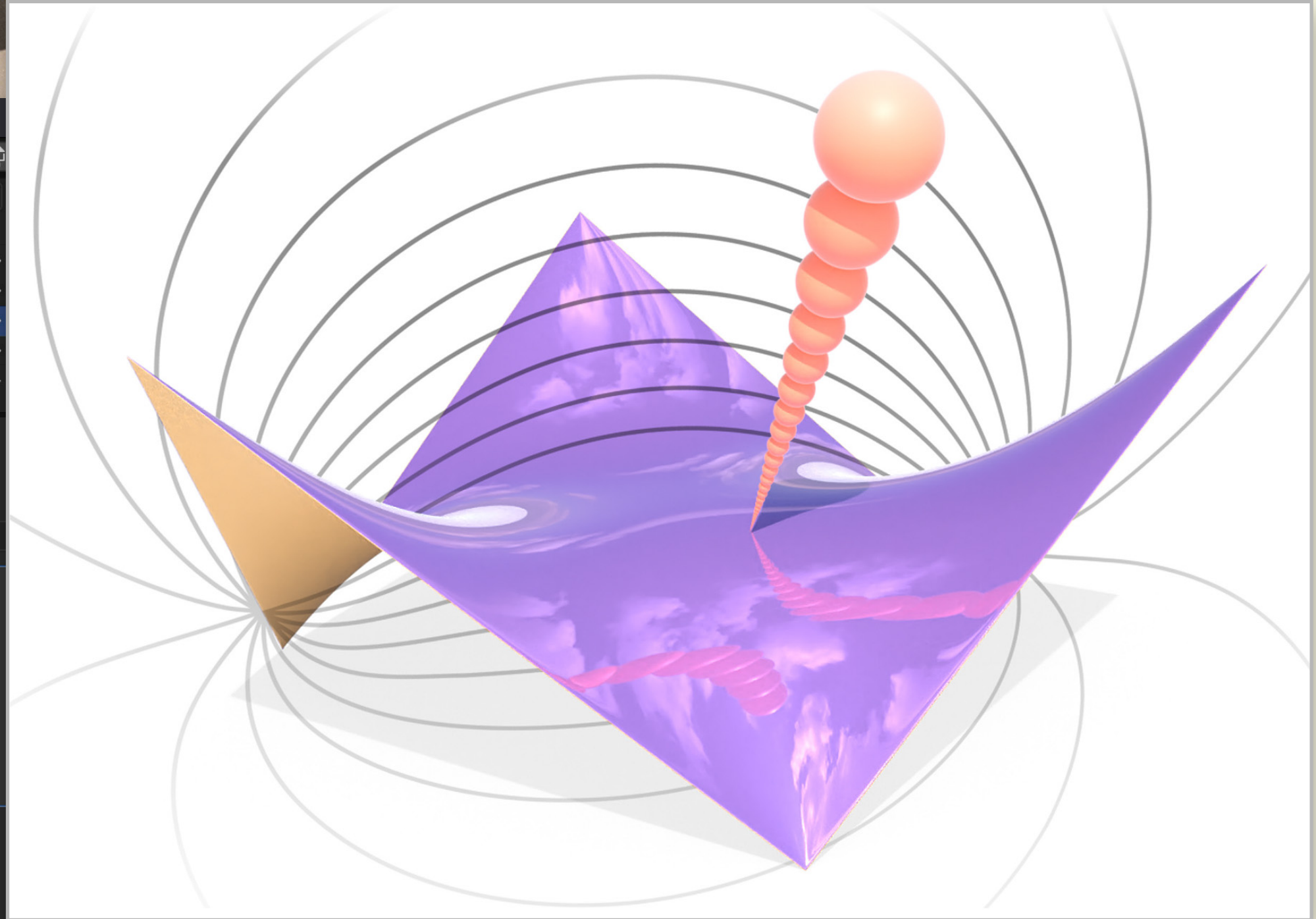
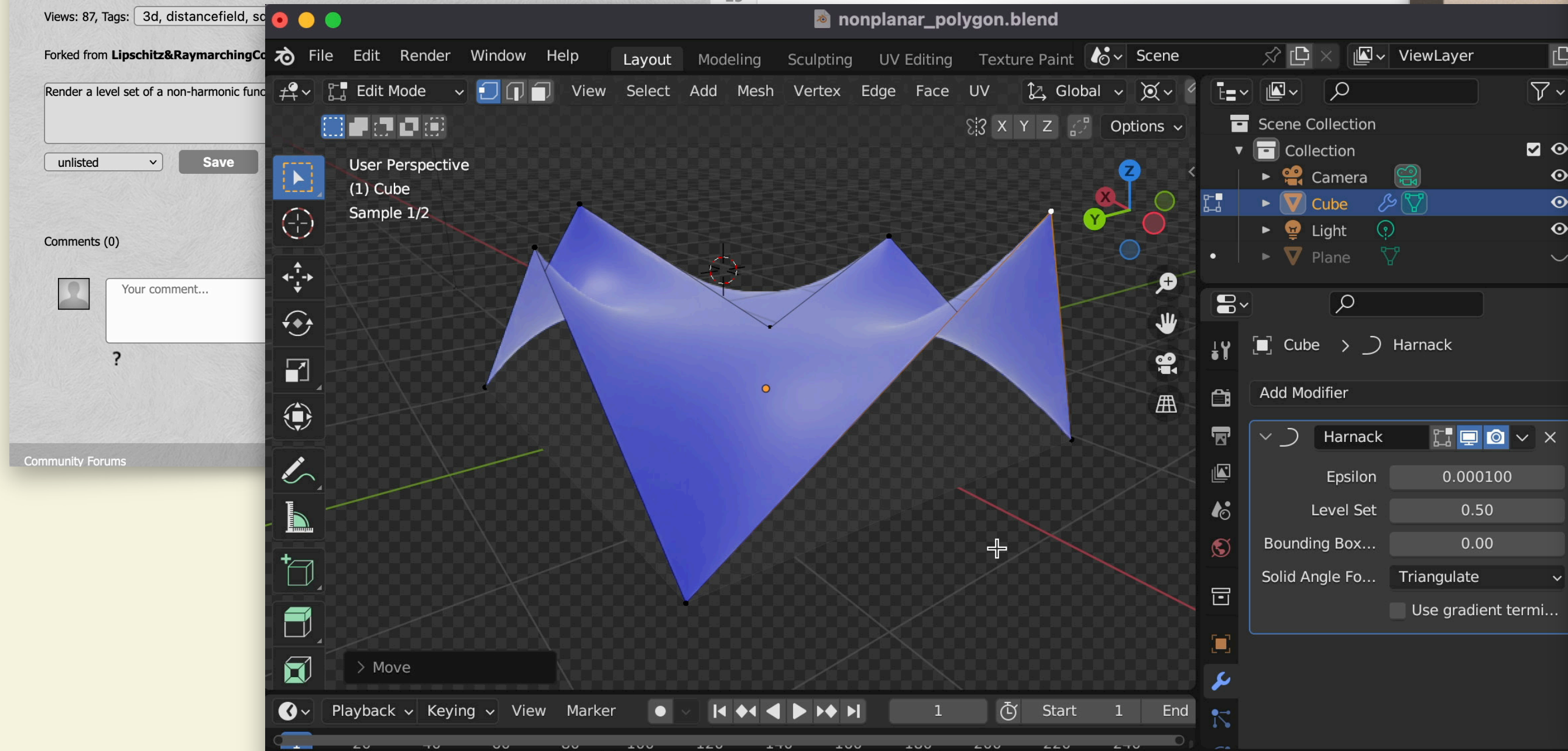
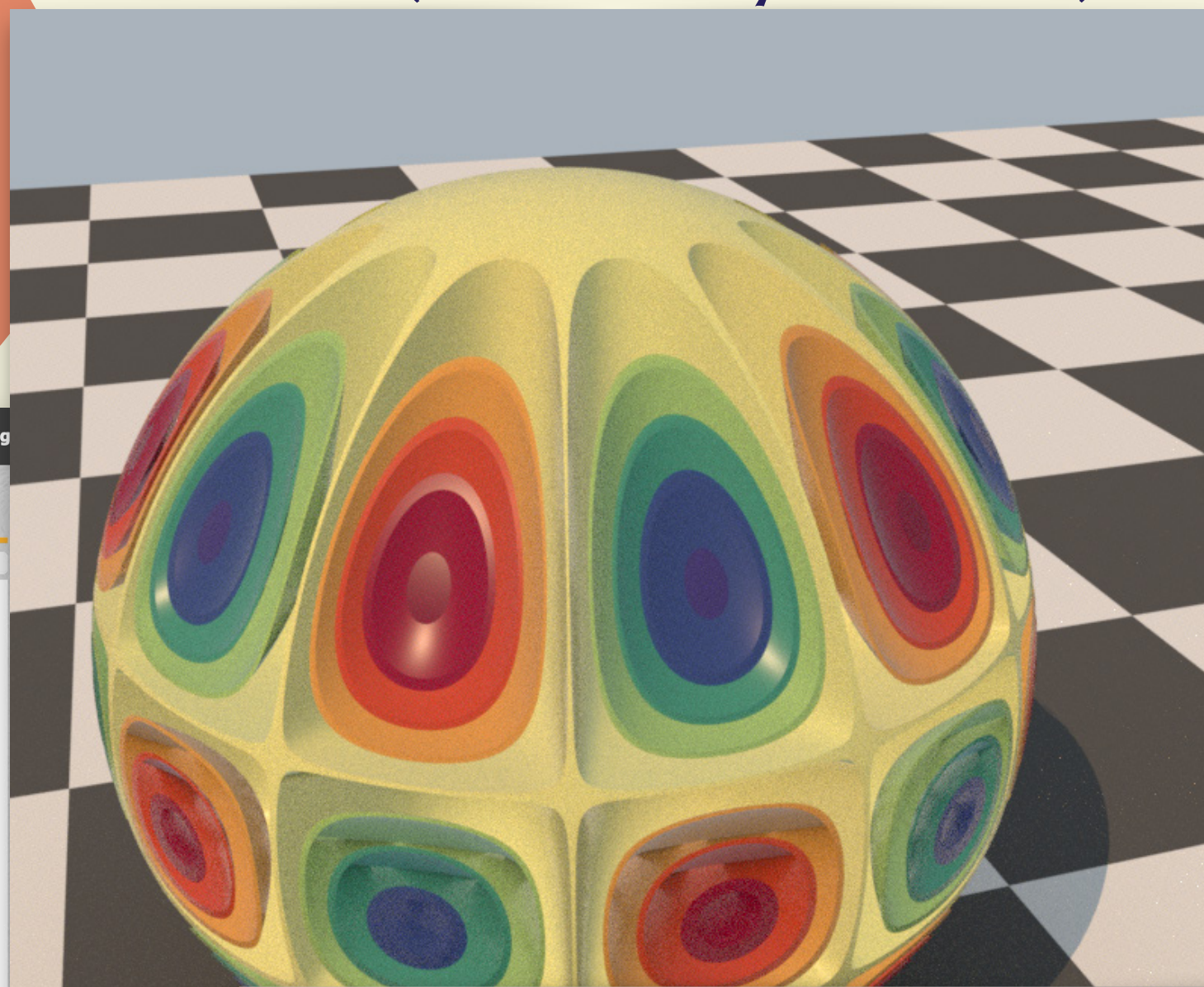
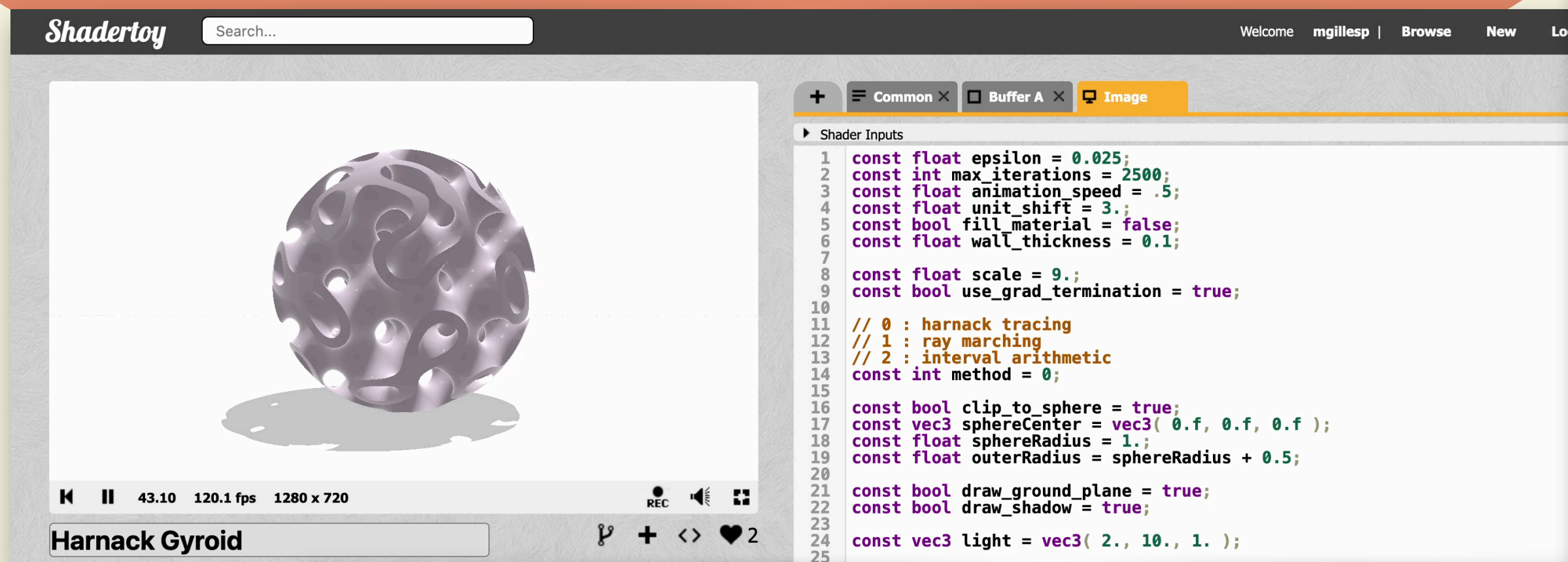
# Angle-valued functions allow for boundaries



# Simple to implement

PBRT (CPU ray tracer)

ShaderToy (WebGL shaders)

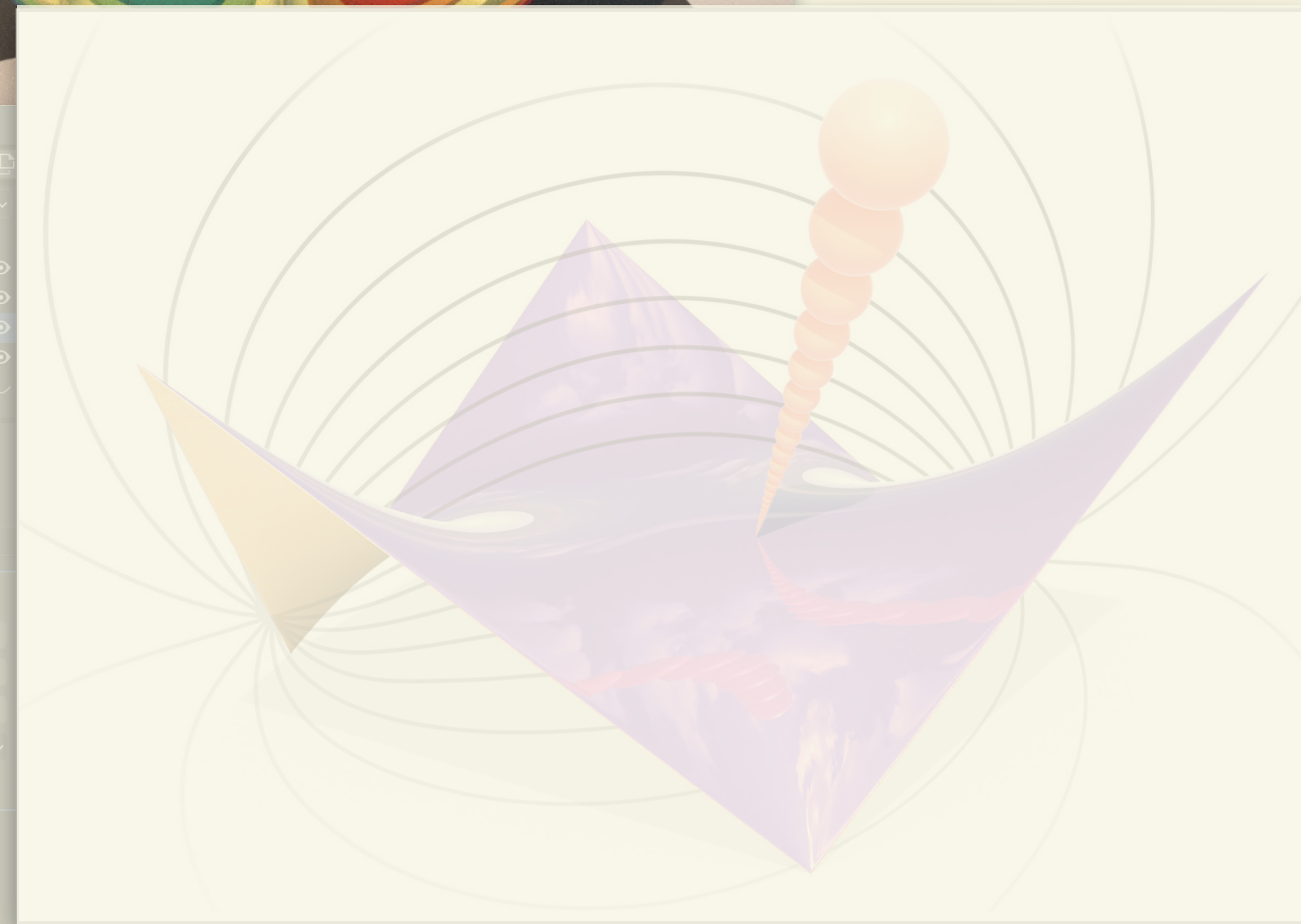
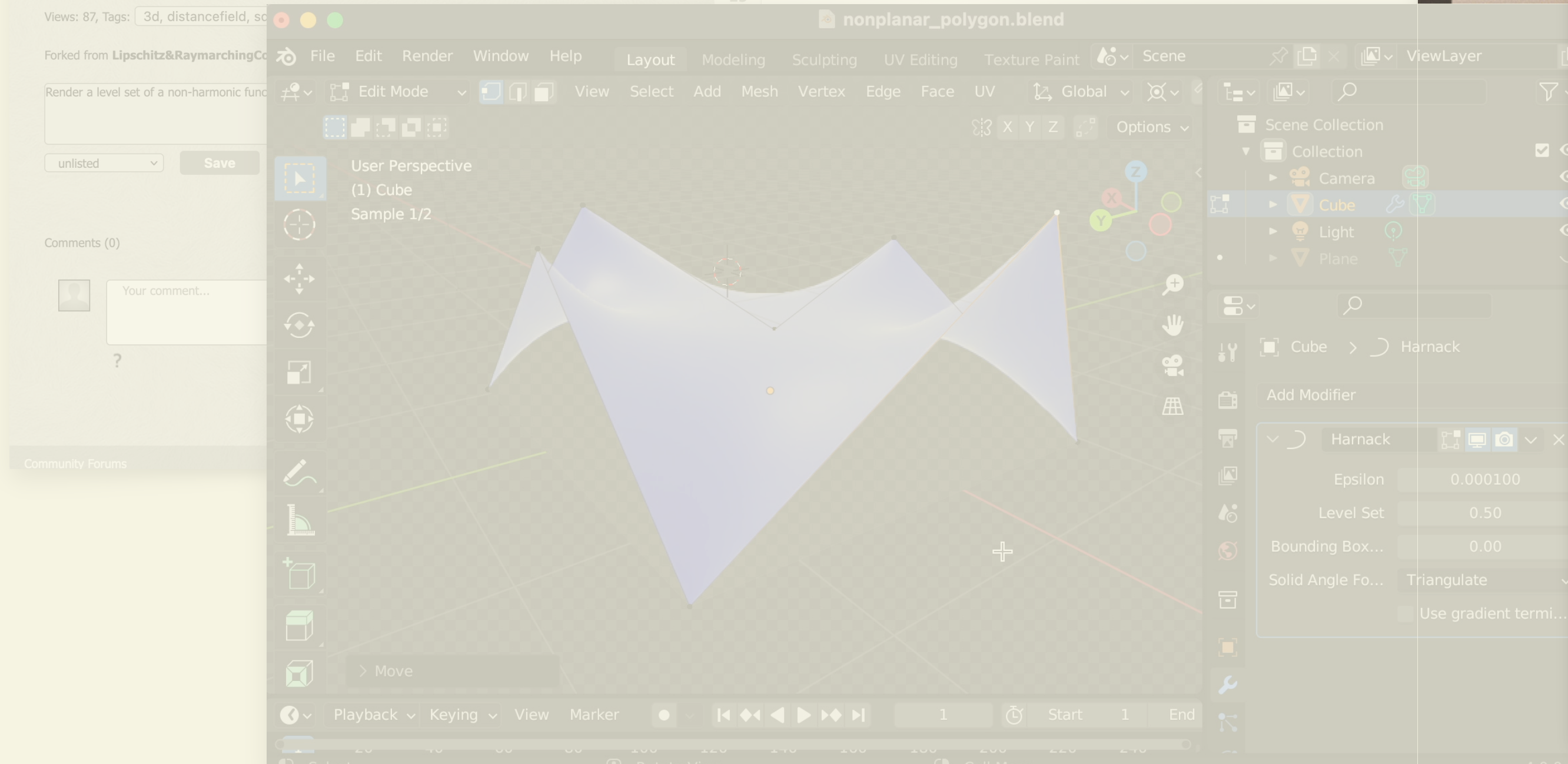
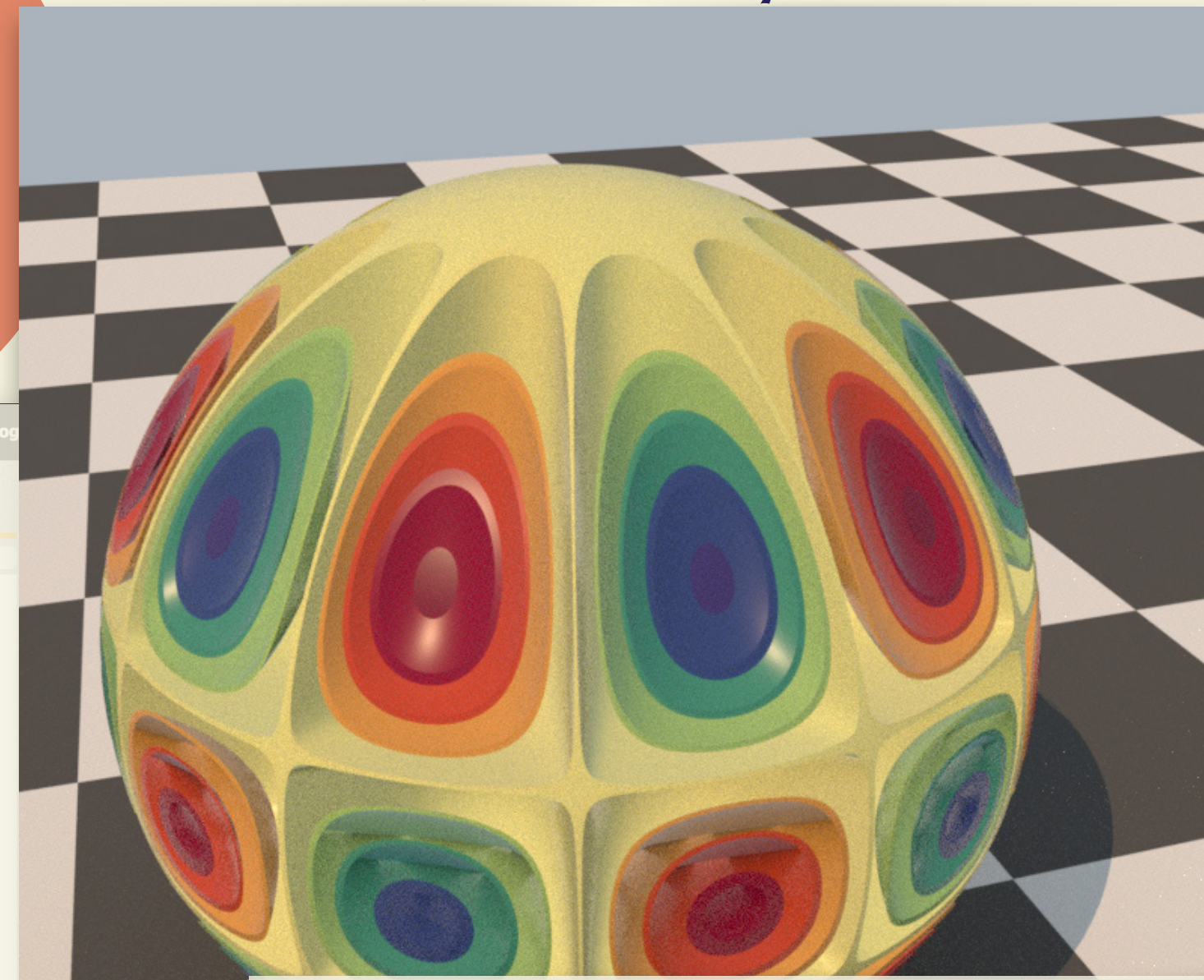
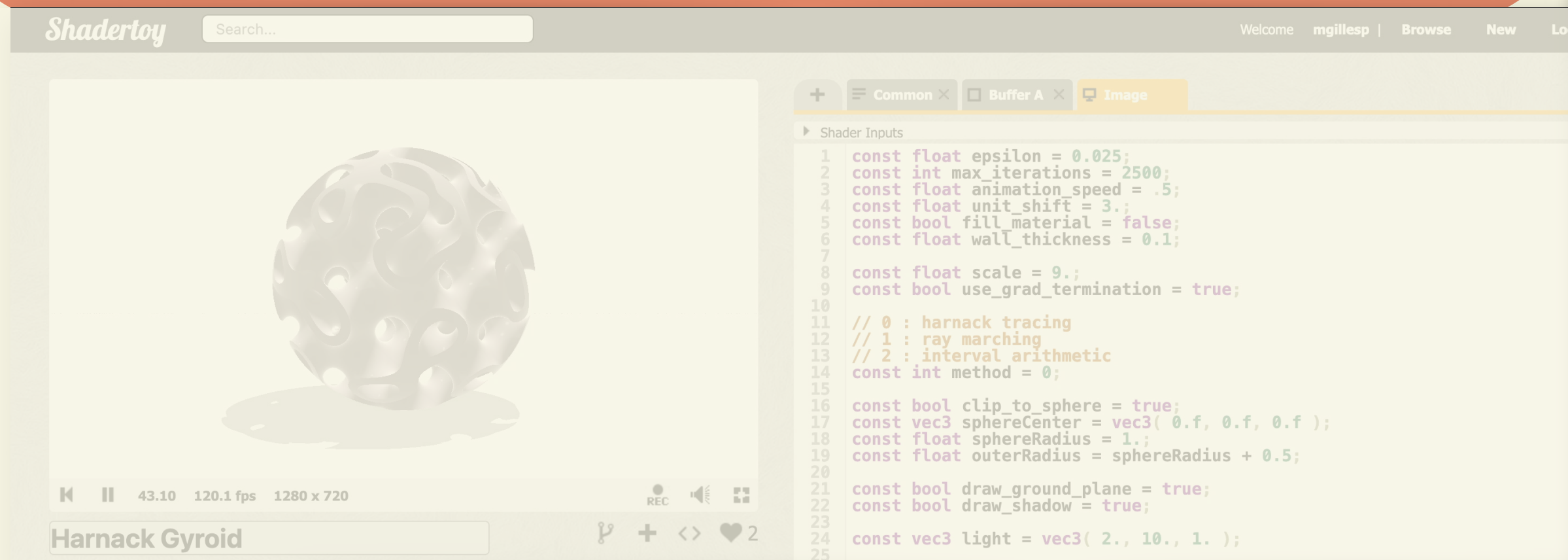


