CGAL Developer Meeting

Ray Tracing Harmonic Functions

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with

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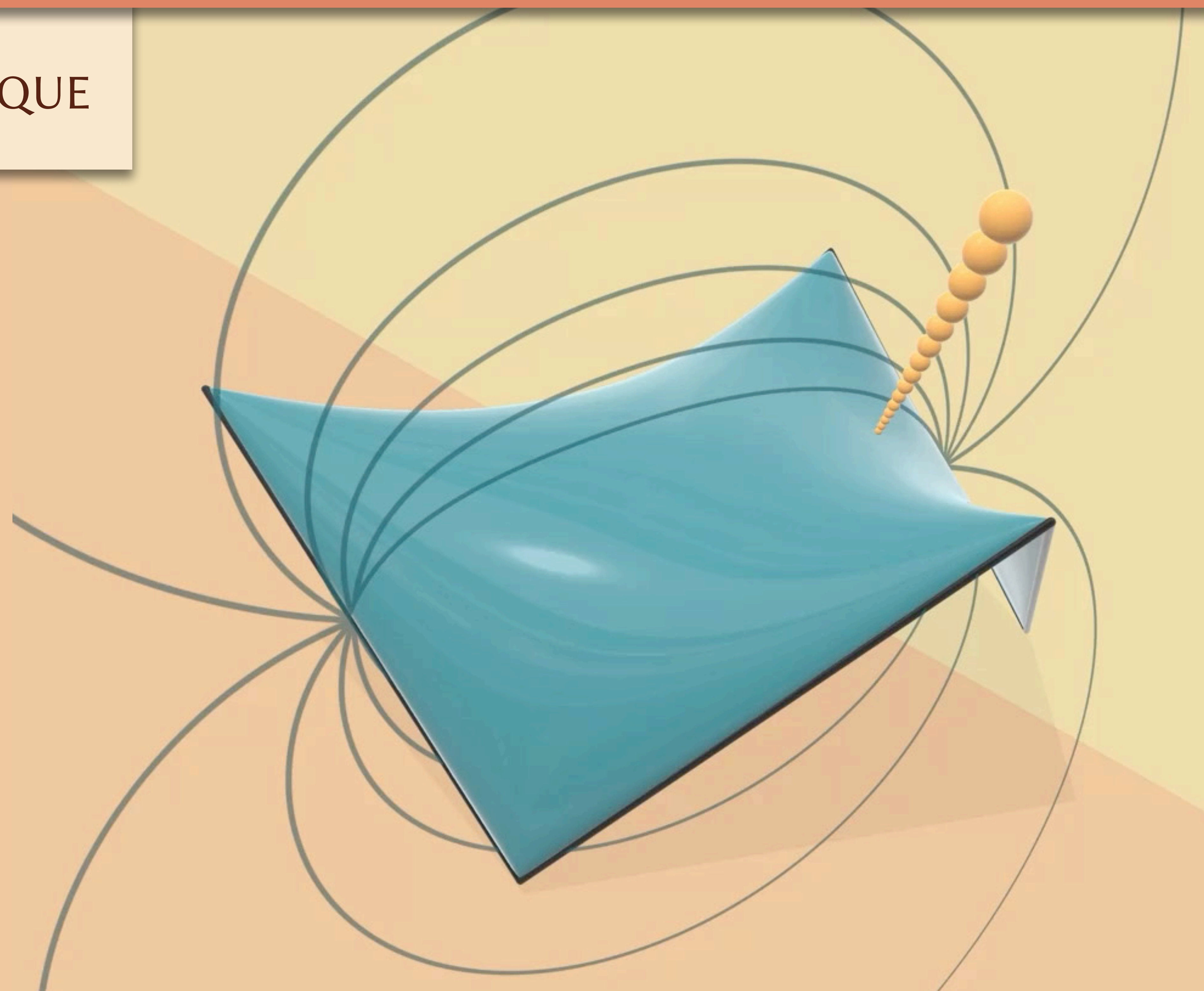
Keenan Crane, Carnegie Mellon University