Blender (CPU ray tracer)

# Simple to implement

PBRT (CPU ray tracer)

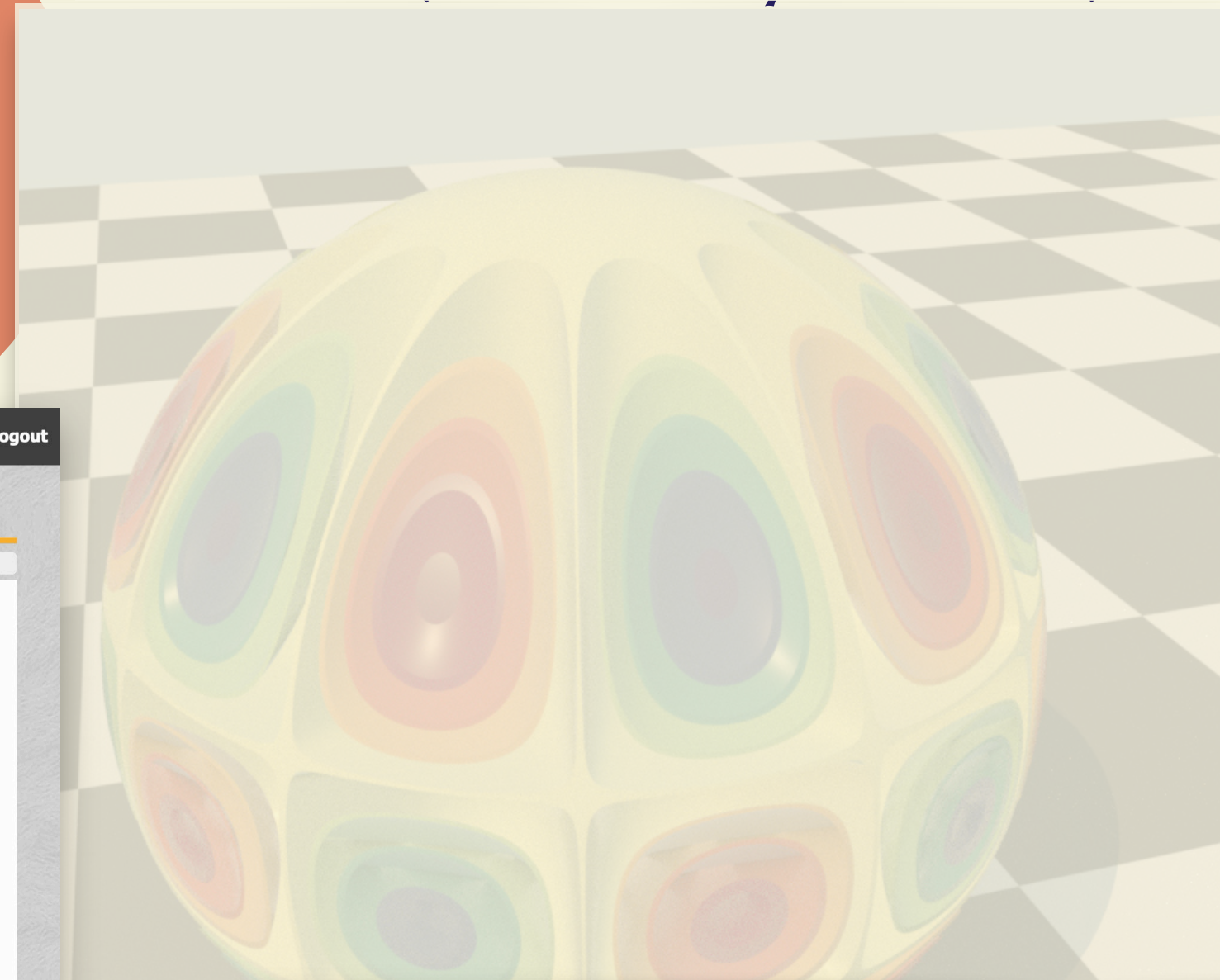
ShaderToy (WebGL shaders)



Blender (CPU ray tracer)

# Simple to implement

PBRT (CPU ray tracer)



ShaderToy (WebGL shaders)

```
1 const float epsilon = 0.025;
2 const int max_iterations = 2500;
3 const float animation_speed = .5;
4 const float unit_shift = 3.;
5 const bool fill_material = false;
6 const float wall_thickness = 0.1;
7
8 const float scale = 9.;
9 const bool use_grad_termination = true;
10
11 // 0 : harnack tracing
12 // 1 : ray marching
13 // 2 : interval arithmetic
14 const int method = 0;
15
16 const bool clip_to_sphere = true;
17 const vec3 sphereCenter = vec3( 0.f, 0.f, 0.f );
18 const float sphereRadius = 1.;
19 const float outerRadius = sphereRadius + 0.5;
20
21 const bool draw_ground_plane = true;
22 const bool draw_shadow = true;
23
24 const vec3 light = vec3( 2., 10., 1. );
25
```

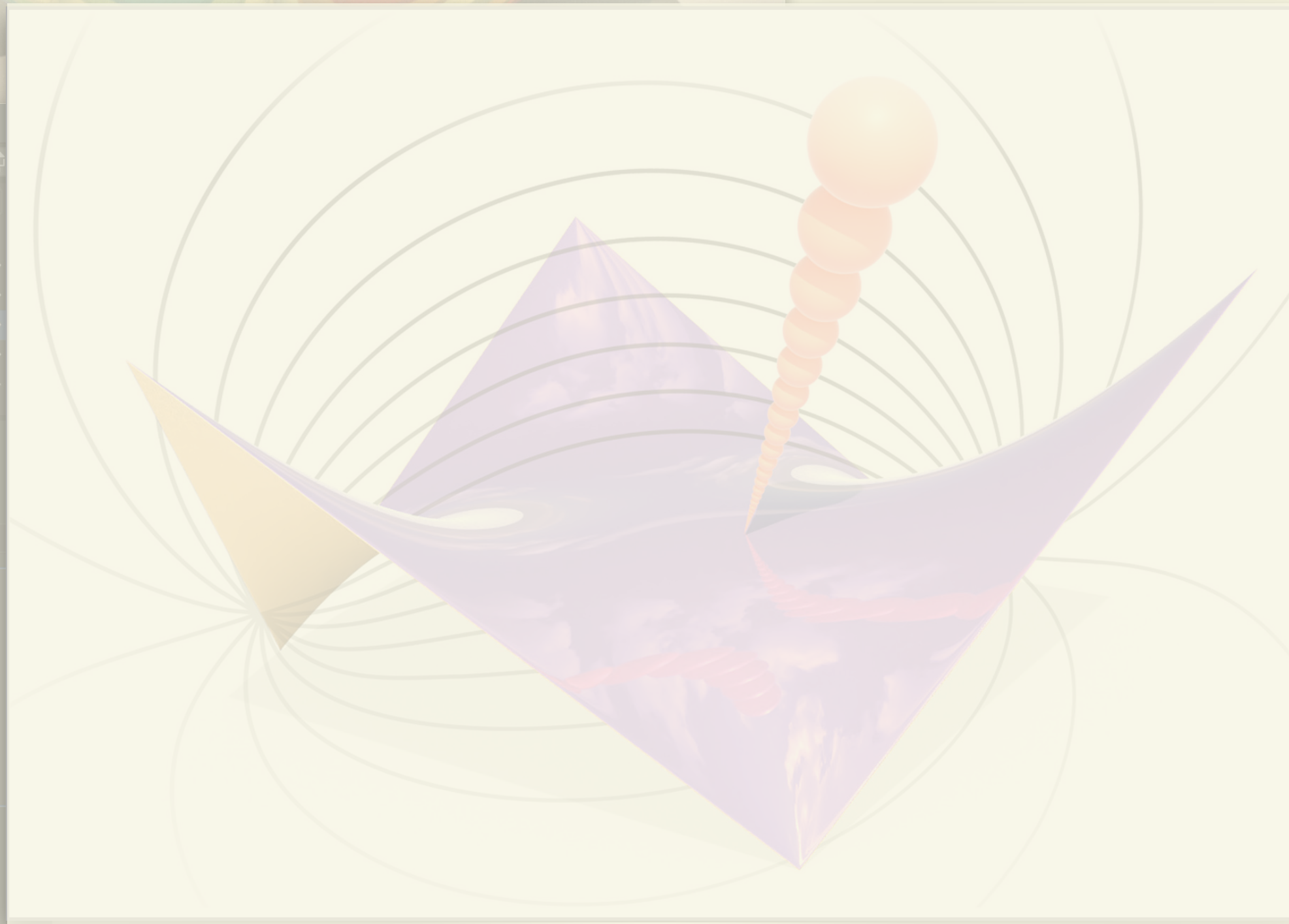
nonplanar\_polygon.blend

Scene Collection

- Collection
- Camera
- Cube
- Light
- Plane

Add Modifier

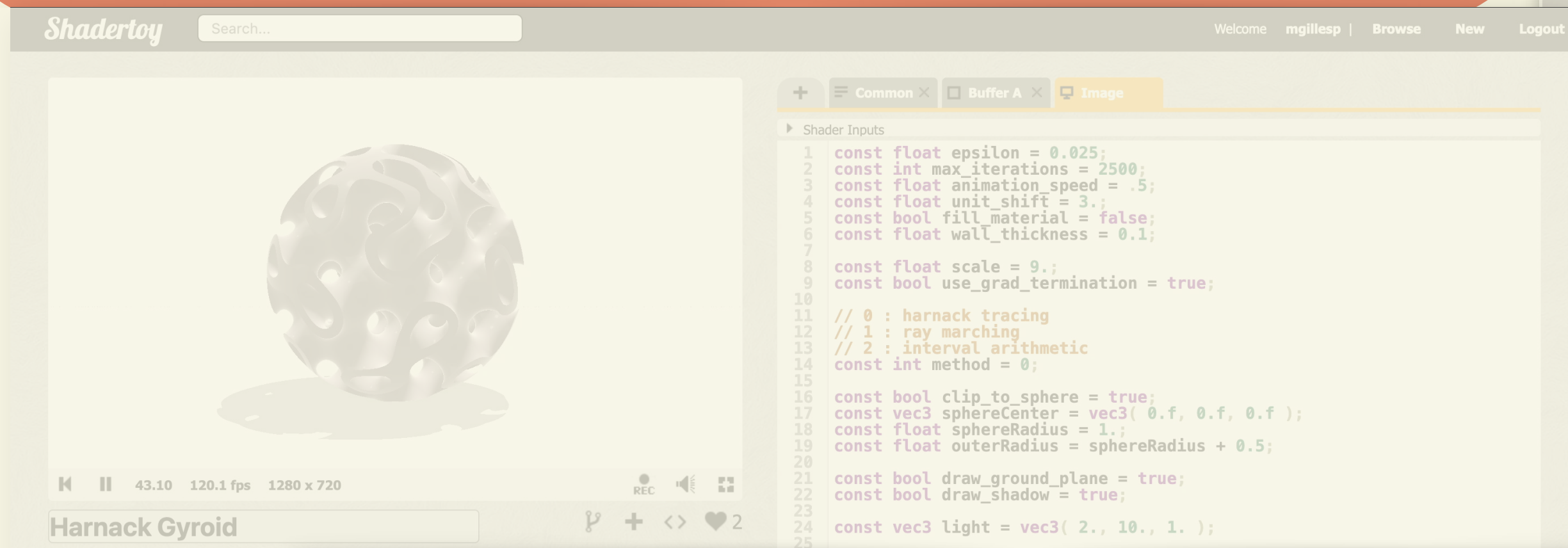
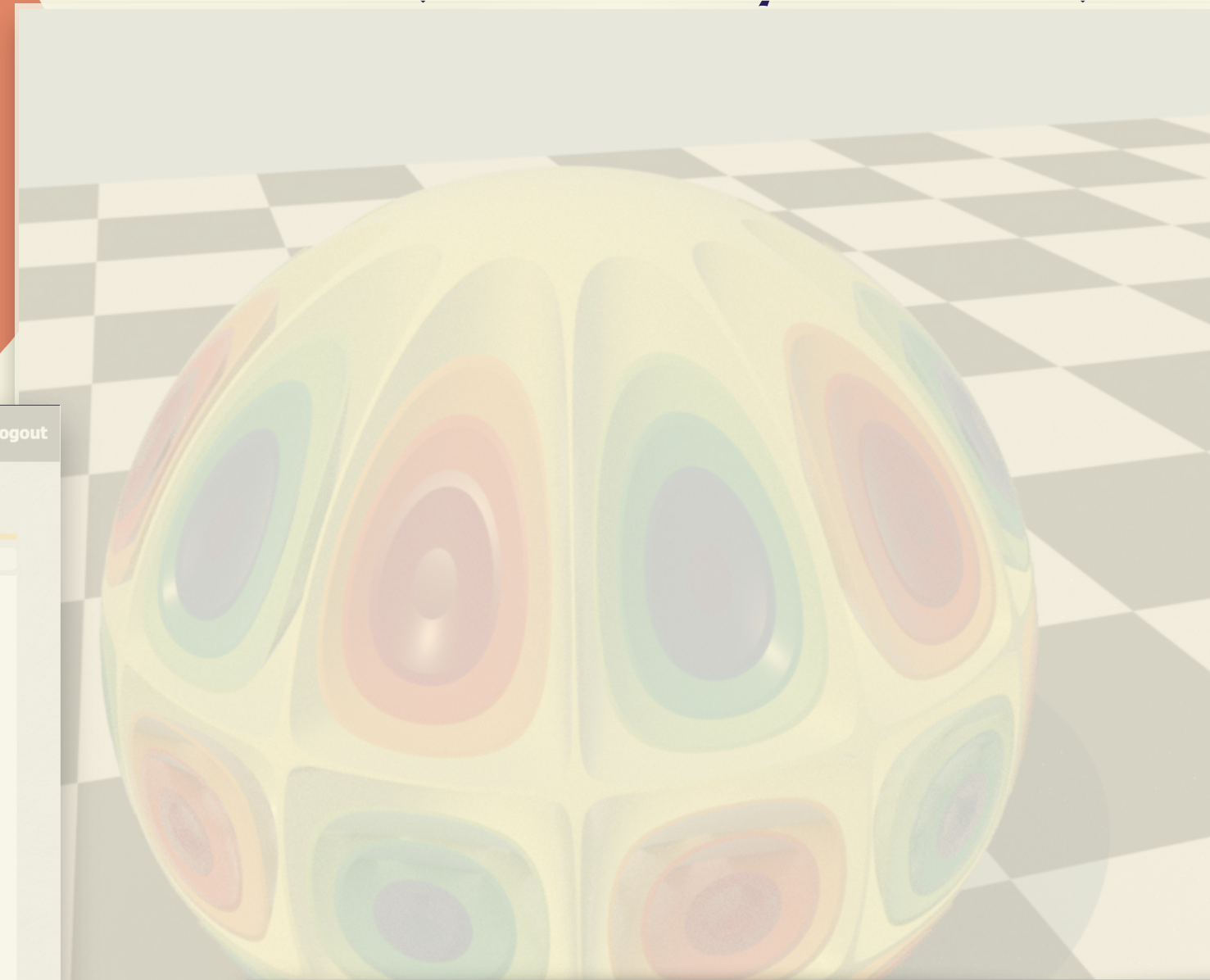
- Harnack
- Epsilon: 0.000100
- Level Set: 0.50
- Bounding Box...: 0.00
- Solid Angle Fo...: Triangulate
- Use gradient termi...



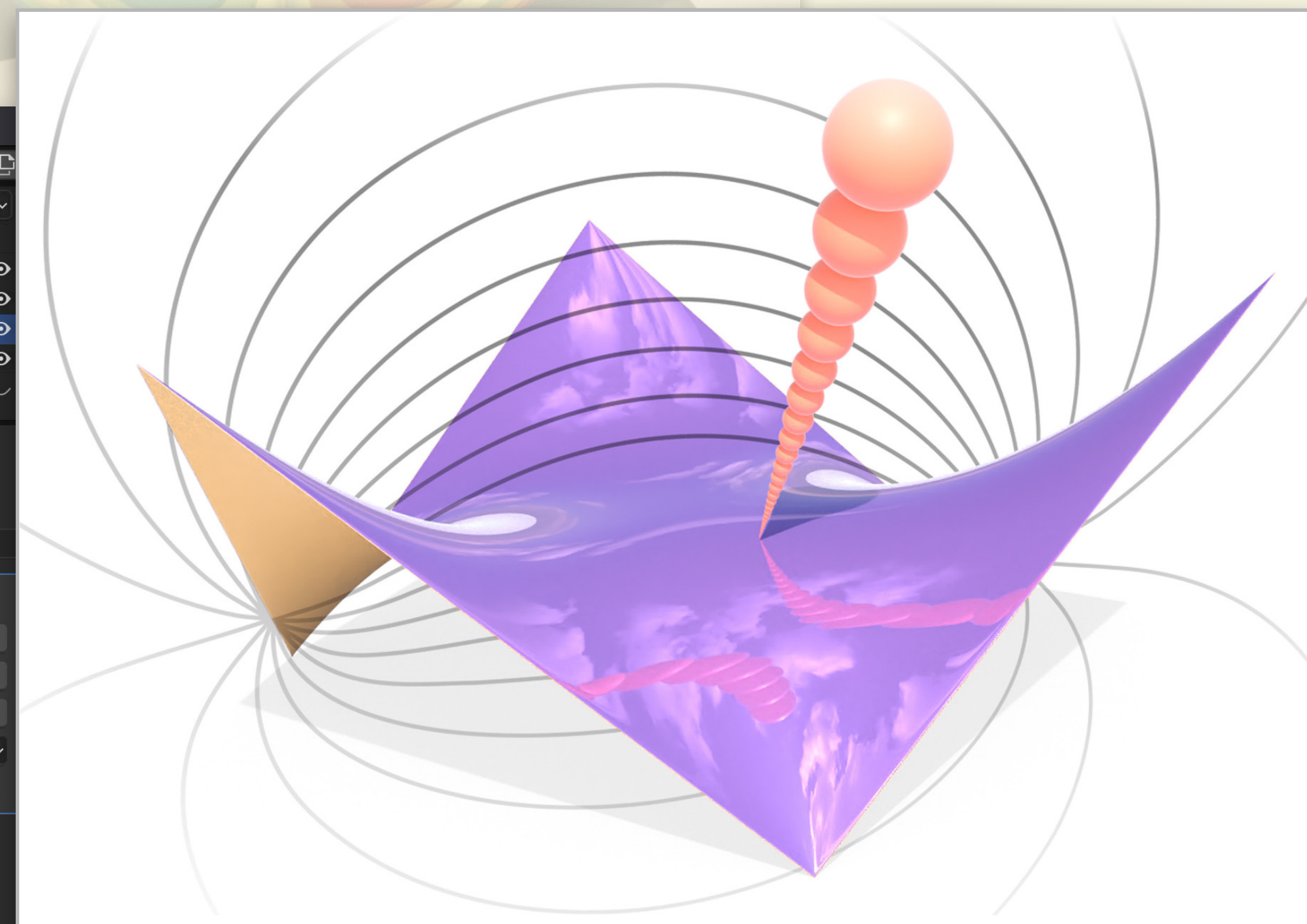
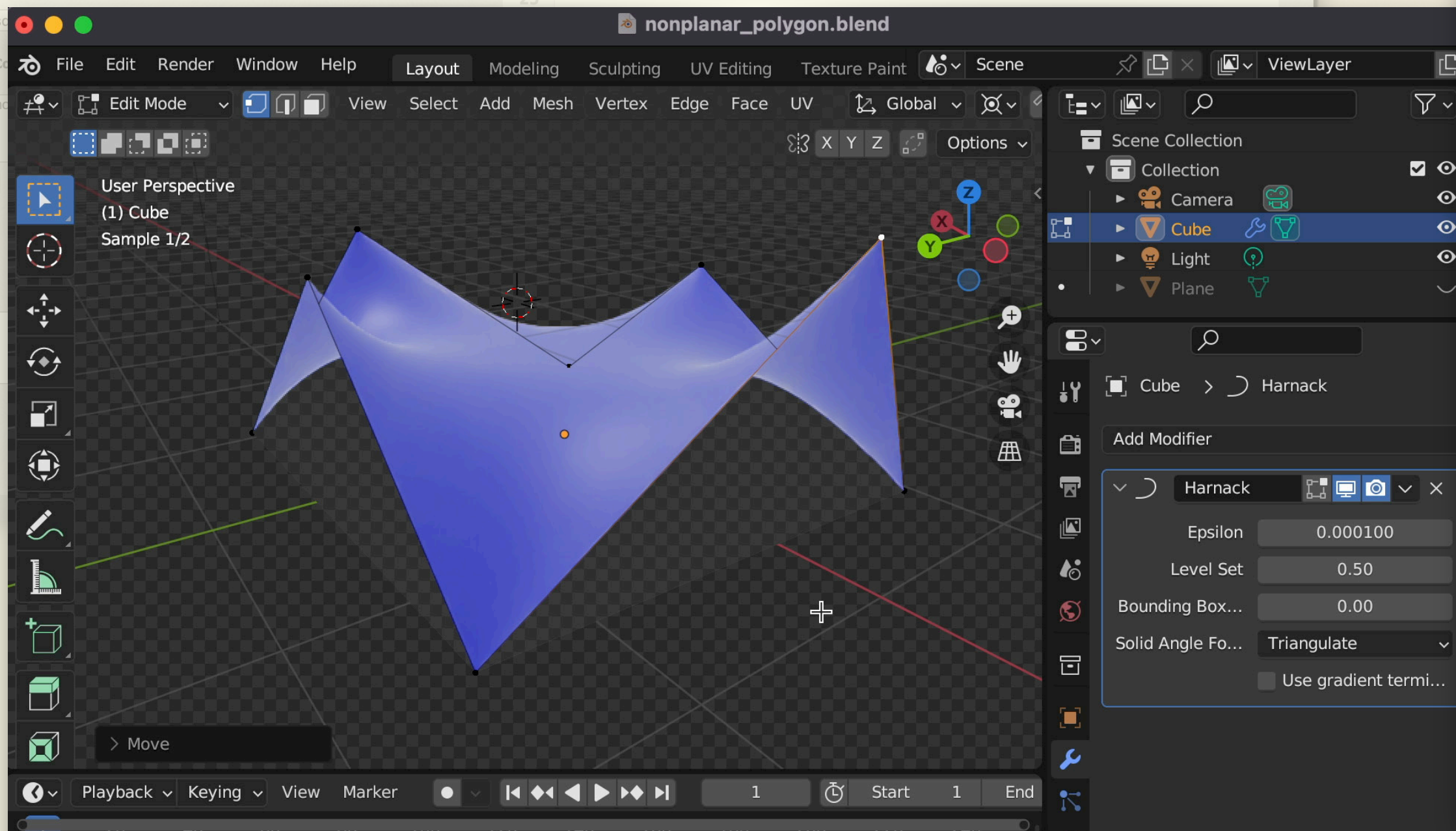
Blender (CPU ray tracer)

# Simple to implement

PBRT (CPU ray tracer)



ShaderToy (WebGL shaders)



Blender (CPU ray tracer)



# III. Examples



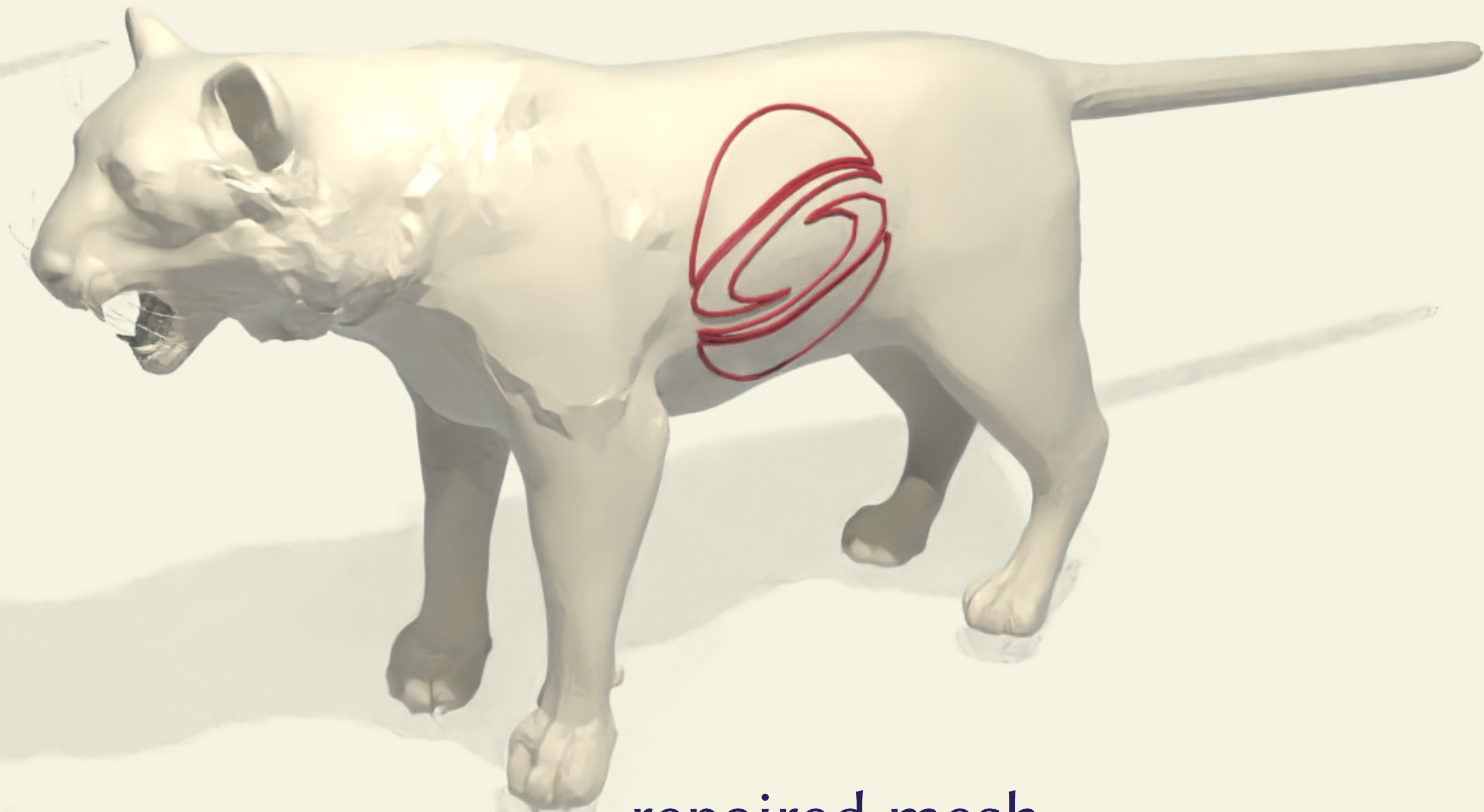
# Generalized winding number

[ Jacobson et al. 2013 ]

a.k.a. *signed solid angle*

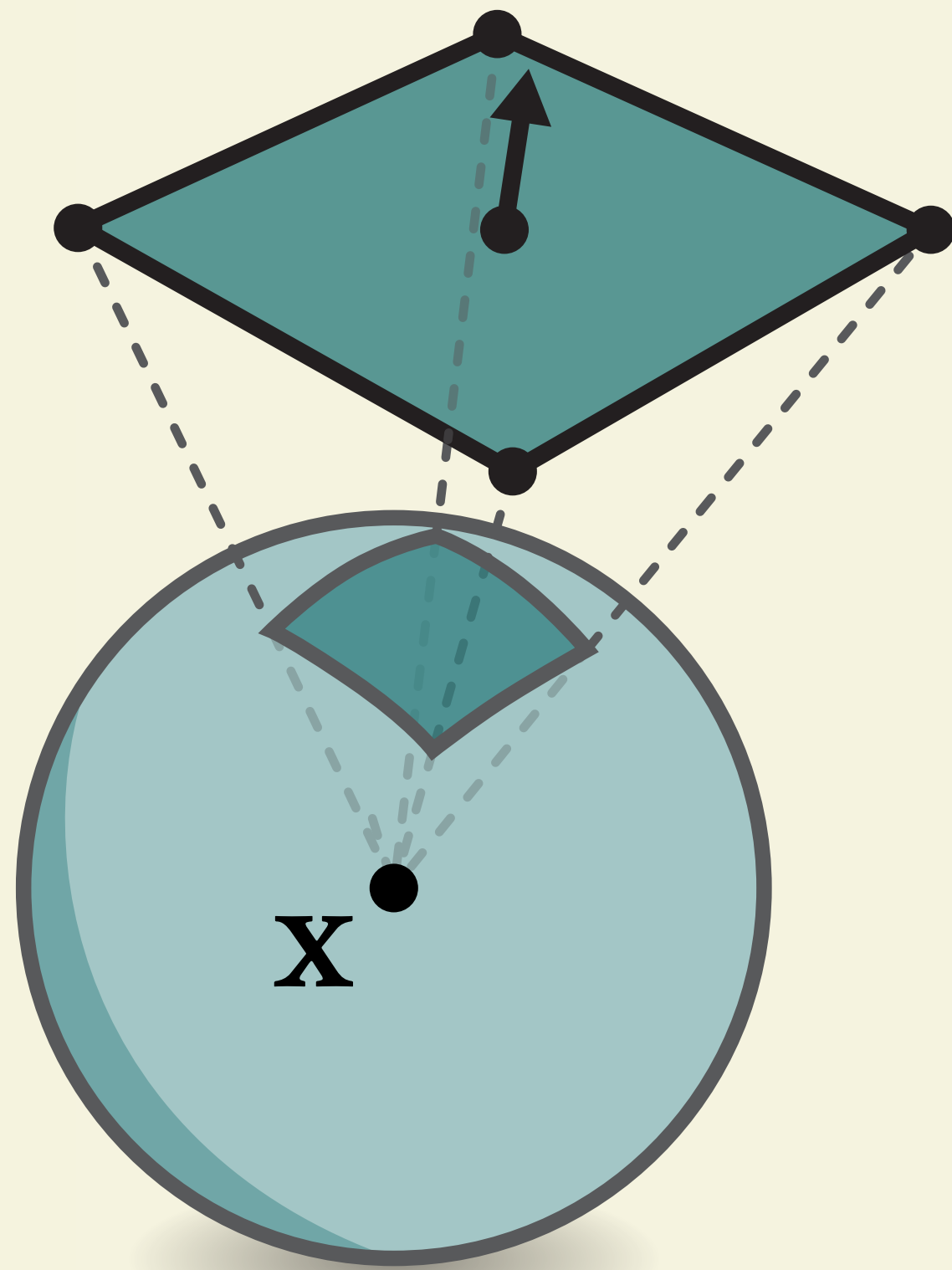


input mesh

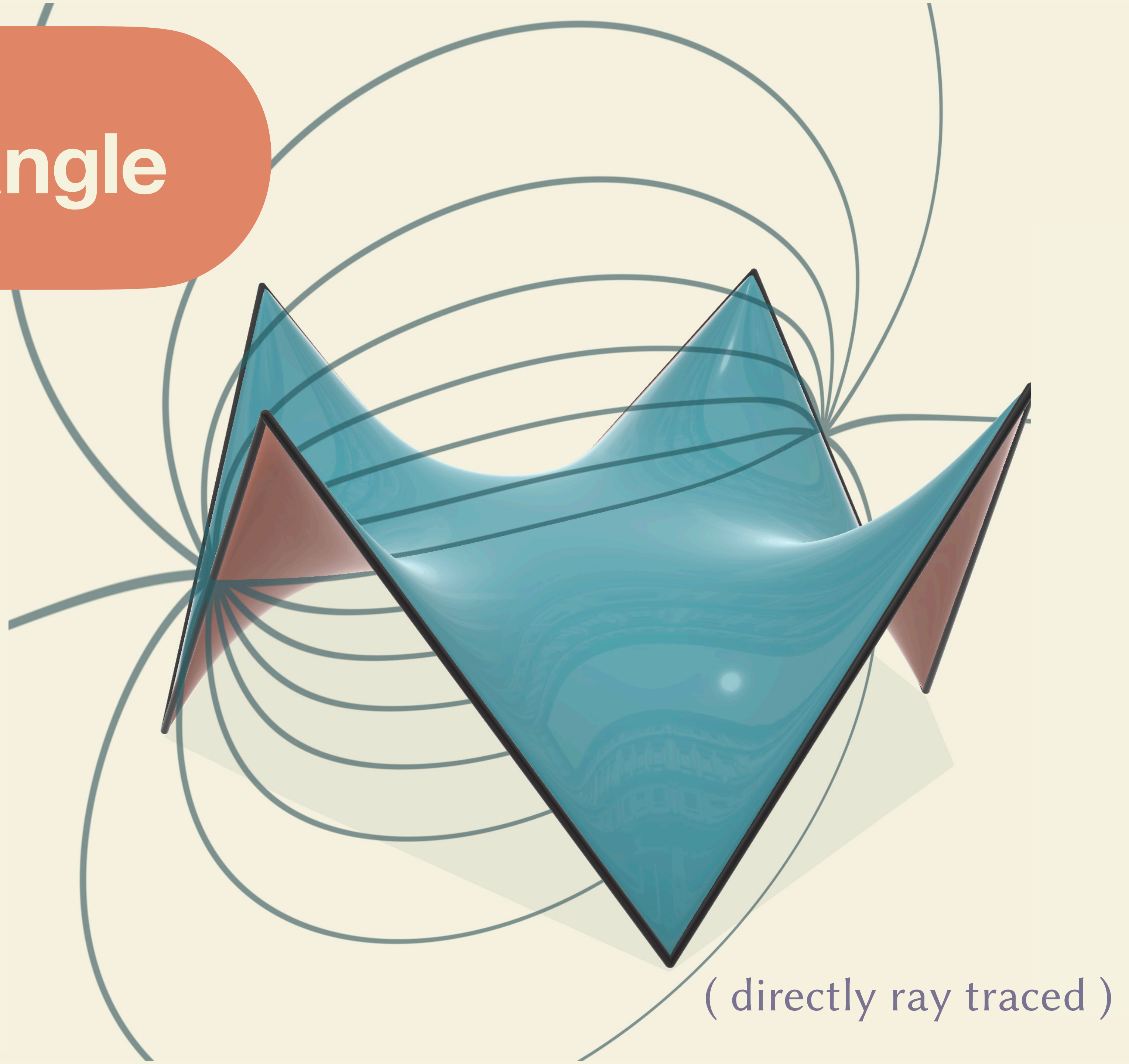


repaired mesh  
( directly ray traced )

# Signed solid angle

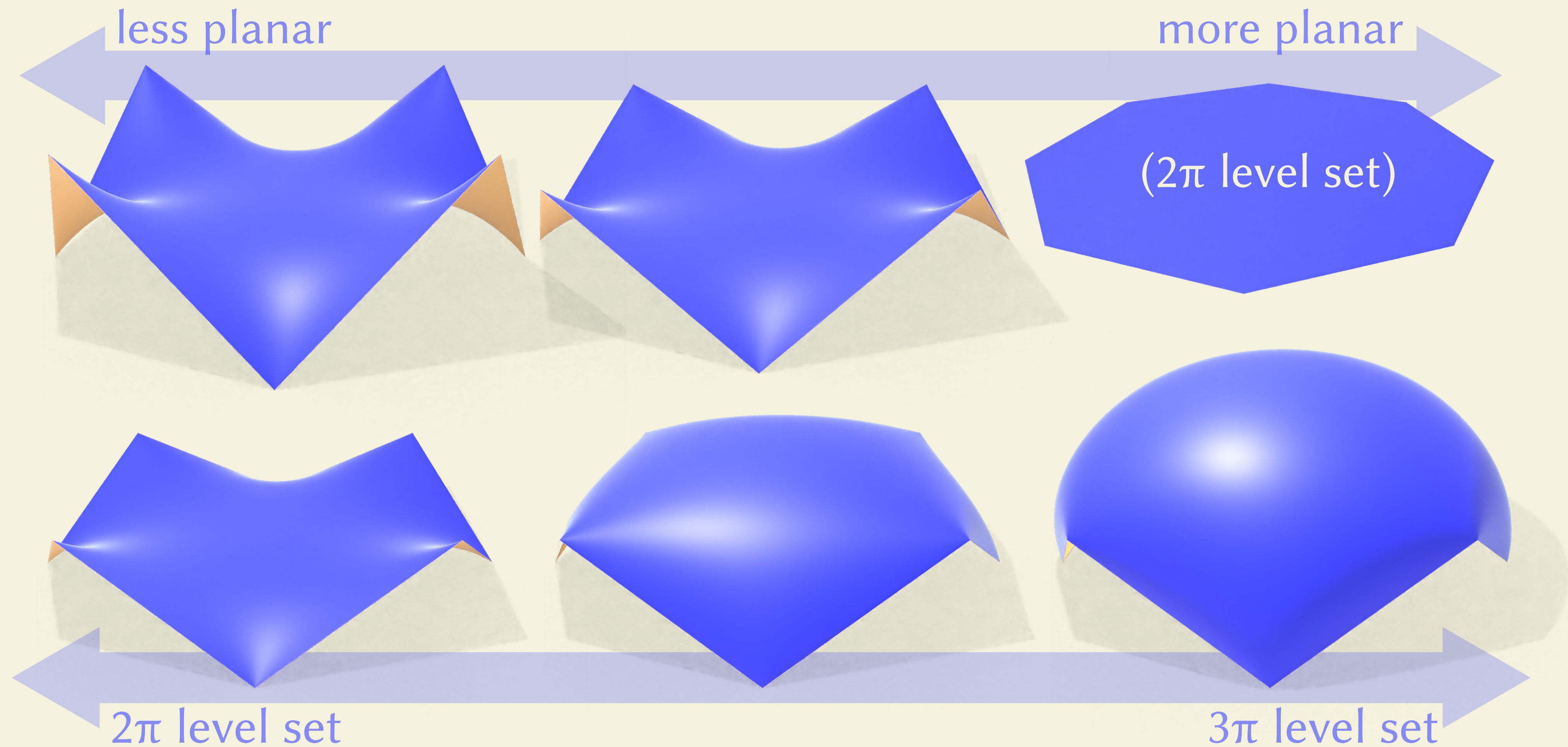


signed solid angle

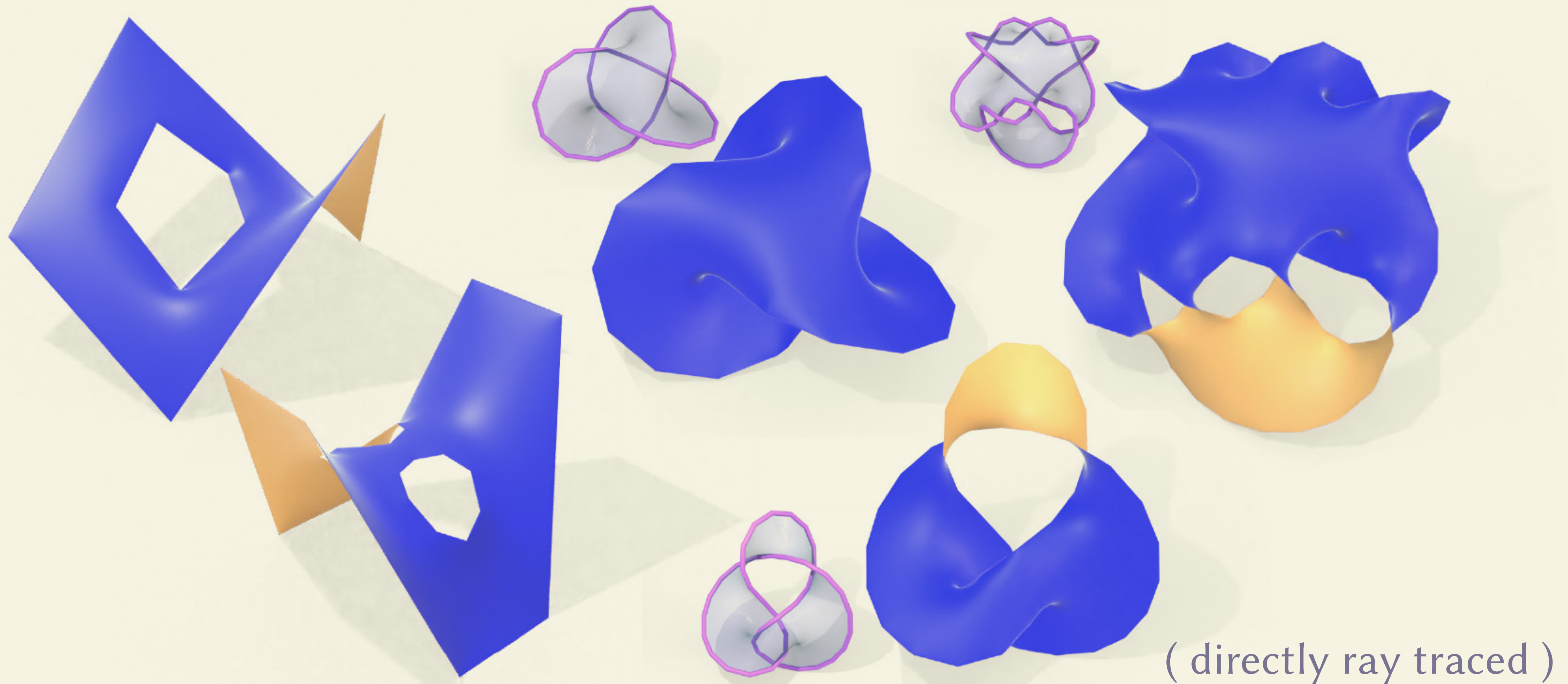


( directly ray traced )

# Signed solid angle



# General nonplanar polygons



( directly ray traced )