Tuesday, 15 April 2025

Harmonic functions

special kind of function

heat transfer

gravitation

$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$$

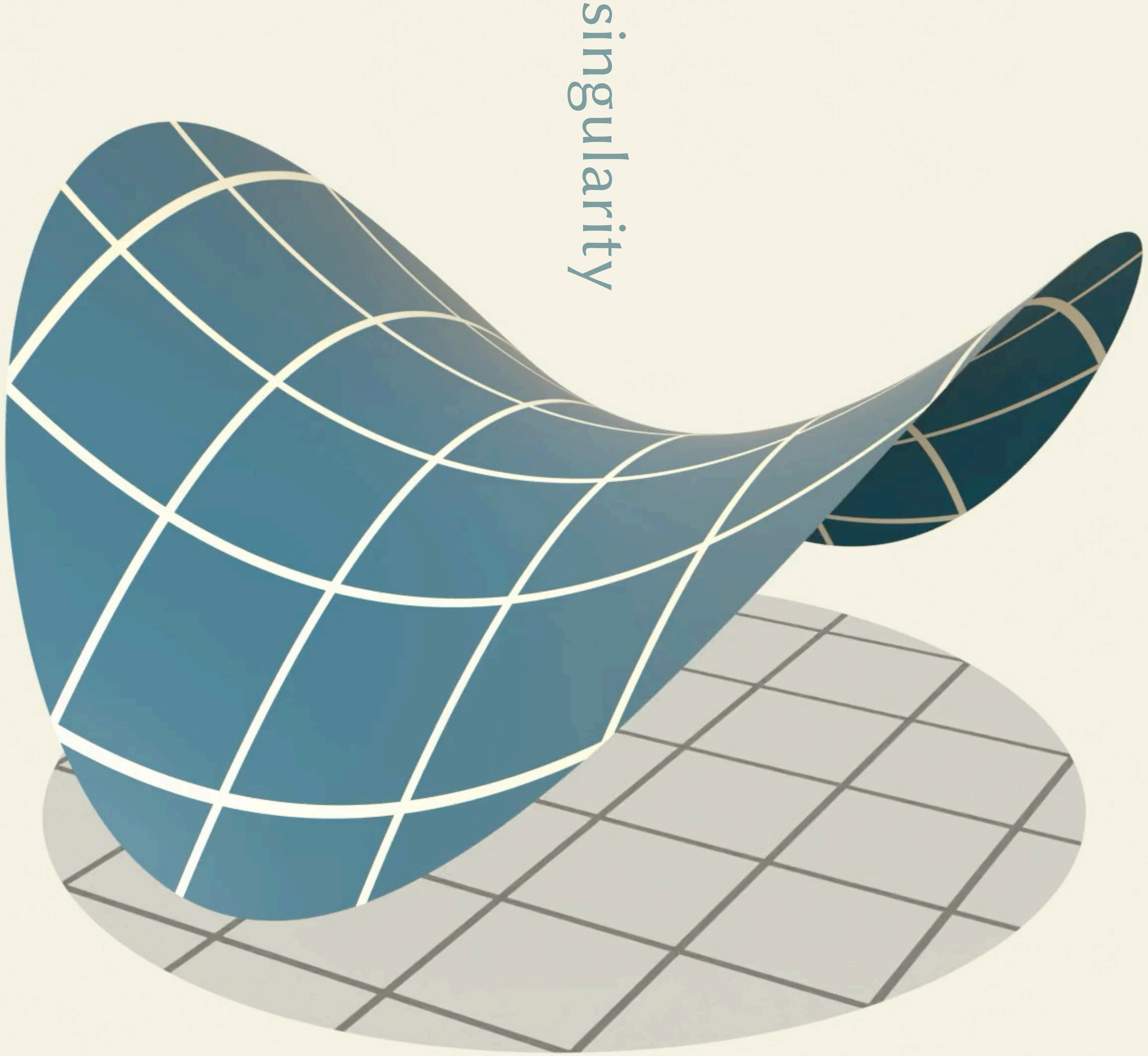