# Interpolating surfaces

*solid angle*

*mean value coordinates*

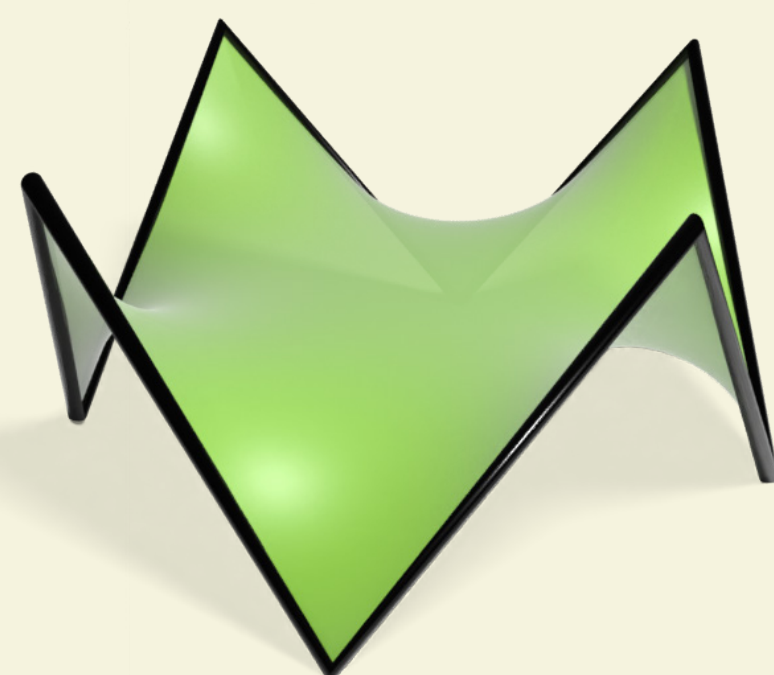
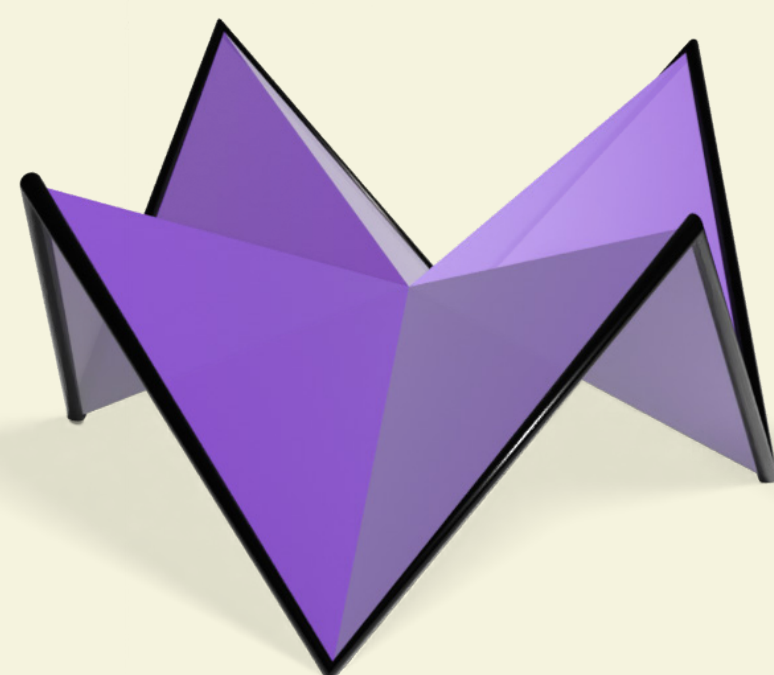
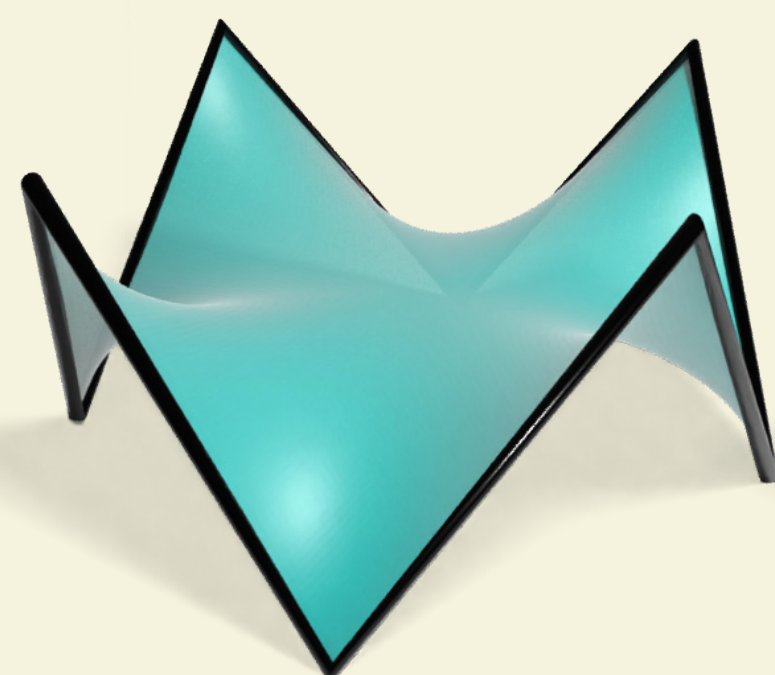
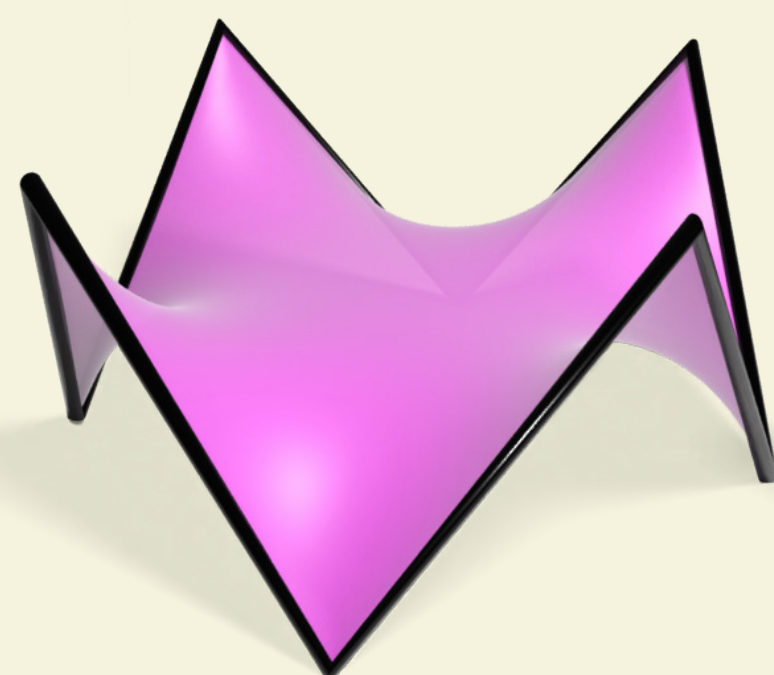
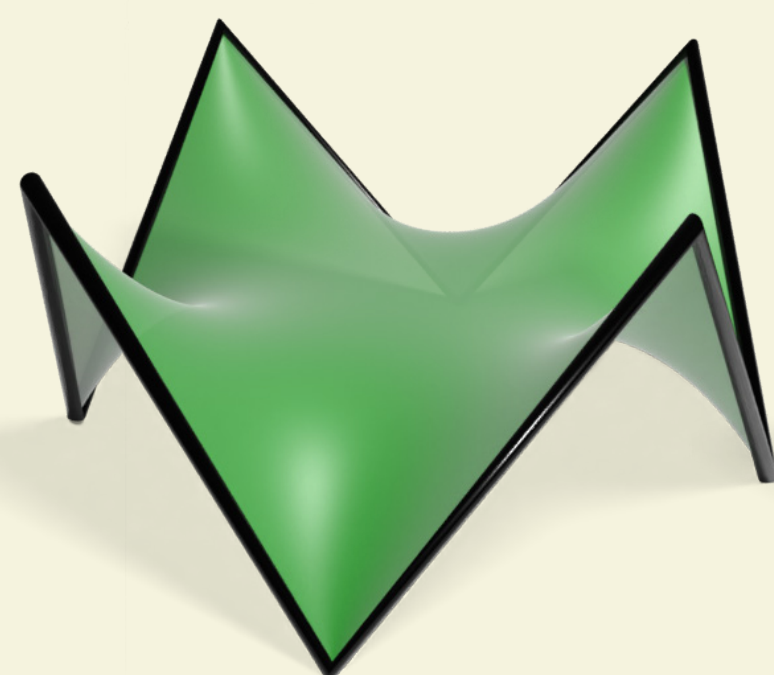
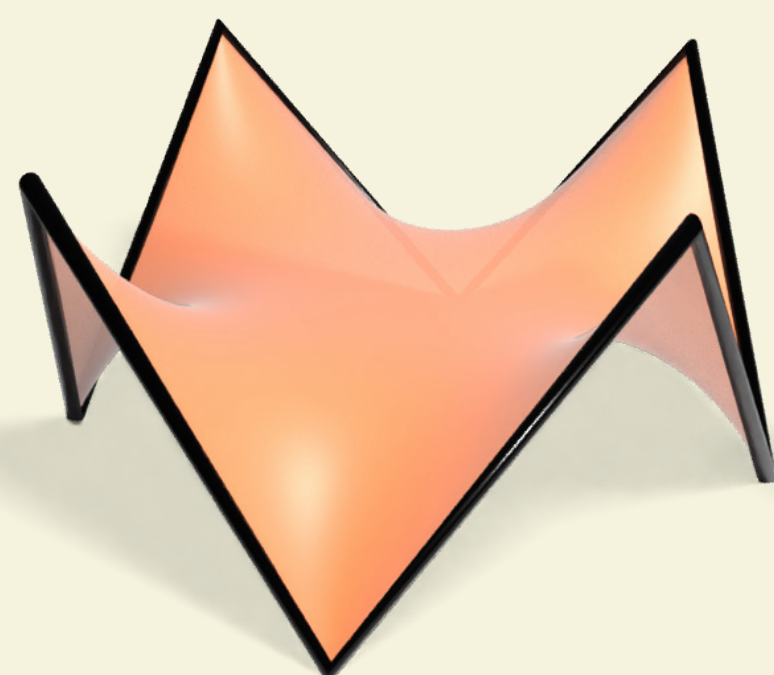
*harmonic coordinates*

*subdivision surface*

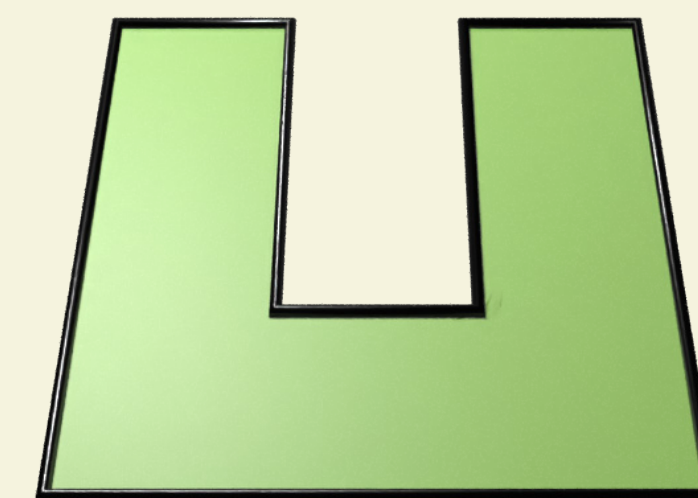
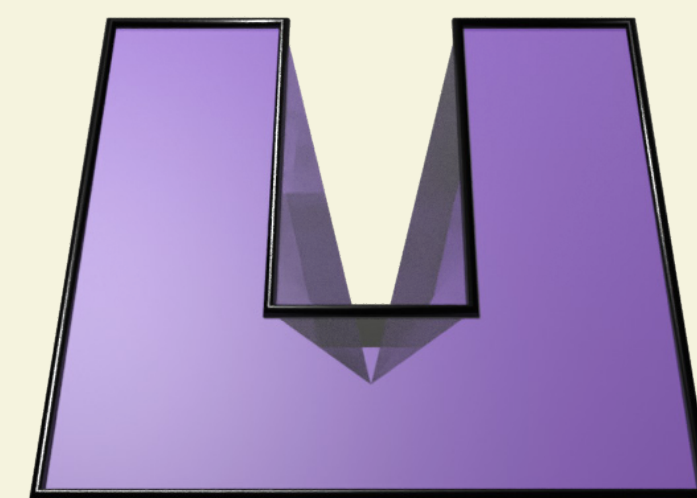
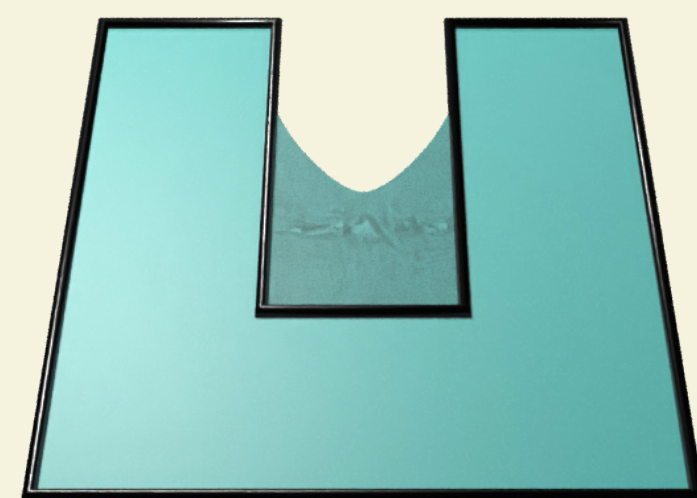
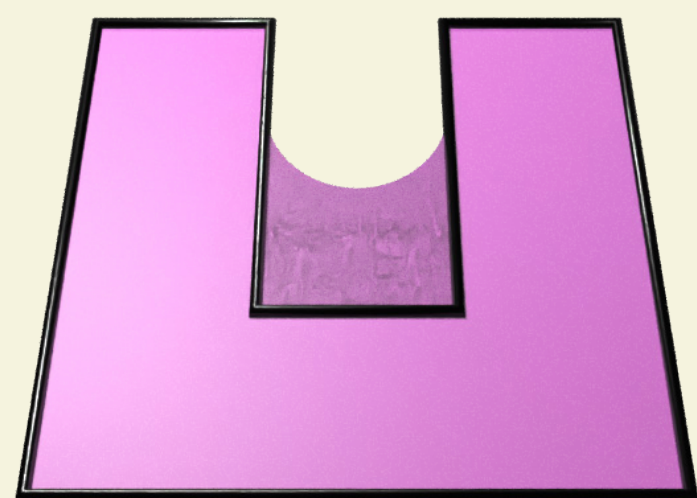
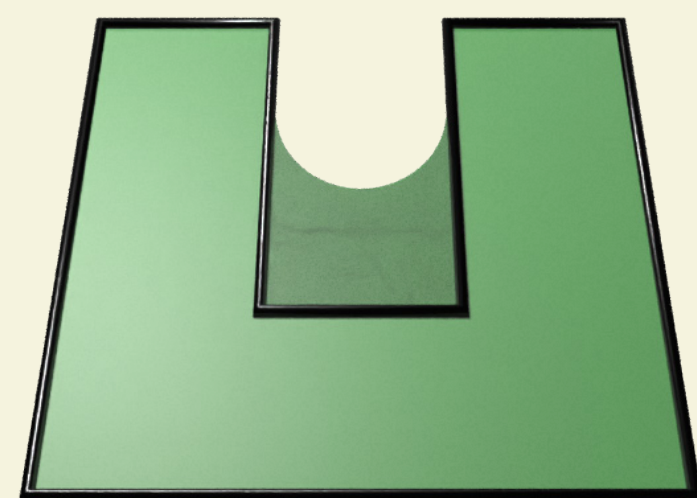
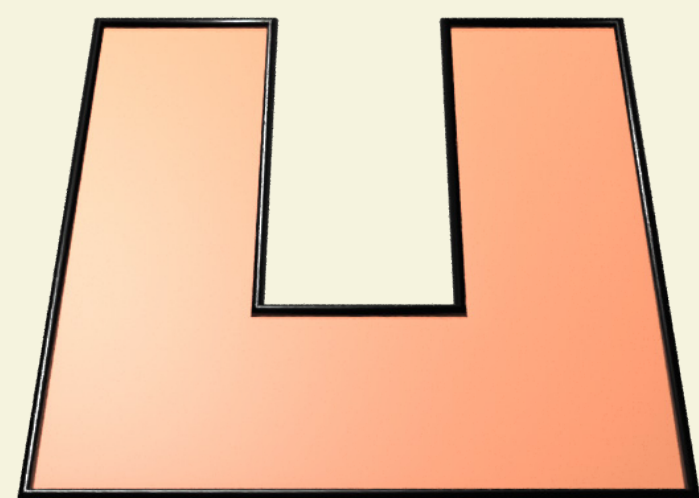
*virtual vertex*

*minimal surface*

*nonplanar polygon*



*nonconvex polygon*



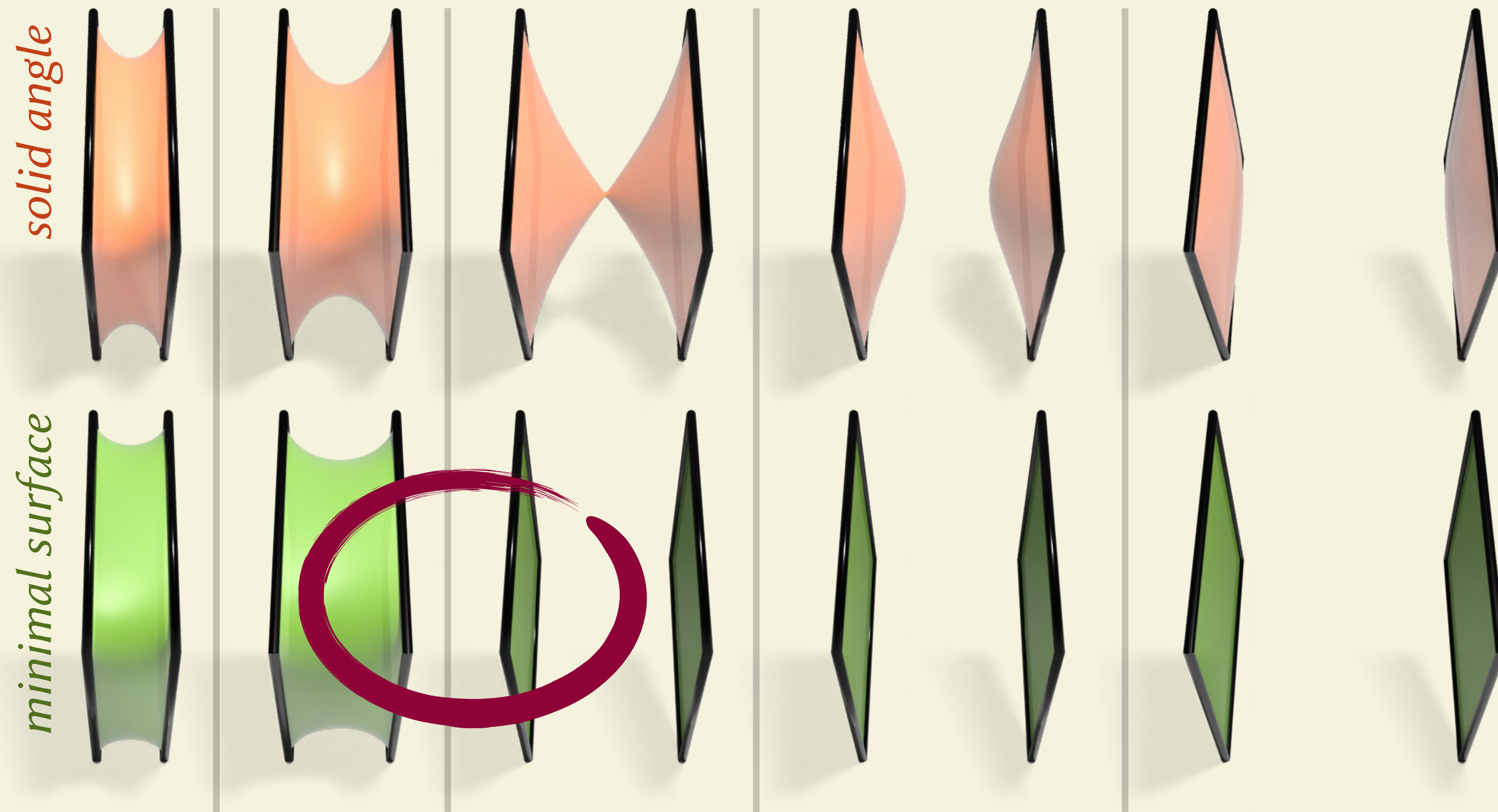
[ Floater 2003 ]

[ Joshi et al. 2007 ]

[ Catmull & Clark 1978 ]

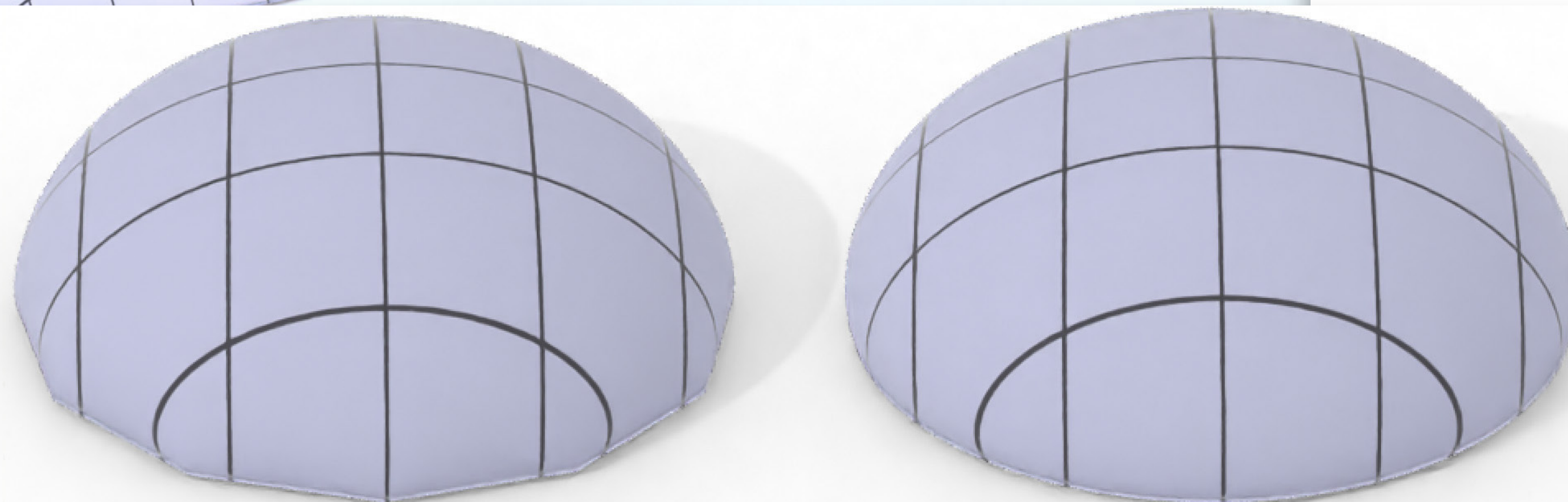
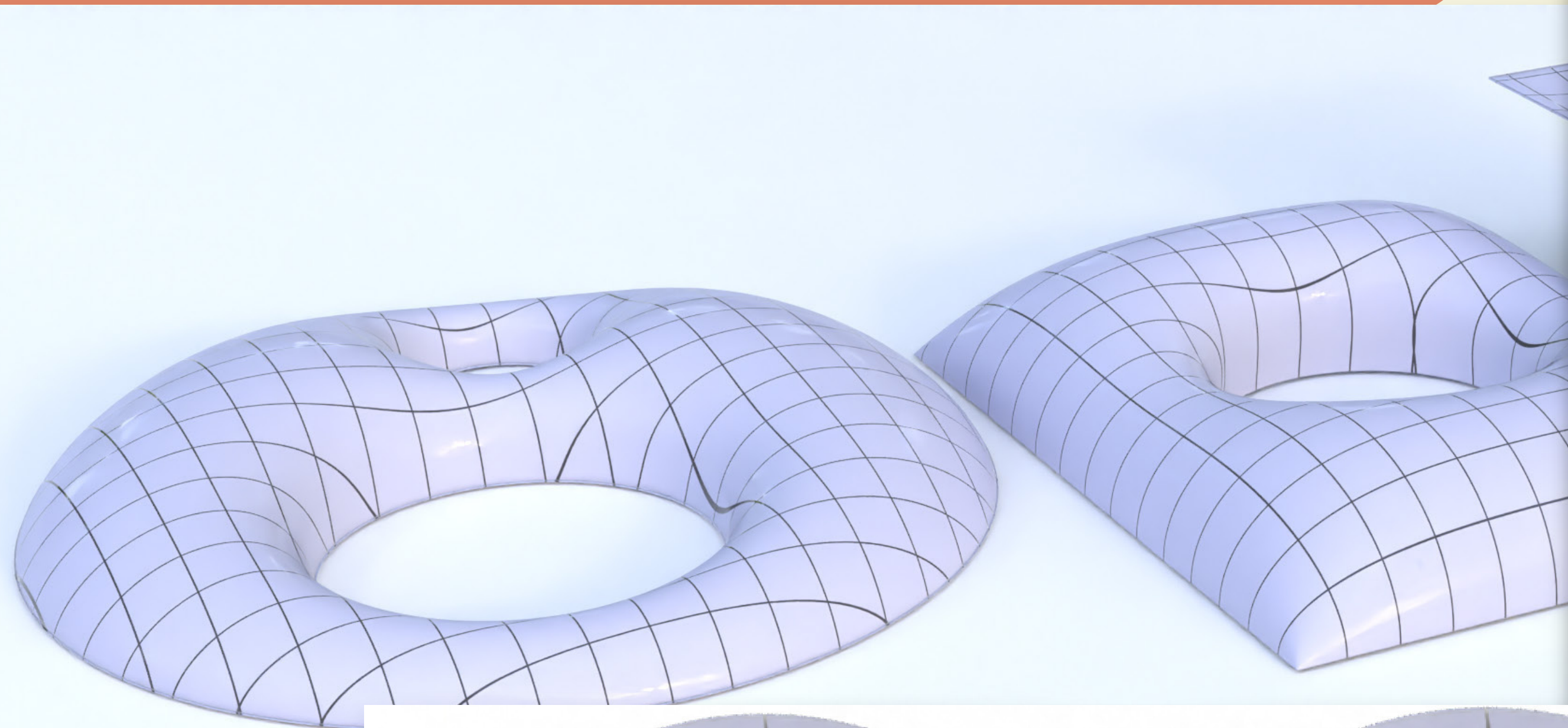
[ Bunge et al. 2020 ]

# Continuous interpolation



# Architectural grid shells

[ Adiels et al. 2022 ]

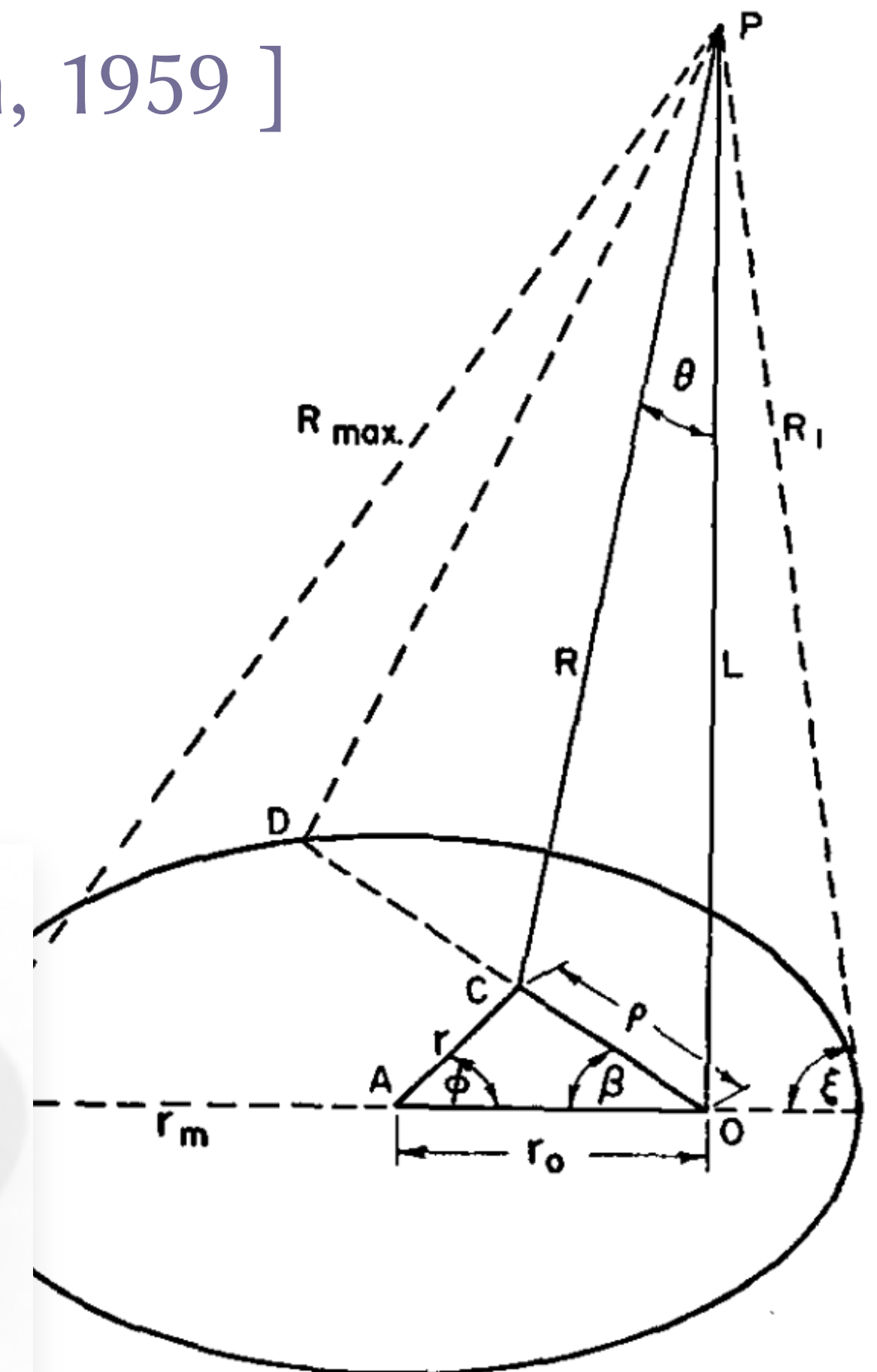


12-sided polygon

circle

$$\Omega = 2\pi - \frac{2L}{R_{\max}} K(k) - \pi \Lambda_0(\xi, k) \quad (19)$$

[ Paxton, 1959 ]



angle subtended at points over the interior or periphery of disk ( $r_0 \leq r_m$ ).

Architectural geometry is the application of geometry to those with curved surfaces like shells and grid shells that carry load mainly through membrane action, making them different from the trusses and beams used today. The complex geometry, combined with modern production, spatial and aesthetic aspects, makes this a new field. Early treatises in architectural geometry include Vitruvius (1512-1570), examining the art of cutting stones in vaults and applications from the field of differential geometry have experimented with various shapes to balance requirements. Other examples include Weingarten surfaces [7], such as minimal surfaces. Additional techniques include form finding [8]

Emil Adiels  
Chalmers University of Technology, Sweden, e-mail: [emil.adiels@chalmers.se](mailto:emil.adiels@chalmers.se)  
Mats Ander  
Chalmers University of Technology, Sweden, e-mail: [mats.ander@chalmers.se](mailto:mats.ander@chalmers.se)  
Chris J. K. Williams  
Chalmers University of Technology, Sweden, e-mail: [christopher.williams@chalmers.se](mailto:christopher.williams@chalmers.se)

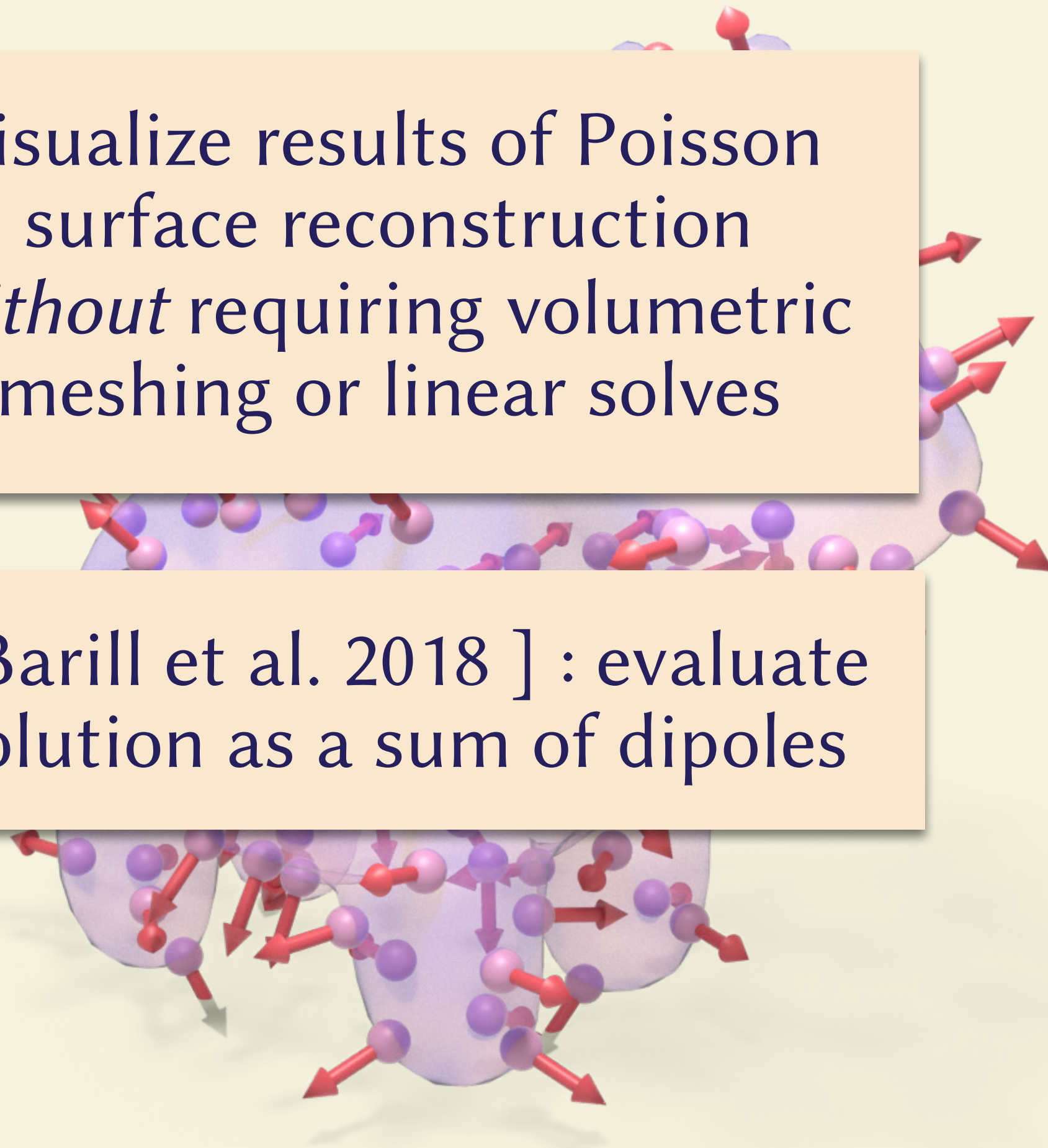


# Surface reconstruction

[ Kazhdan et al. 2006 ]

visualize results of Poisson  
surface reconstruction  
*without* requiring volumetric  
meshing or linear solves

[ Barill et al. 2018 ] : evaluate  
solution as a sum of dipoles



( directly ray traced )

# Riemann surfaces

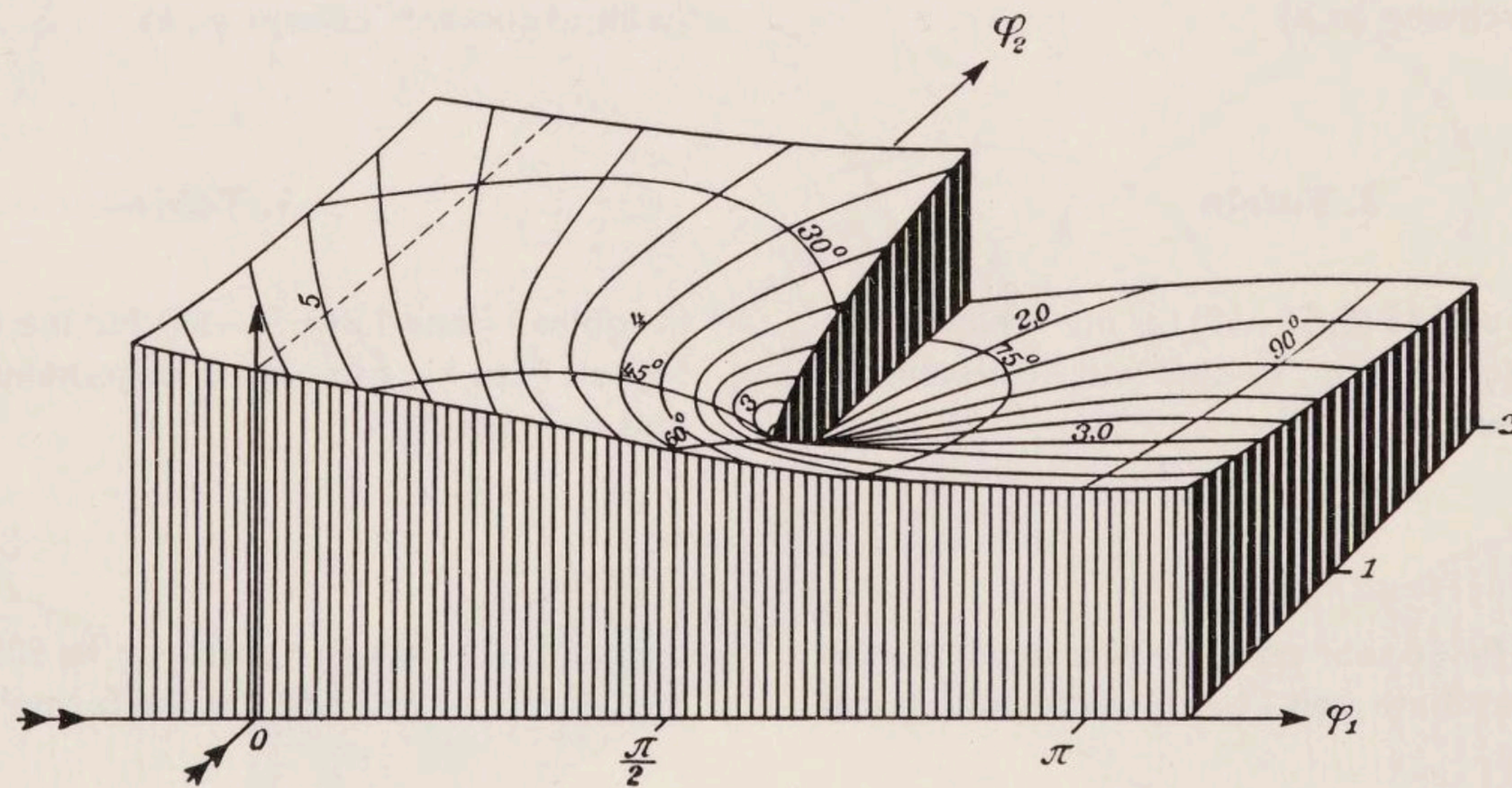
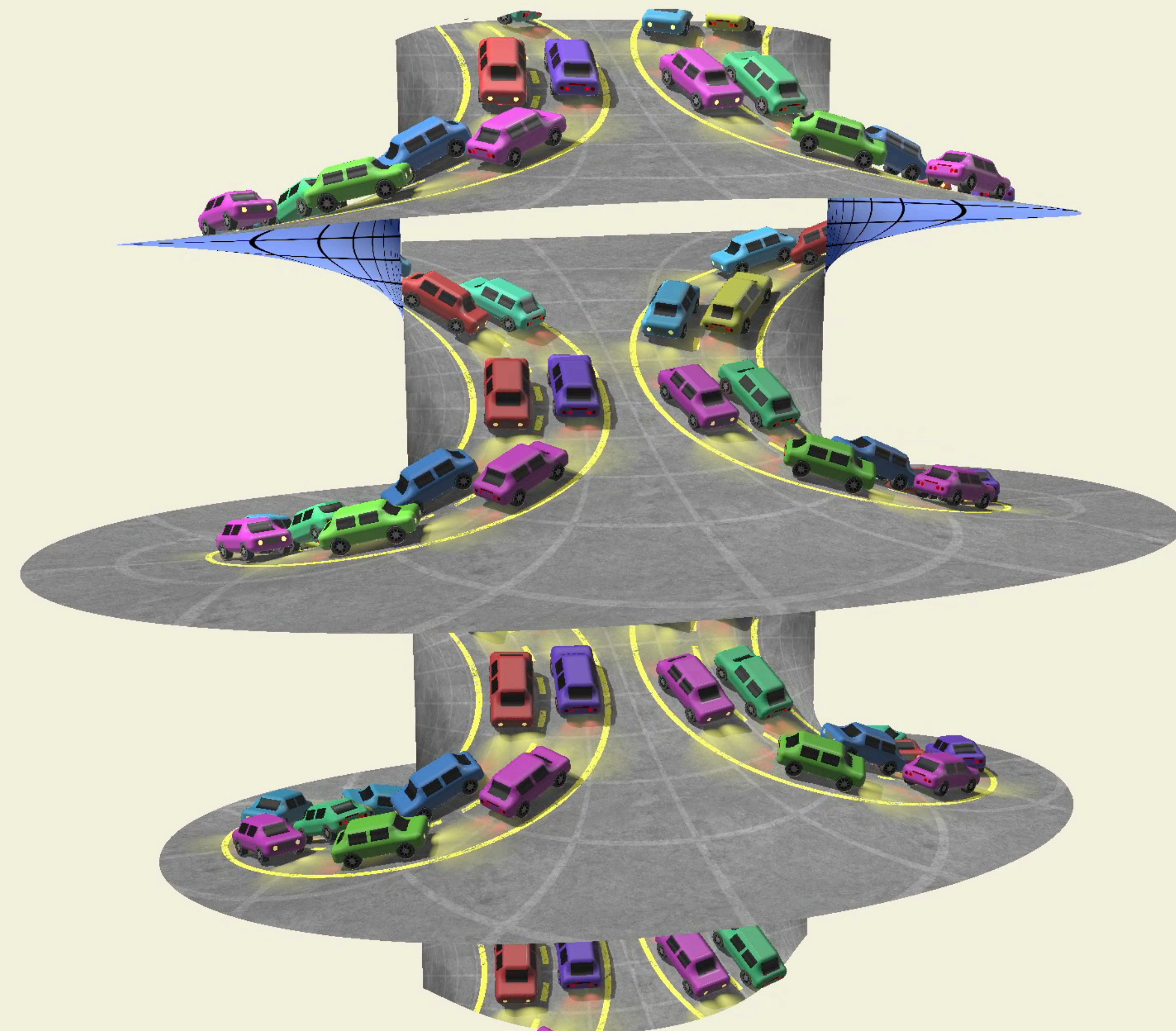


Fig. 24 Relief des 2. Zweiges der Funktion  $F(\varphi, k)$  mit  $k = 0,8$ . ( $\varphi = \varphi_1 + i\varphi_2$ )

Fig. 24 Relief of the 2<sup>nd</sup> branch of the function  $F(\varphi, k)$  with  $k = 0,8$ . ( $\varphi = \varphi_1 + i\varphi_2$ )



( directly ray traced ) 66

[ Jahnke, Emde & Lösch 1960 ]

# Riemann surfaces as graphs

graph

$$z = f(x, y)$$



( directly ray traced ) 67

# Riemann surfaces as graphs

$$f(x, y) - z = 0$$

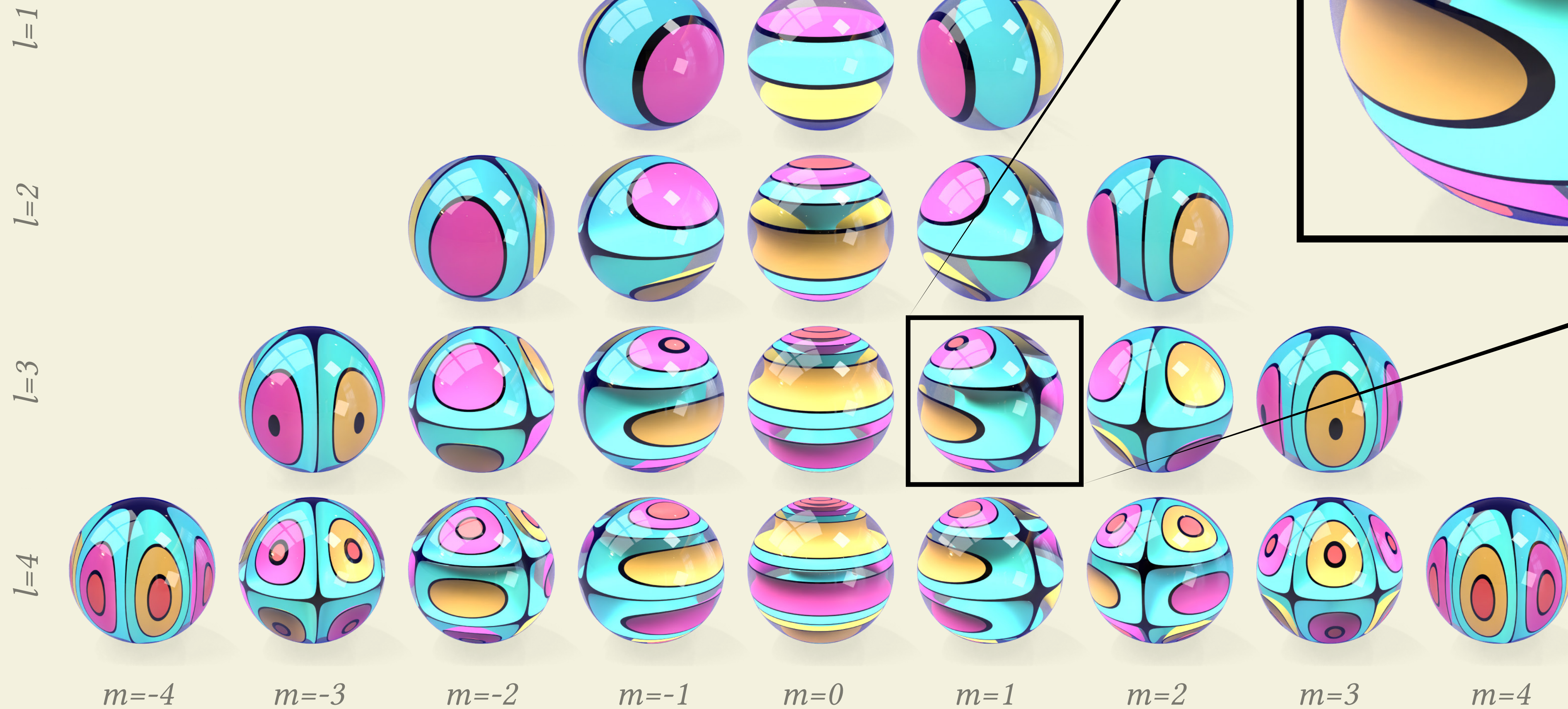
level set



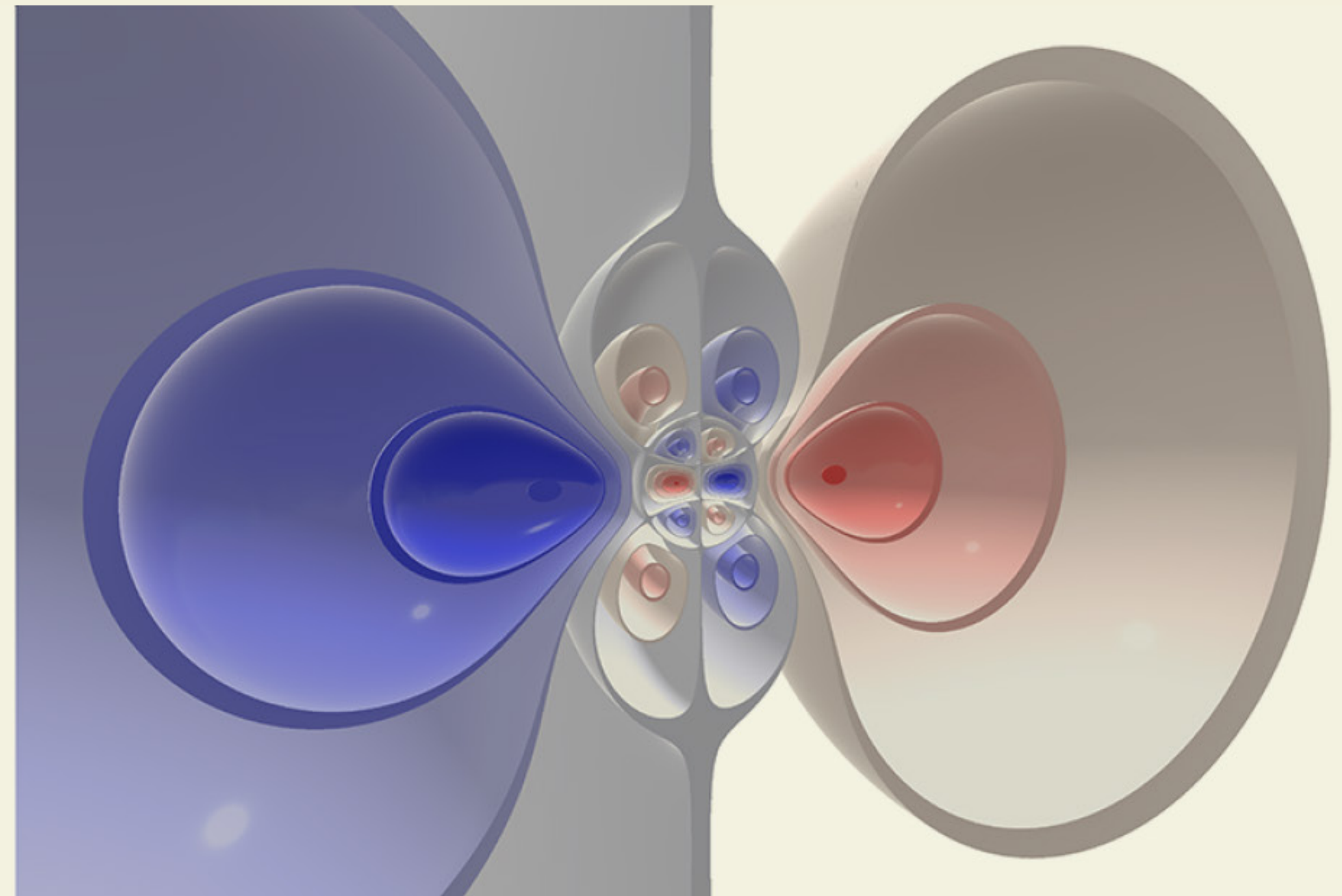
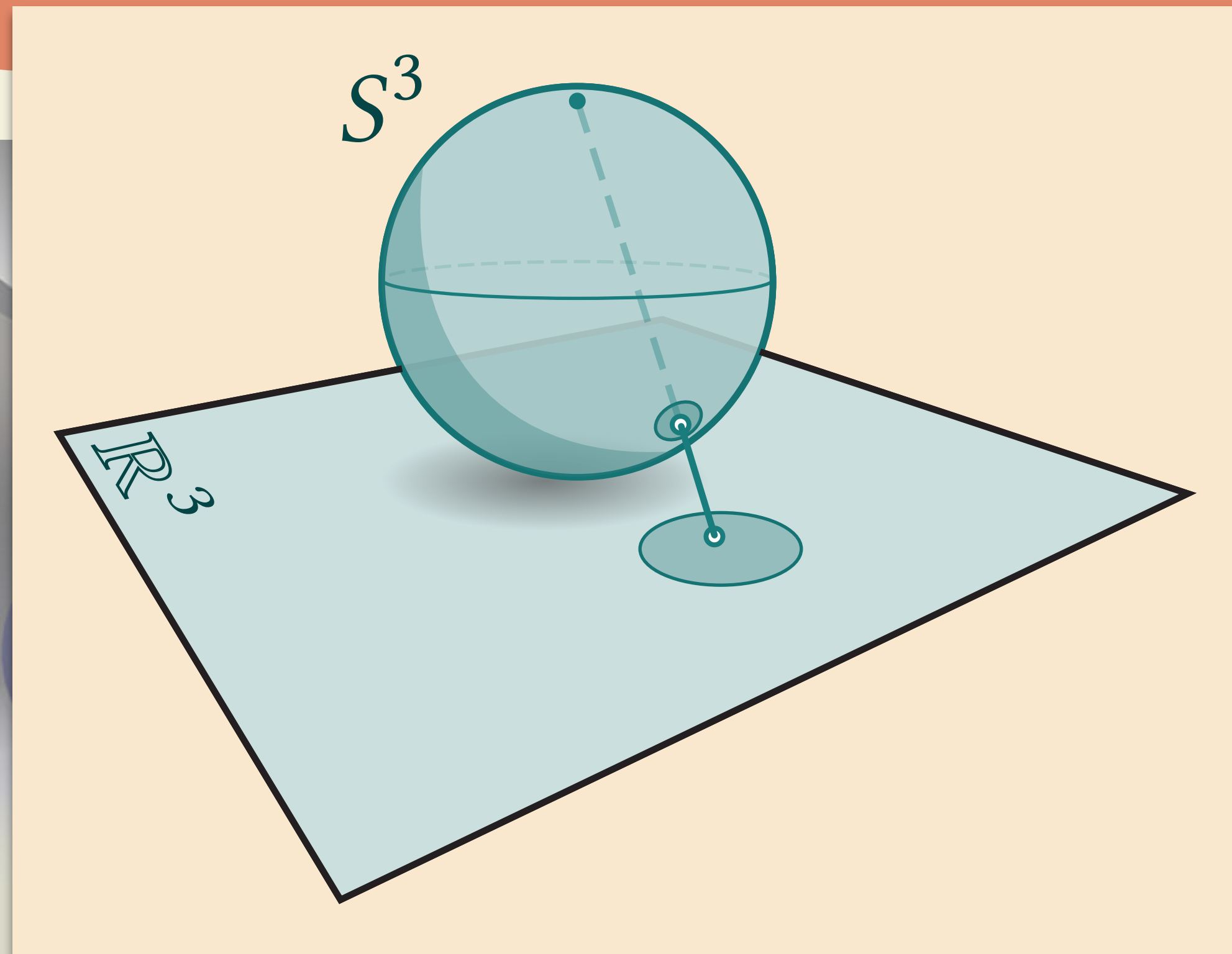
( directly ray traced ) 68