electrostatics

complex analysis

well-studied

Harmonic functions

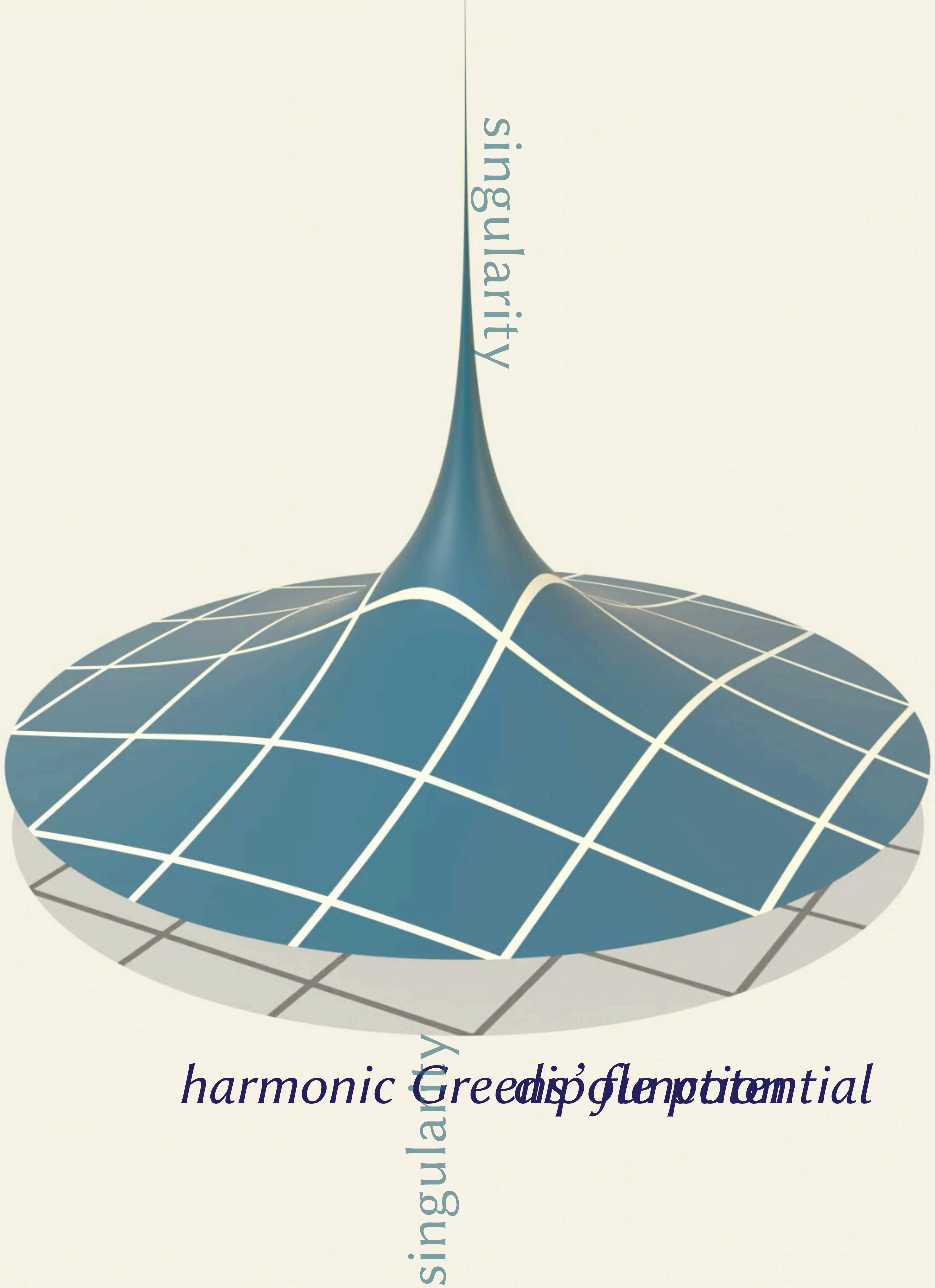
$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$$



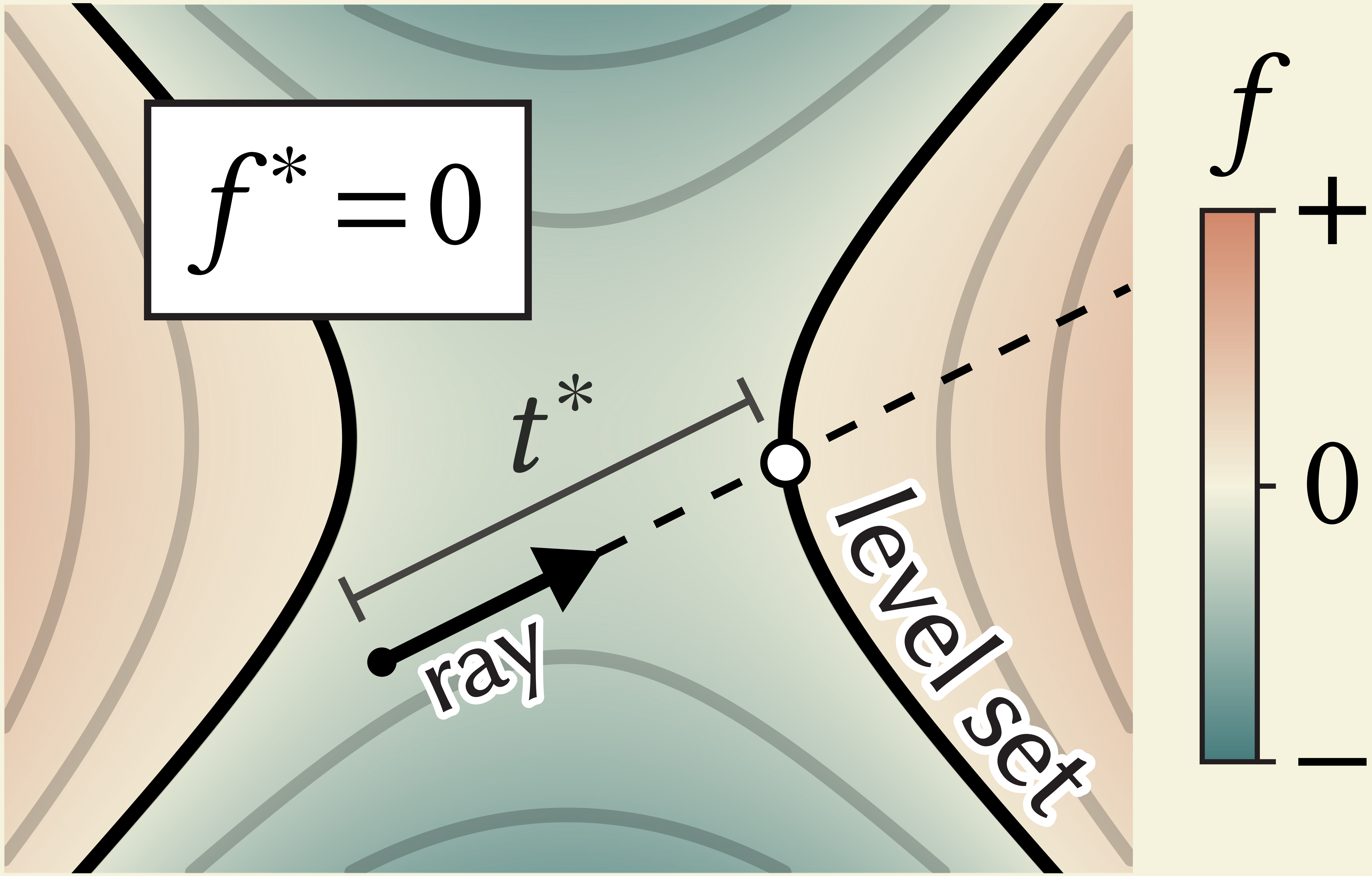
harmonic Green's function $f(x,y) = x^2 - y^2$