# Spherical harmonics

( directly ray traced )



# Hyperspherical harmonics



$$f(x, y, z, w) = y^3 - 3yz^2$$

$$f(x, y, z, w) = x^3 y + xy^3 - 3xyw^2 - 3xyz^2$$

( directly ray traced ) 70

# The gyroid

[ Diegel 2021 ]

**not** a harmonic function in 3D  
... but is a *slice* of a harmonic function in 4D

Metal AM heat exchanger design workflow

| contents | news | ever

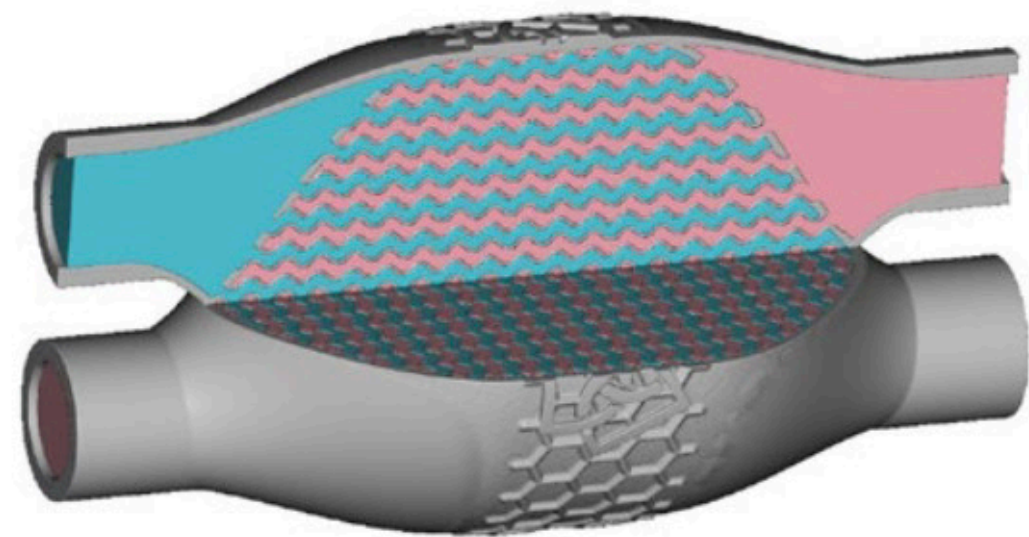
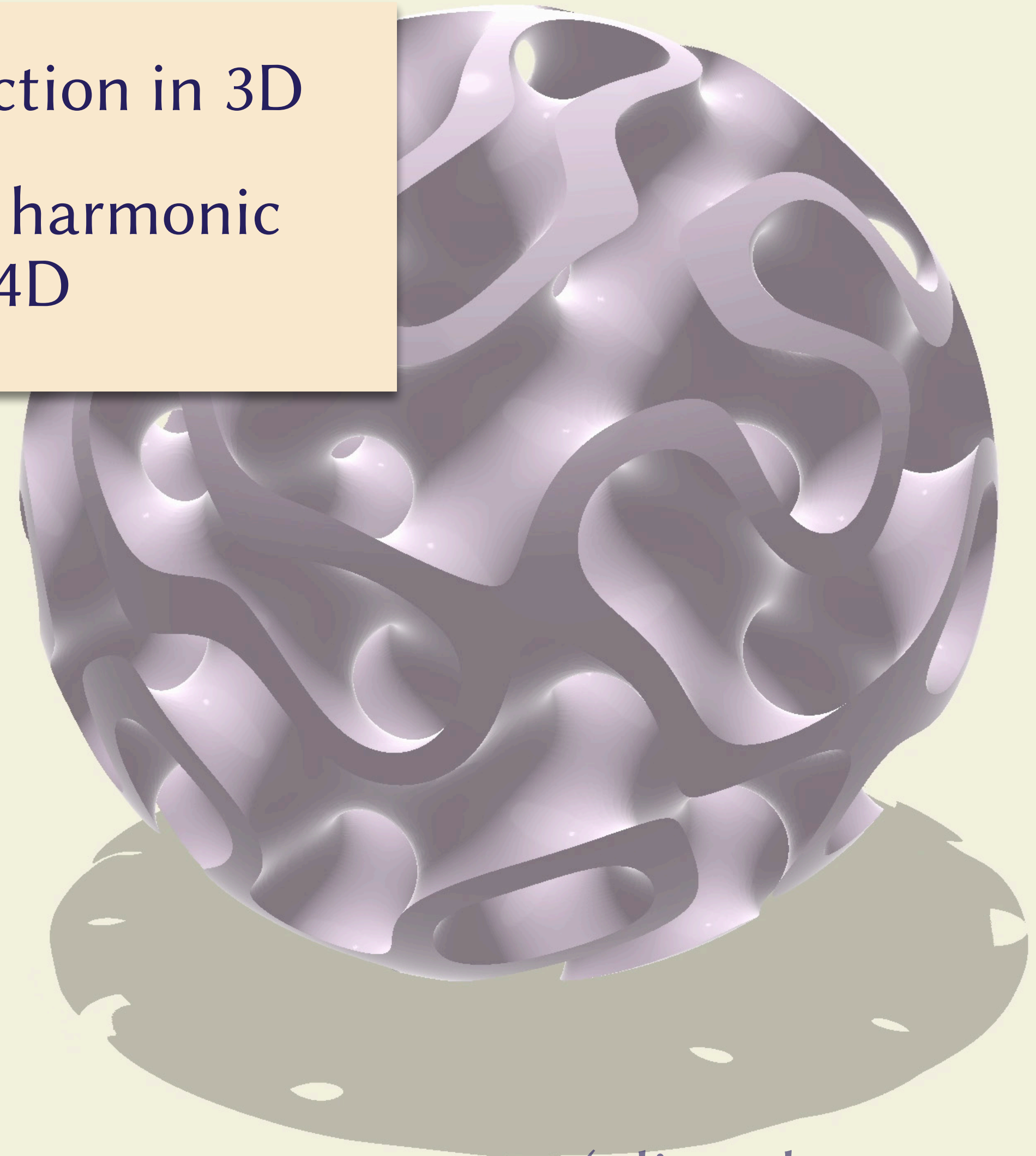
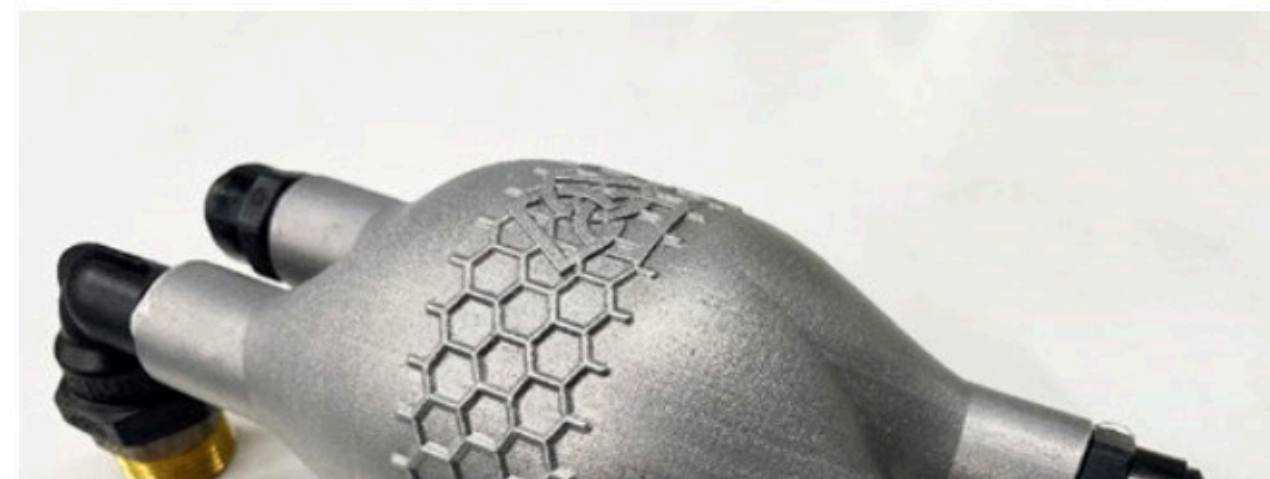


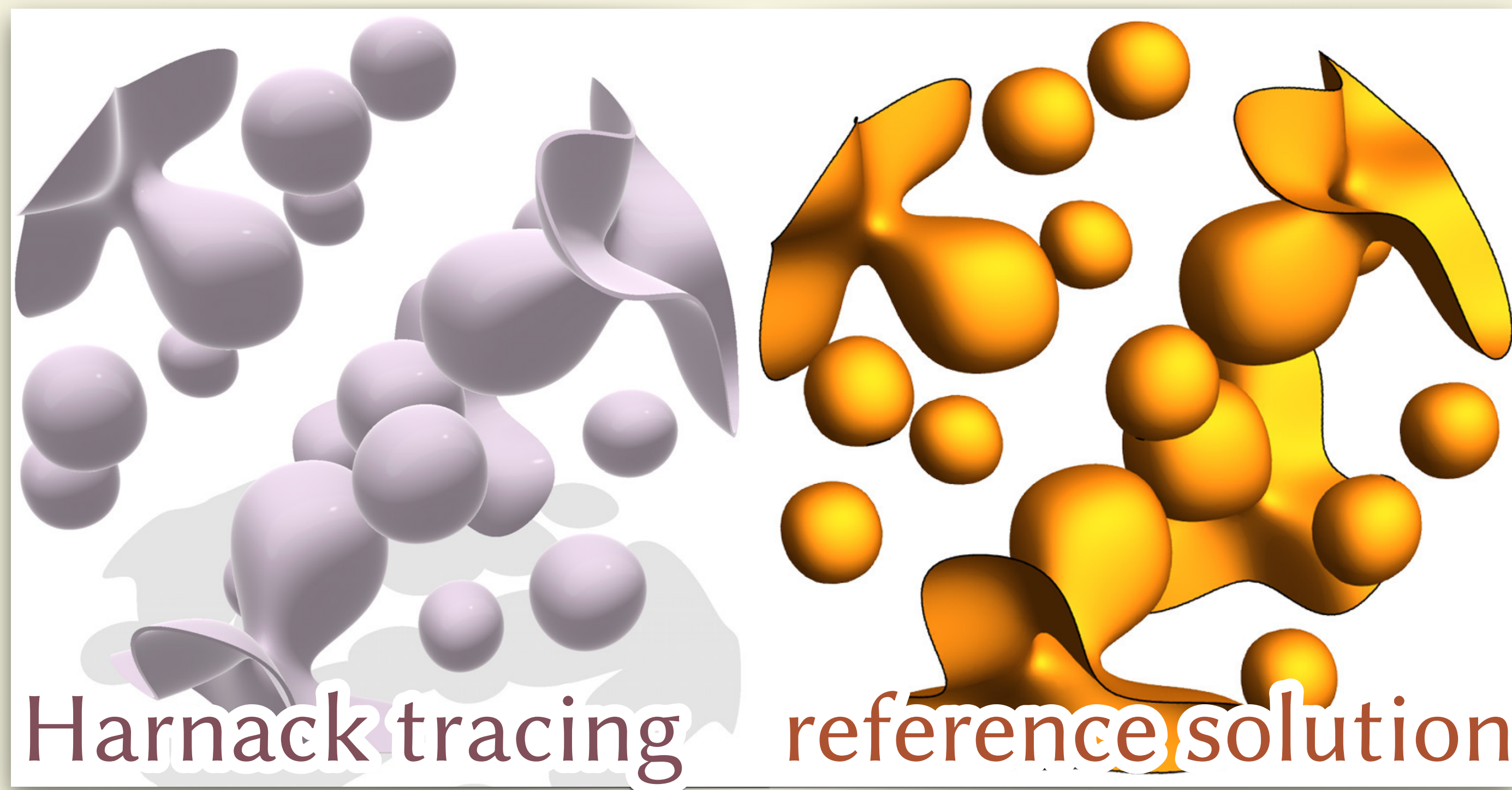
Fig. 6 Section view of completed heat exchanger, including hot and cold fluid zones (left), and the printed part showing minimal support material requirements (right).



( directly ray traced ) 71

# Laplacian Eigenfunctions

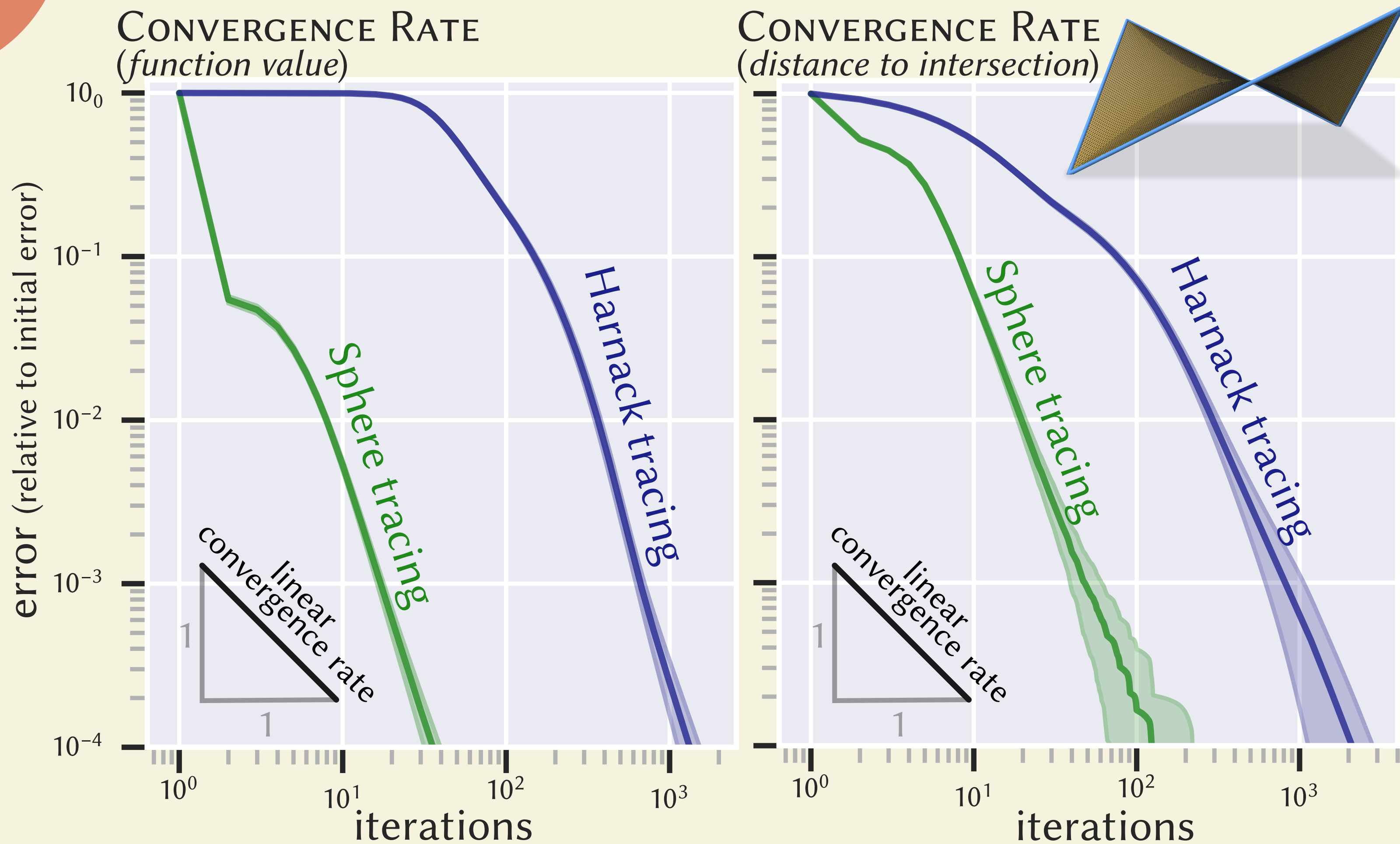
$$\Delta_{\mathbb{R}^3} \phi(x, y, z) = \lambda \phi(x, y, z) \implies \Delta_{\mathbb{R}^4} \left( e^{w\sqrt{-\lambda}} \phi(x, y, z) \right) = 0$$





# Convergence

Same asymptotic rate  
as sphere tracing



# IV. Future Work

# Subharmonic functions

harmonic:  $\Delta f = 0$

subharmonic:  $\Delta f \leq 0$

less than the harmonic function  
with the same boundary values

obeys *upper* bounds on  
harmonic functions

superharmonic:  $\Delta f \geq 0$

greater than the harmonic function  
with the same boundary values

obeys *lower* bounds on  
harmonic functions

Can we apply Harnack tracing?

Warning: this slide uses the *positive-semidefinite Laplacian* where  $\Delta f = - \sum_i \frac{\partial^2}{\partial x_i^2} f$

# Functions with bounded Laplacian

if  $|\Delta f| \leq \lambda$ , then  $f(x) - \frac{\lambda}{2d} \|x\|_{\mathbb{R}^d}^2$  is superharmonic

and  $f(x) + \frac{\lambda}{2d} \|x\|_{\mathbb{R}^d}^2$  is subharmonic

Warning: this slide uses the *positive-semidefinite Laplacian* where  $\Delta f = -\sum_i \frac{\partial^2}{\partial x_i^2} f$

# Harnack tracing for other PDEs

Harnack inequalities exist for many PDEs

But positivity becomes harder to enforce!

# Optimization

## Signed Distance Functions

eikonal condition  $\|\nabla f\| = 1$

✗ nonconvex, nondifferentiable

✗ insufficient to ensure  $f$  is an SDF

[ Xie et al. 2022, Marschner et al. 2023 ]

## Harmonic Functions

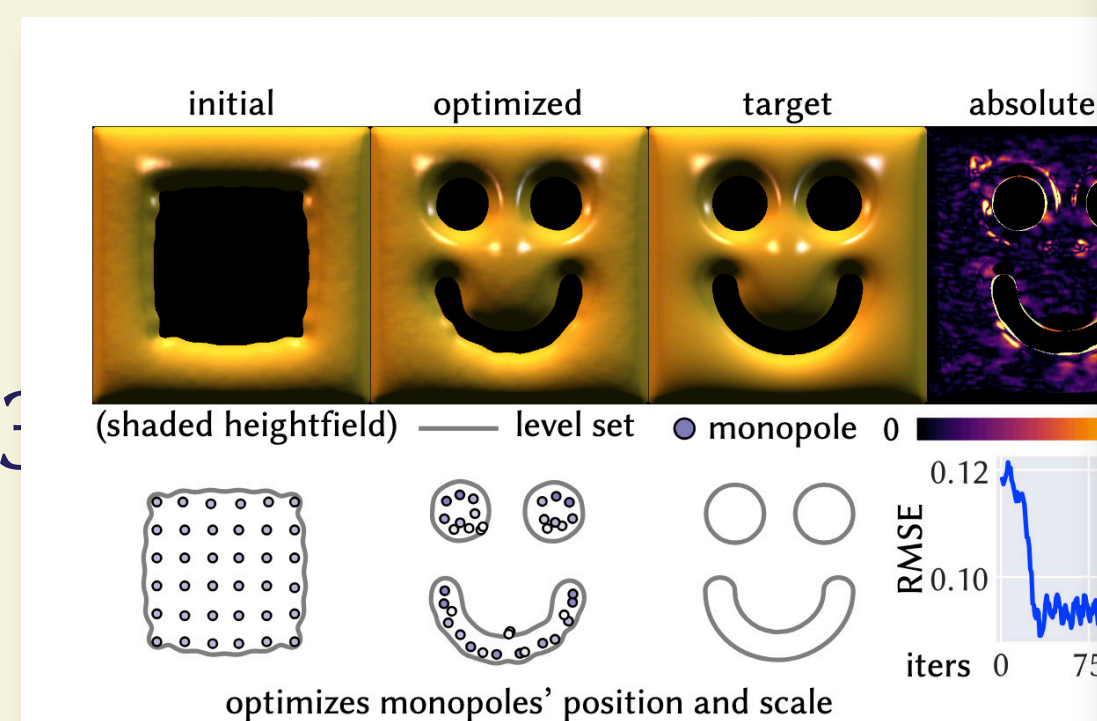


Figure 12. Baran and Lehtinen [2009] define smooth heightfields domain as the solution to a Poisson equation with zero Dirichlet boundary conditions. We optimize an implicit boundary, specifically the position and scale of harmonic radial basis functions, to achieve a target heightfield visualized as a shaded surface (top row).