Harmonic functions

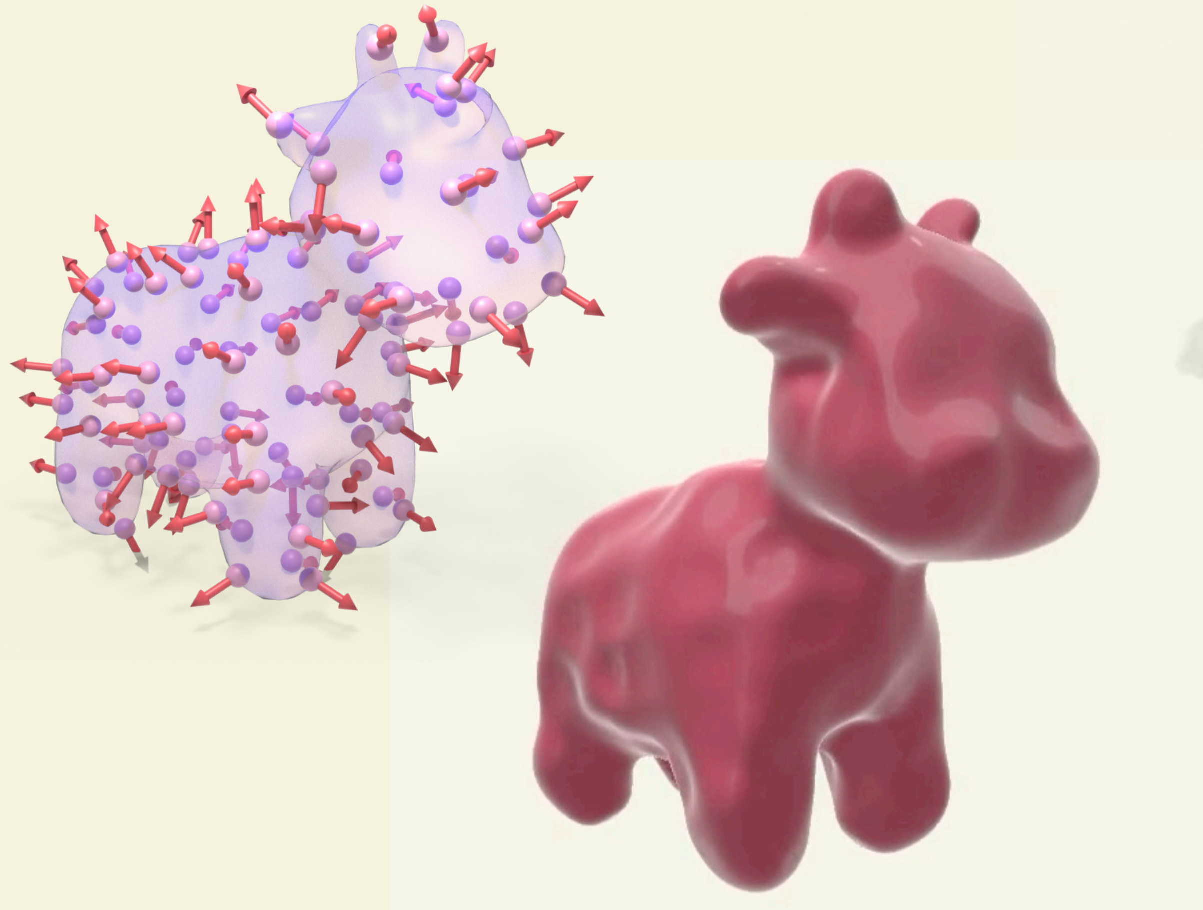
$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$$