[ Miller et al. 2024 ]

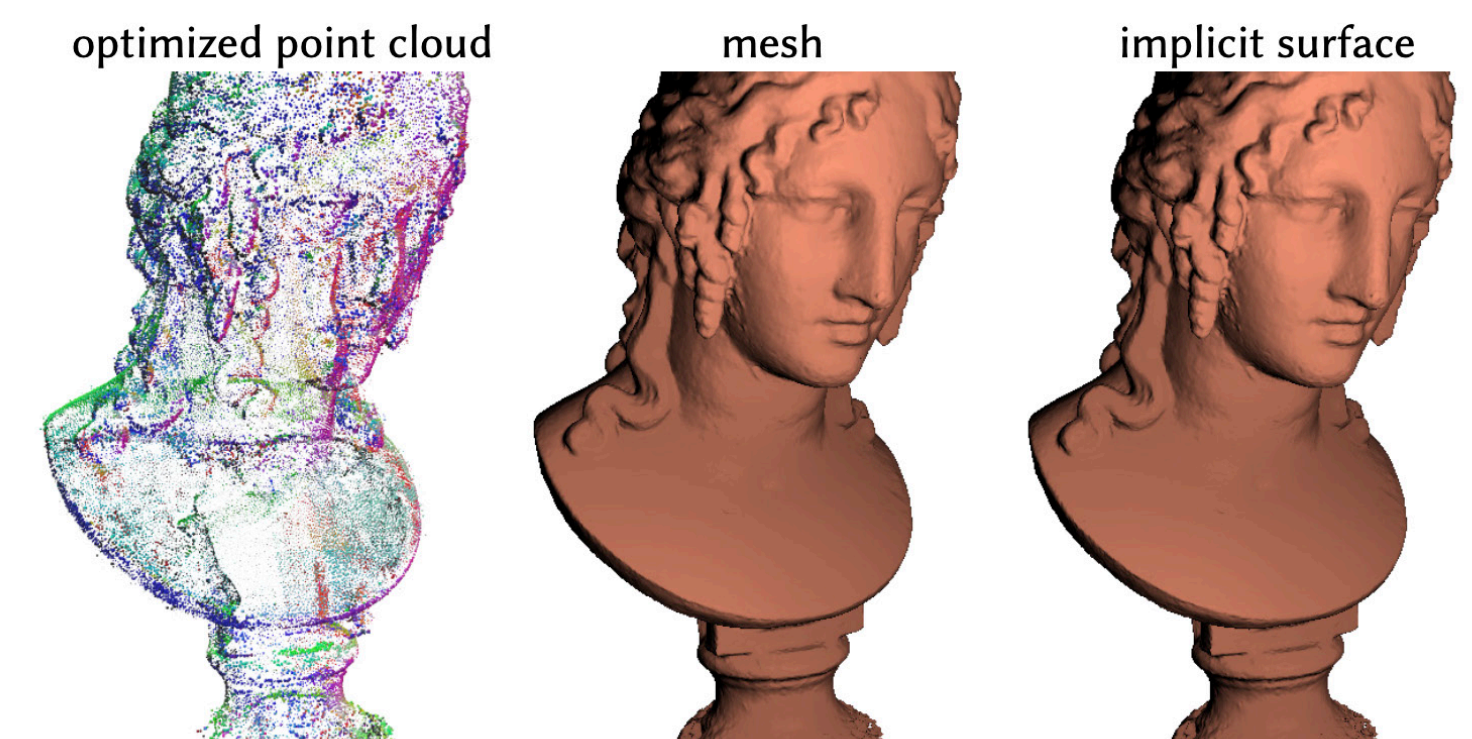
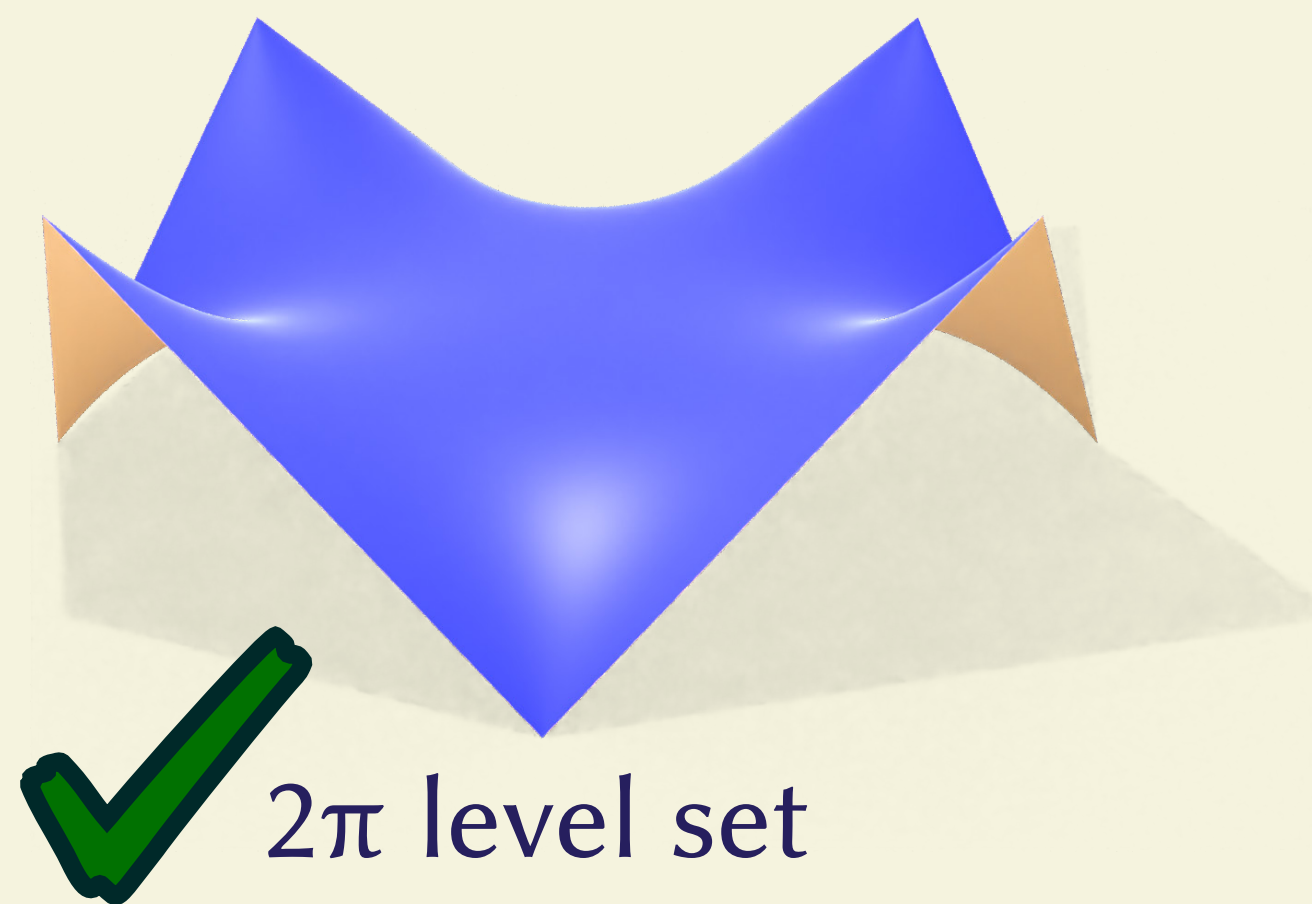


Figure 5. Our regularized dipole sum representation allows us to directly ray trace the optimized point cloud (where we use color to visualize normals, and size to visualize geometry attributes), achieving the same results as ray tracing a mesh without the need to extract one.

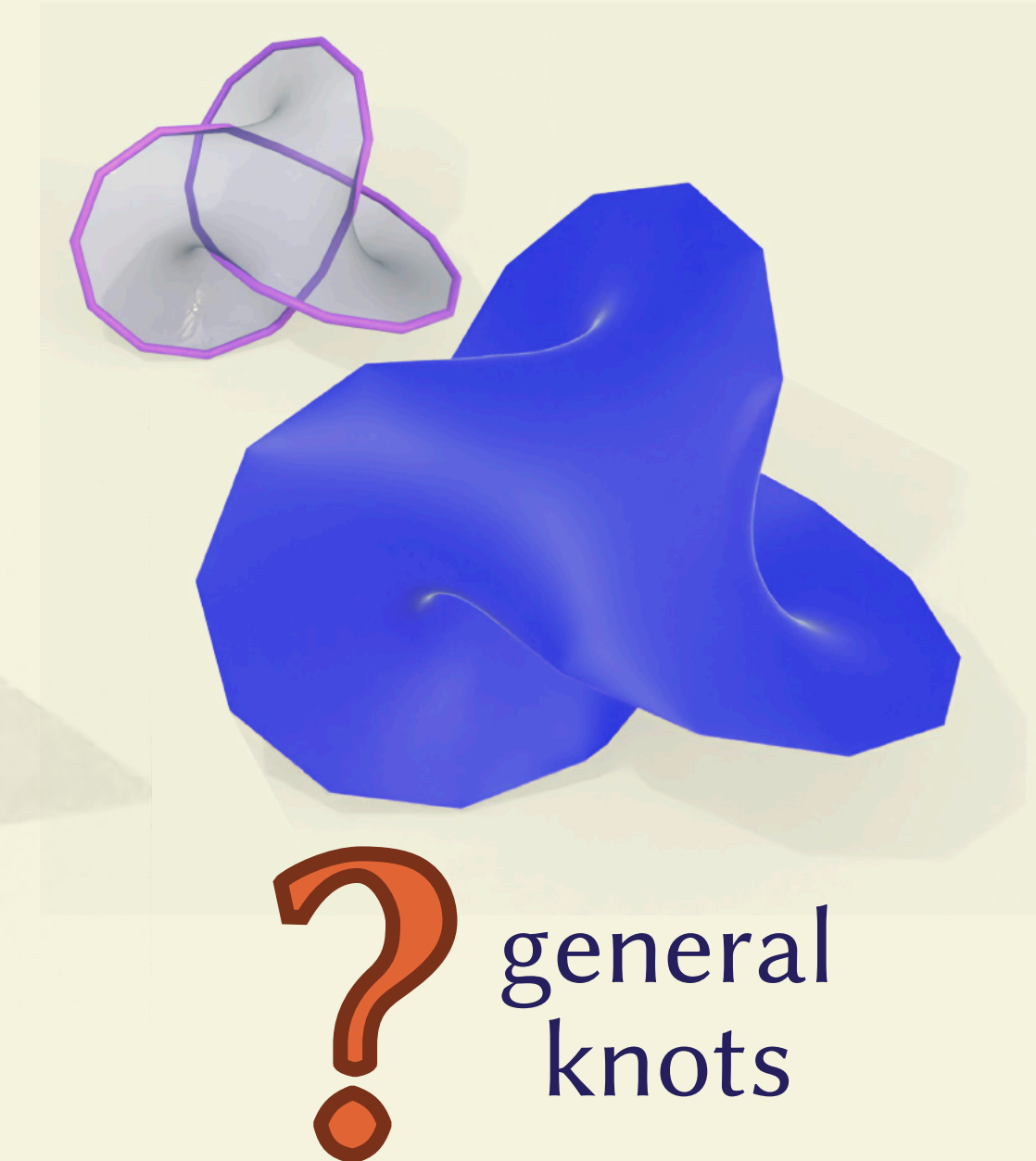
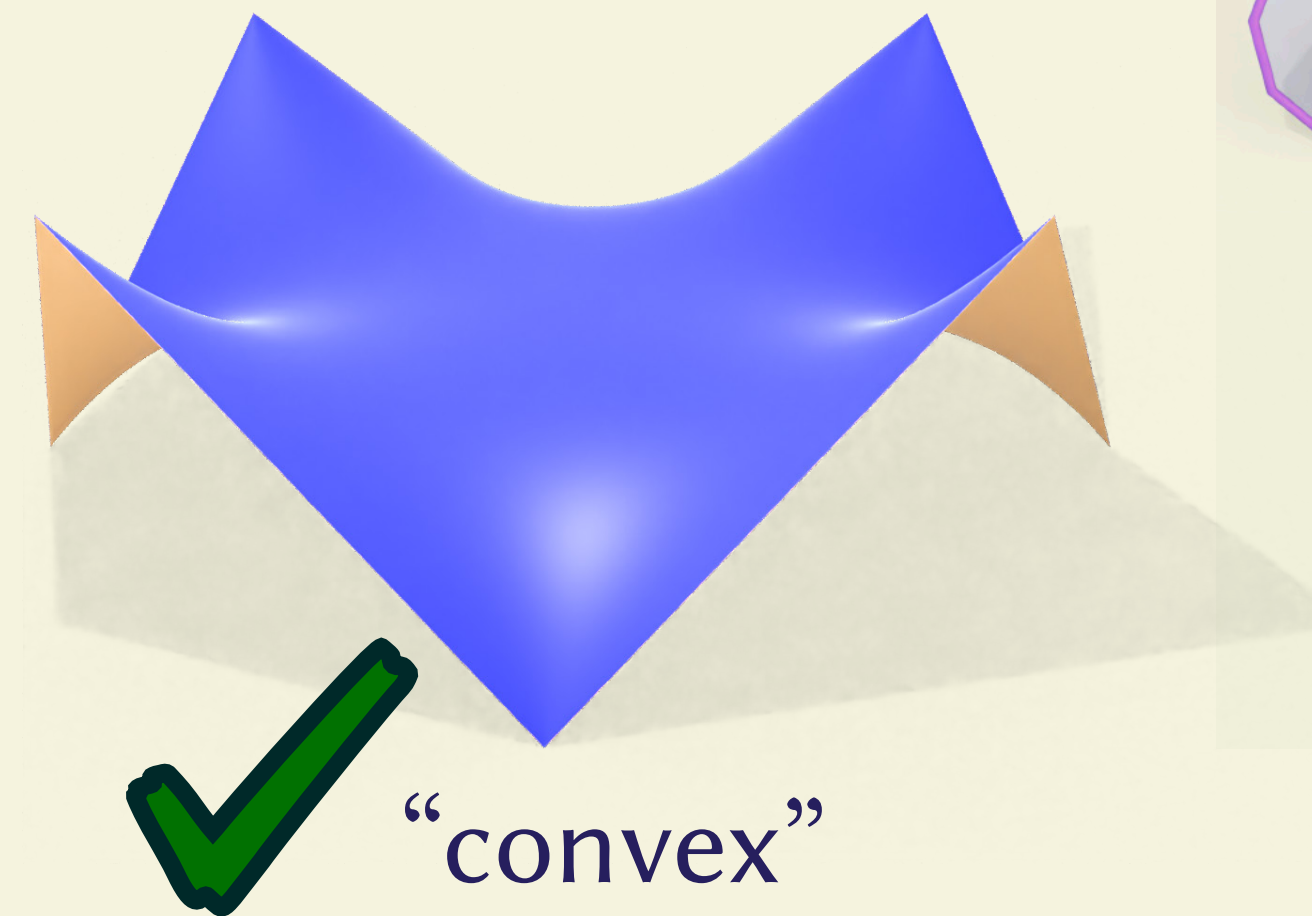
[ Chen et al. 2024 ]

# Solid angle bounds

spatial extent of level sets

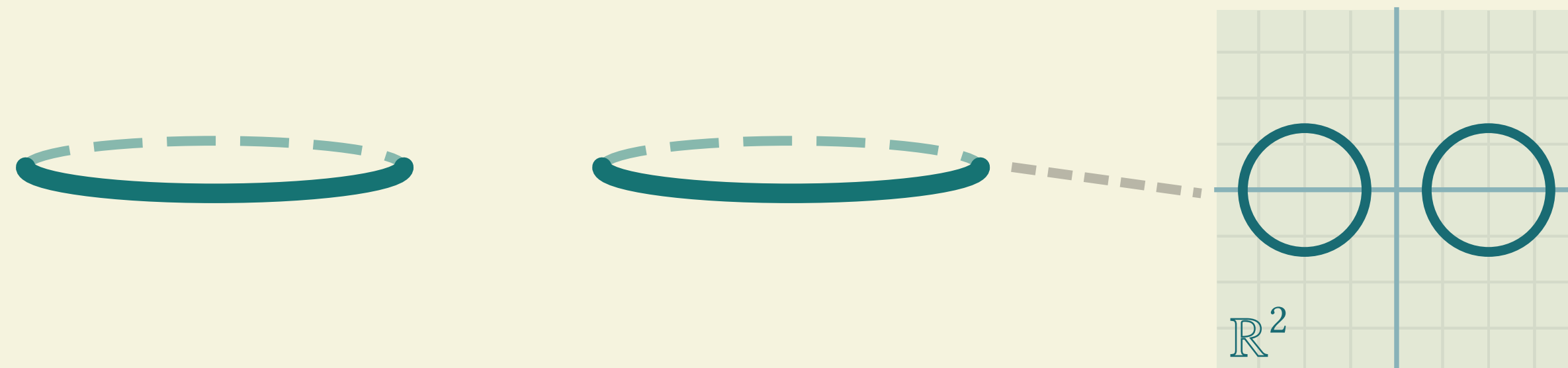


function value



# Solid angle in 4D

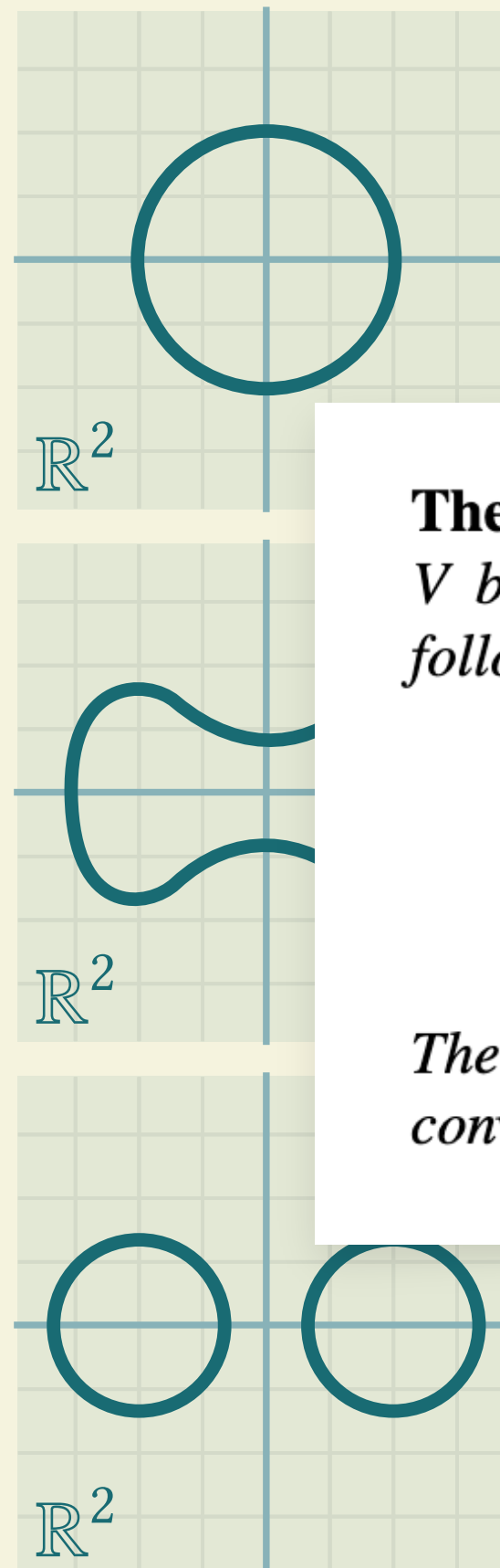
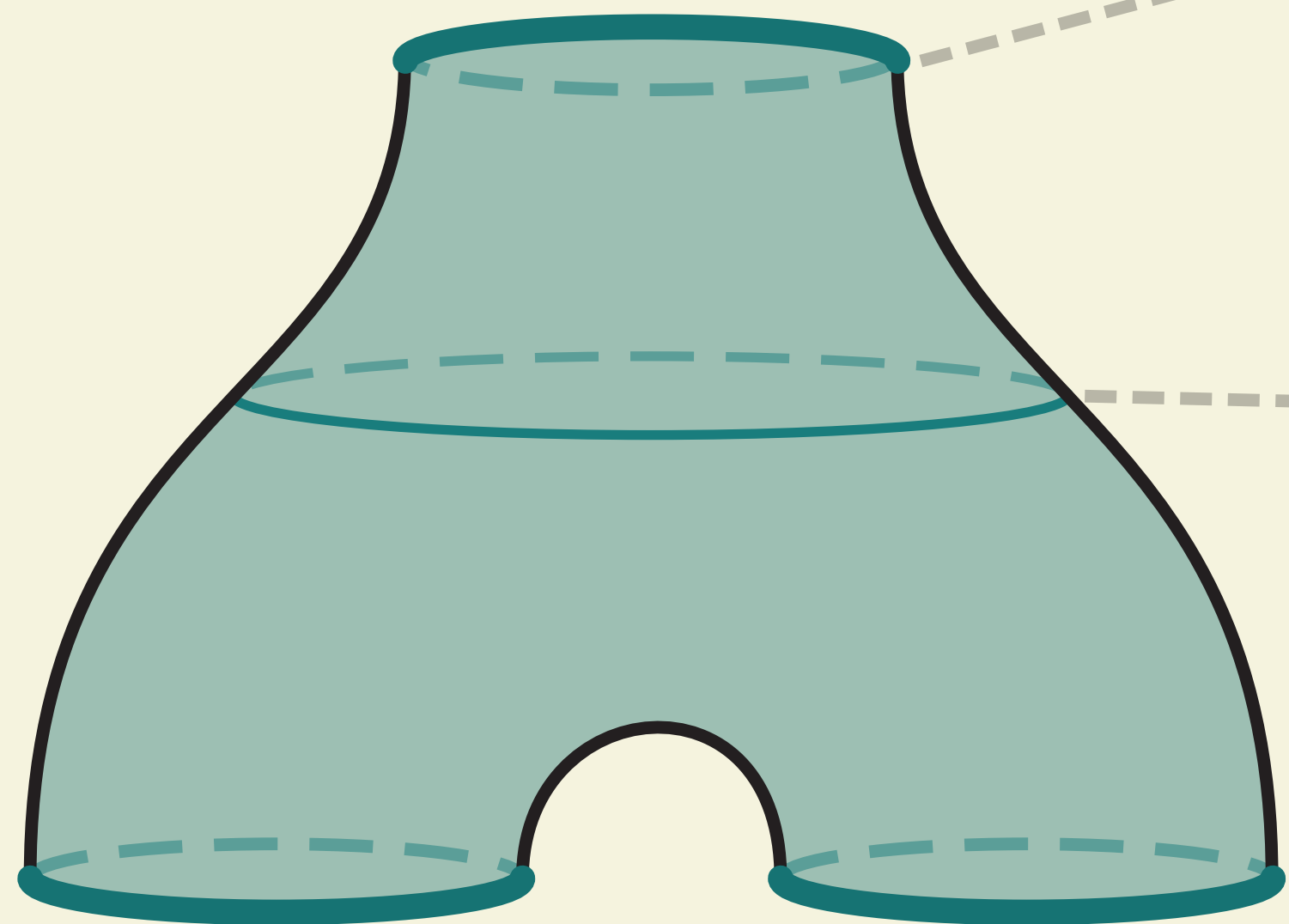
Shape interpolation via solid angle





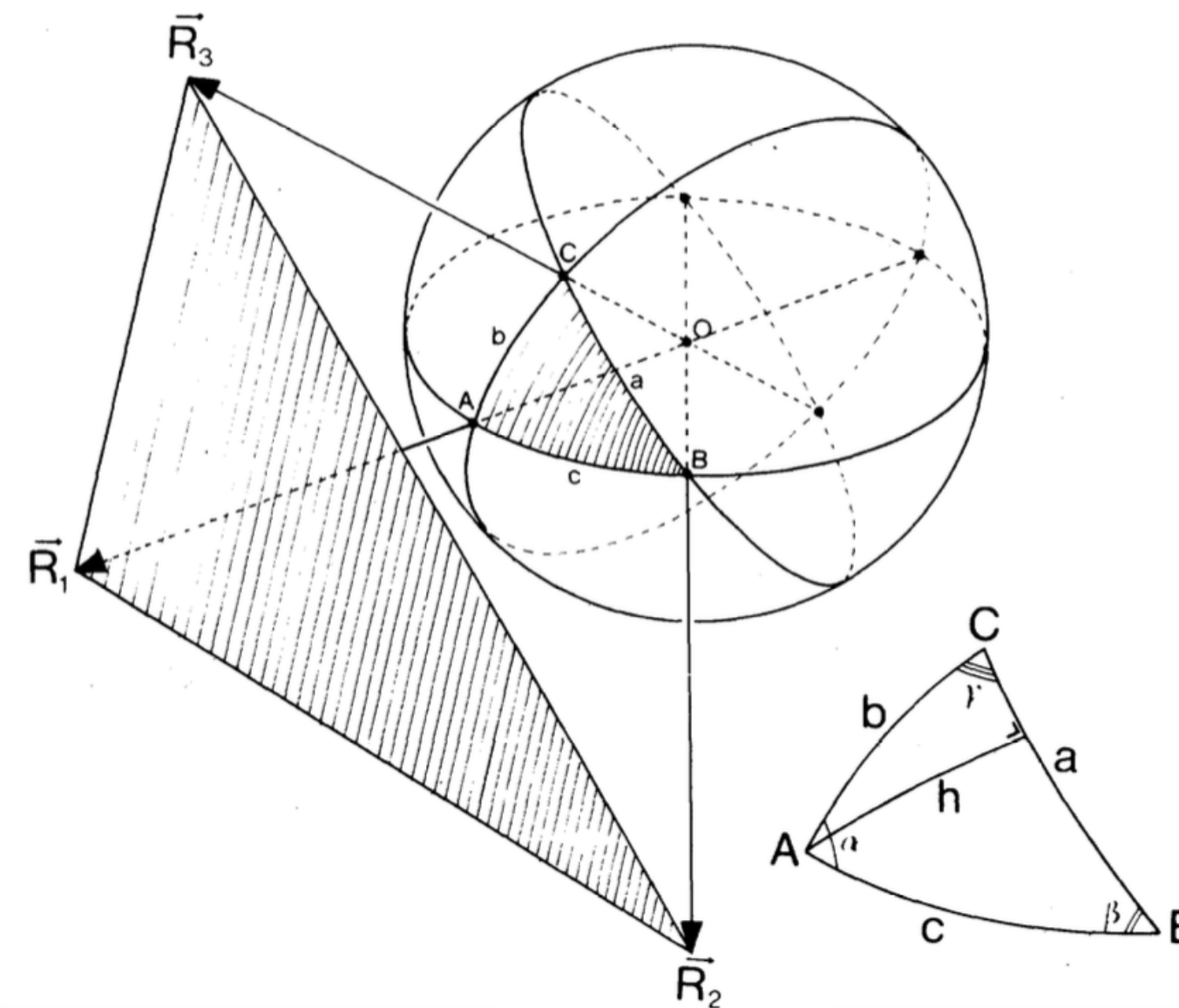
# Solid angle in 4D

Shape interpolation via solid angle



## 3D solid angle formula

[ van Oosterom and Strackee 1983 ]



**Theorem 2.2.** Let  $\Omega \subseteq \mathbb{R}^n$  be a solid angle spanned by unit vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ , let  $V$  be the matrix whose  $i$ th column is  $\mathbf{v}_i$ , and let  $\alpha_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$ , as above. Let  $T_\alpha$  be the following infinite multivariable Taylor series:

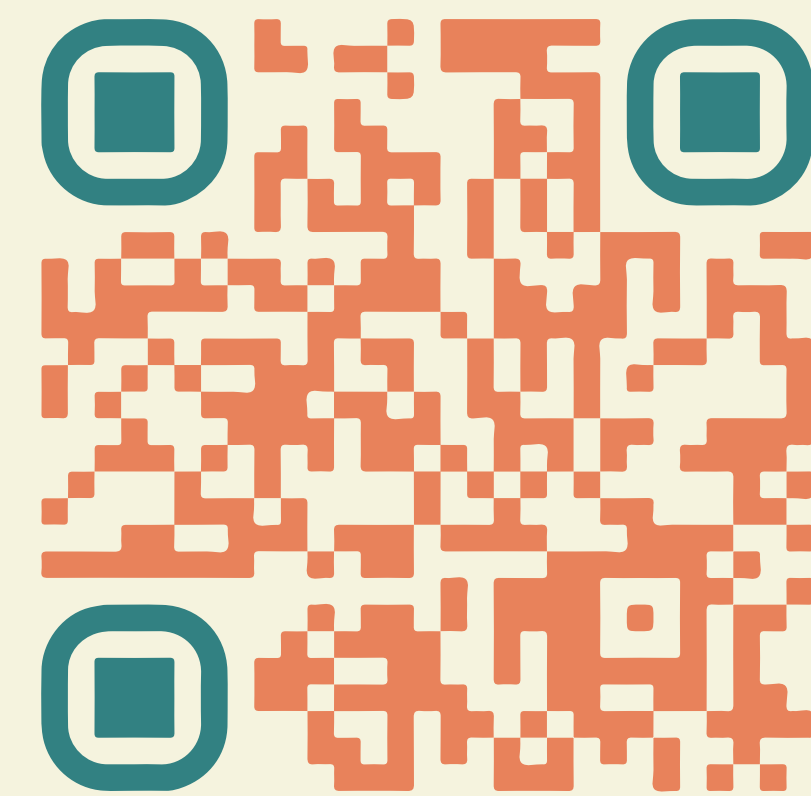
$$T_\alpha = \frac{|\det V|}{(4\pi)^{n/2}} \sum_{\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}} \left[ \frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_i \Gamma \left( \frac{1 + \sum_{m \neq i} a_{im}}{2} \right) \right] \alpha^{\mathbf{a}}.$$

The series  $T_\alpha$  agrees with  $\tilde{V}_\Omega$ , the normalized measure of solid angle  $\Omega$ , wherever  $T_\alpha$  converges.