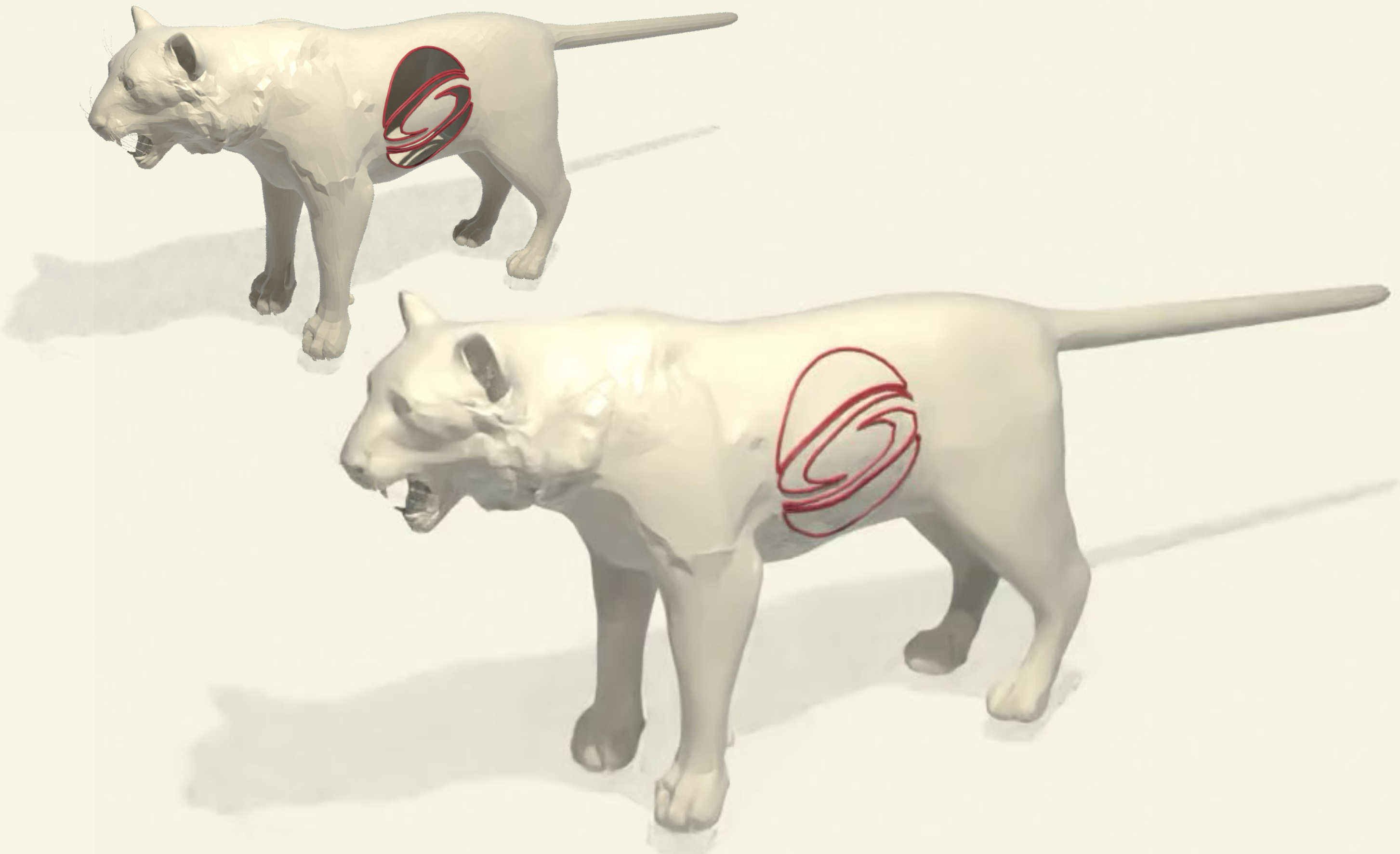
Intersecting a ray with a level set