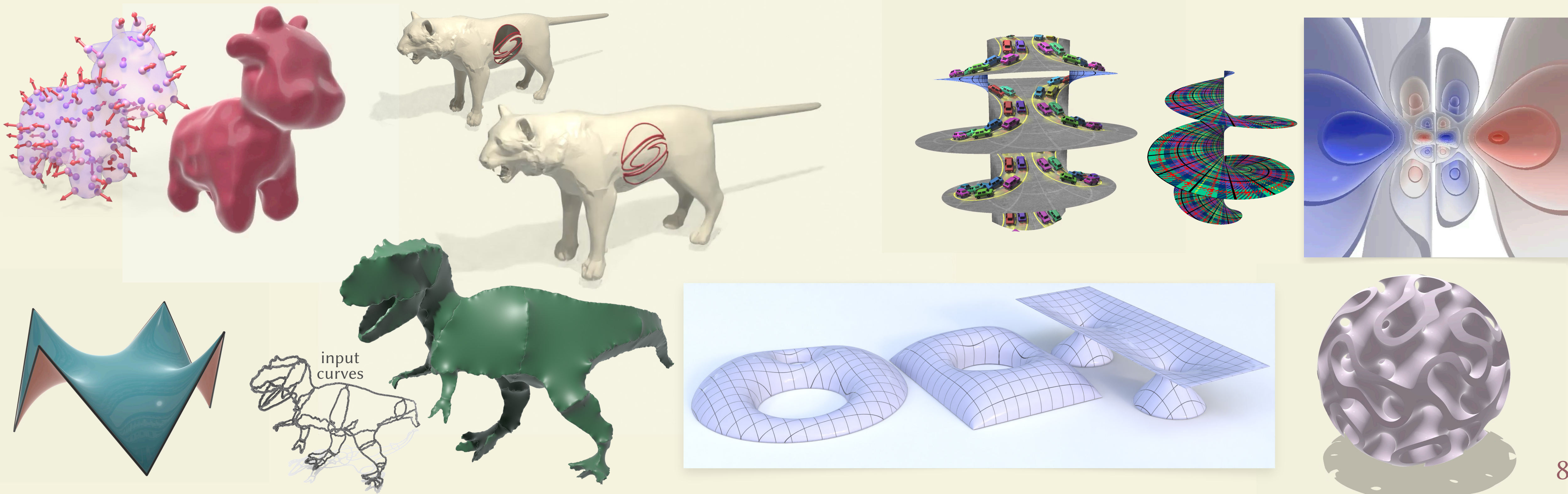
[ Ribando 2006 ]

4D solid angle formula?

# Thanks for listening

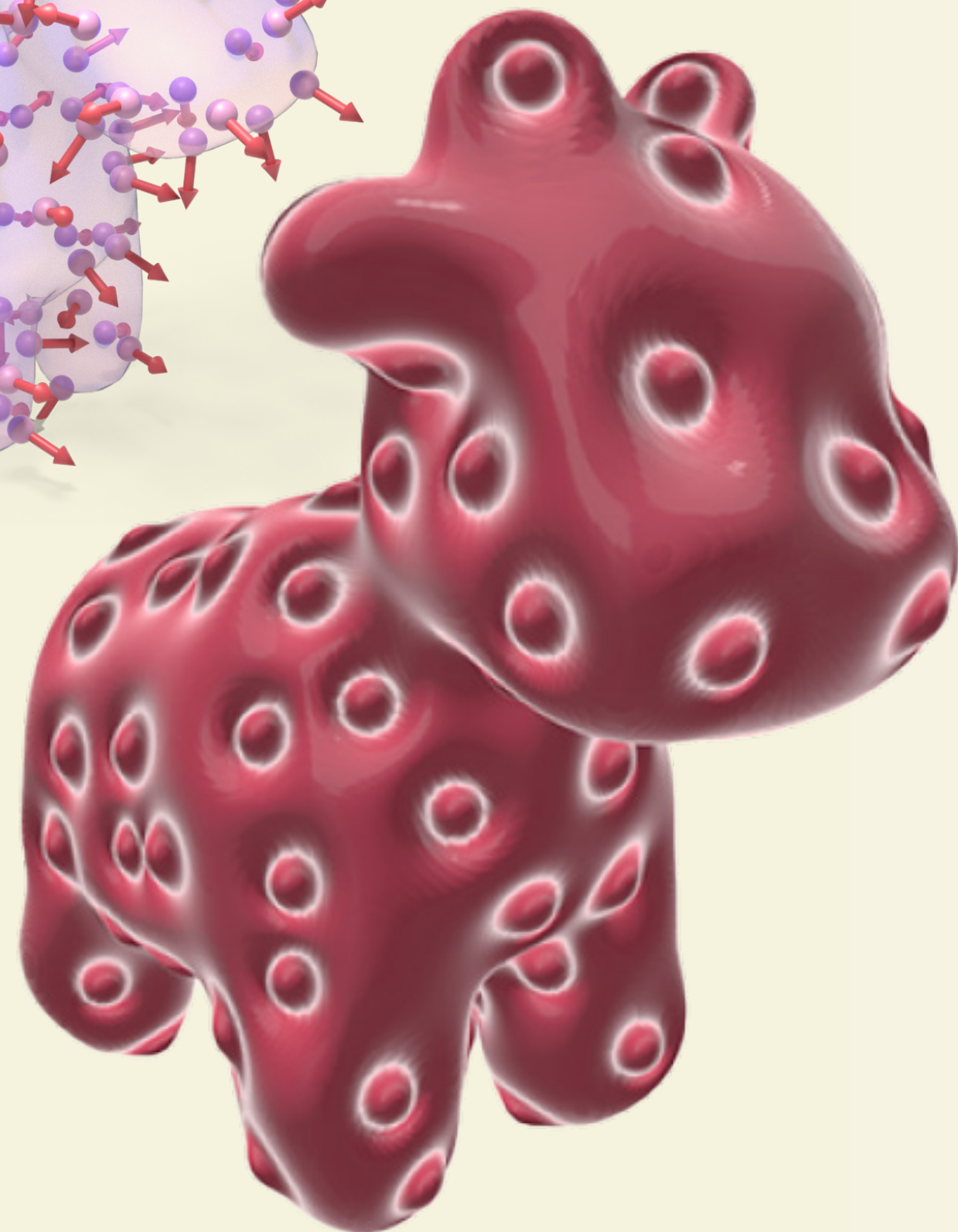
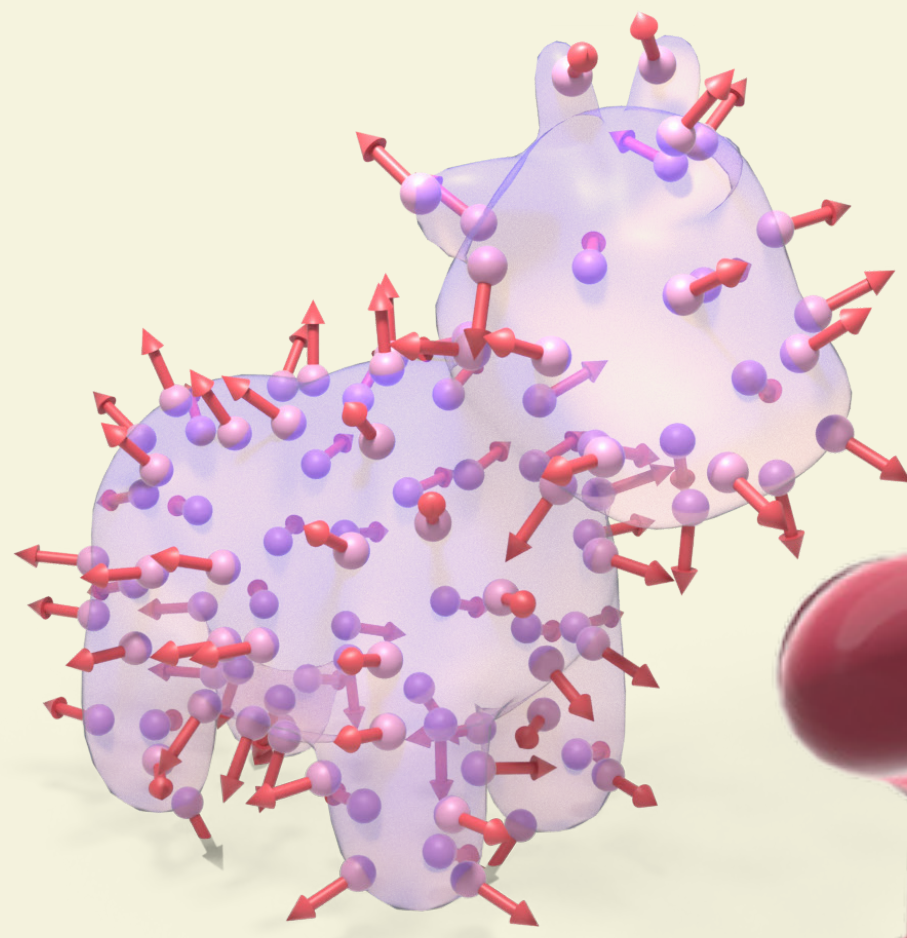


Links to Blender code and ShaderToy examples can be found at:  
[www.markjgillespie.com/Research/harnack-tracing](http://www.markjgillespie.com/Research/harnack-tracing)



# Supplemental Slides

# Poisson gradient termination artifacts



gradient termination

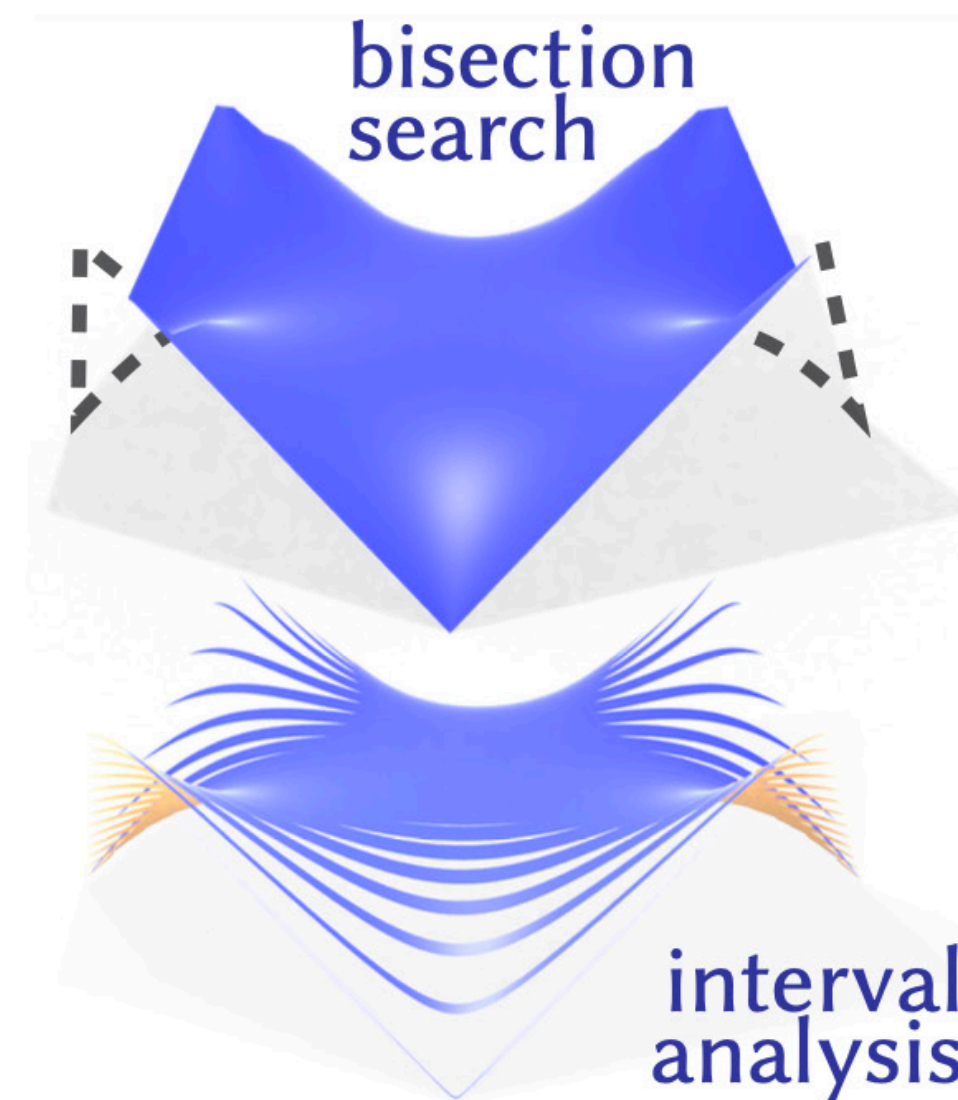


fixed termination

$$f(\vec{x}) := \sum_{i=1}^k a_i \frac{(\vec{p}_i - \vec{x}) \cdot \vec{n}_i}{\|\vec{p}_i - \vec{x}\|^3}$$
$$\nabla f(\vec{x}) = \sum_{i=1}^k a_i \left( 3 \frac{(\vec{p}_i - \vec{x}) \cdot \vec{n}_i}{\|\vec{p}_i - \vec{x}\|^5} (\vec{p}_i - \vec{x}) - \frac{1}{\|\vec{p}_i - \vec{x}\|^3} \vec{n}_i \right)$$

# Filtering out spurious intersections

Moreover, for angle-valued functions, one may detect discontinuities in  $f(\mathbf{x})$ , rather than true geometric intersections. For interval-based methods, one can try to “patch” this issue by, *e.g.*, checking whether the value of  $\phi(t)$  at the center of the interval is within  $\varepsilon$  of zero, but this sort of modification voids any guarantees—causing new artifacts (see inset). For level sets of simple functions, like the globally continuous gyroid in Figure 27 (*top right*), interval analysis can reliably compute the first intersection. But for the broader class of surfaces handled by Harnack tracing, significant artifacts were visible in all root finding methods we tried (Figure 27).



# Solid angle numerics

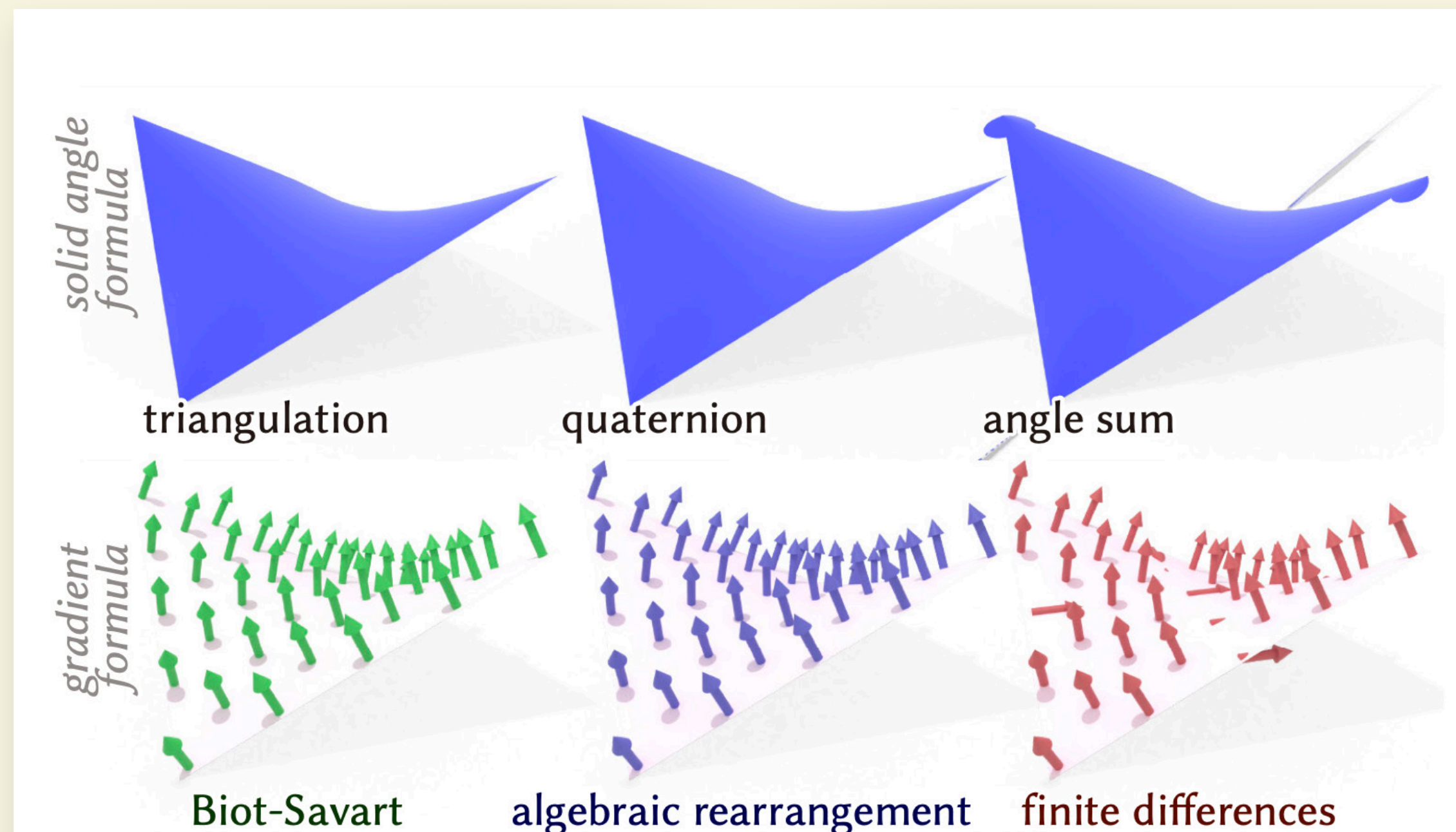


Fig. 12. Not all expressions for the solid angle or its derivative provide accurate results in floating point. *Top*: the solid angle formulas based on triangulation and quaternions work well, but the expression based on angle sums suffers from numerical instability. *Bottom*: The Biot-Savart law and its rearrangement by Adiels et al. [2022] both yield accurate normals, but finite differences give incorrect results due to jumps in the angle-valued function.

# Off-centered envelopes

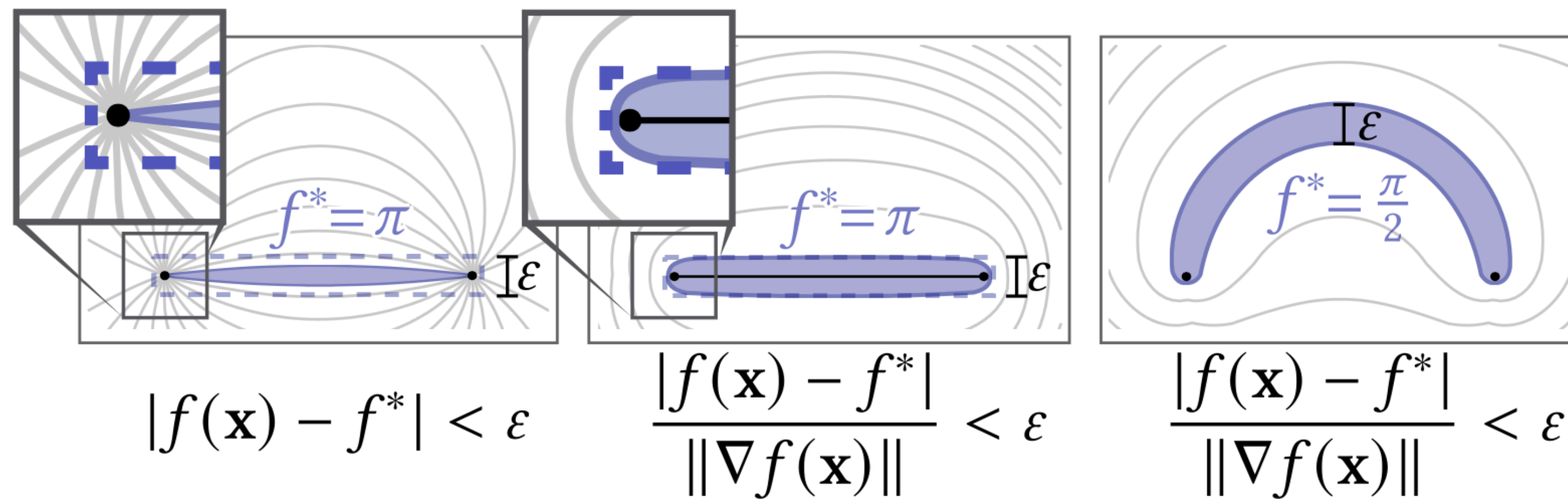


Fig. 5. If  $f(\mathbf{x})$  is a signed distance function, then terminating intersection queries when  $|f(\mathbf{x}) - f^*| < \epsilon$  ensures that  $\mathbf{x}$  is within  $\epsilon$  of the chosen level set. But, when  $f(\mathbf{x})$  is a general function, this condition loses its geometric meaning and produces an uneven profile along the target surface (*left*). We can obtain a more meaningful stopping condition using the gradient  $\nabla f(\mathbf{x})$ , to relate changes in function value to changes in position (*center, right*).