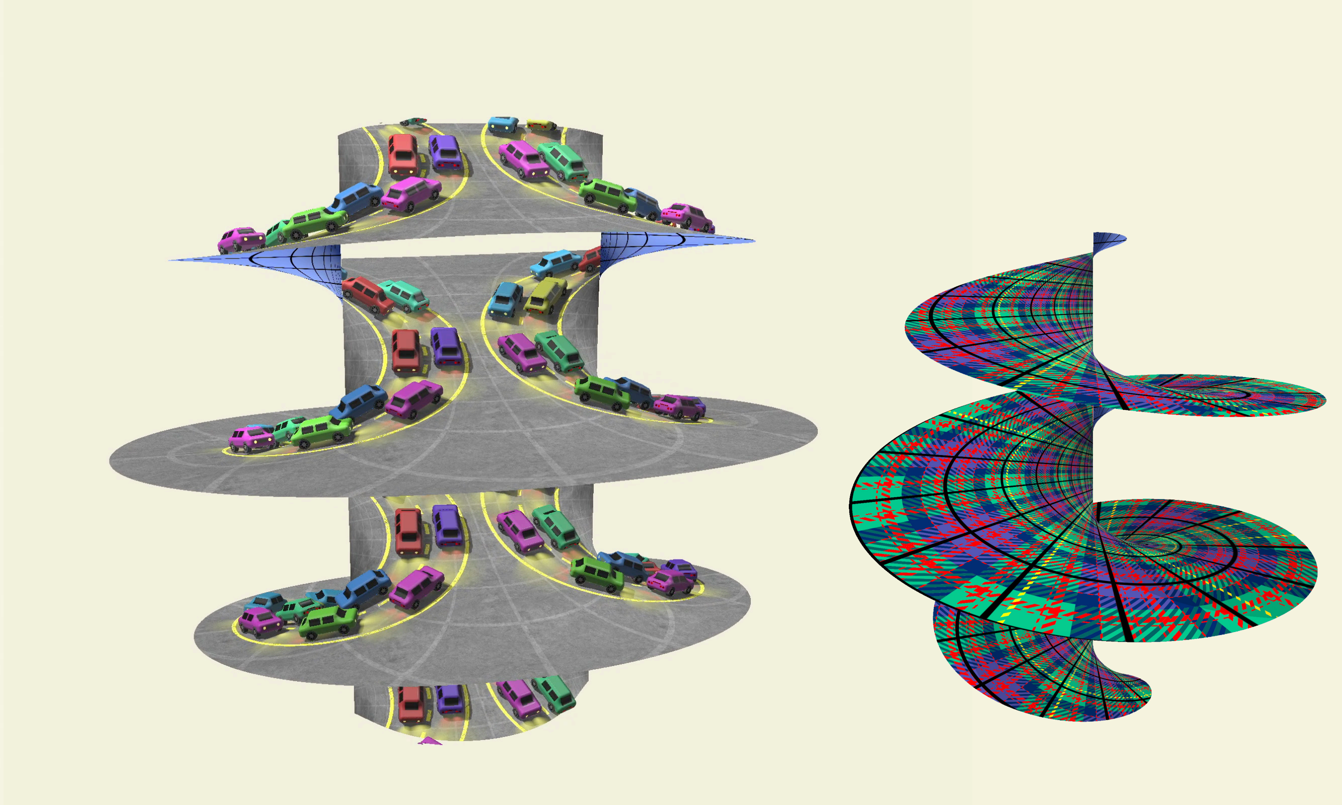
Level sets of harmonic functions show up everywhere



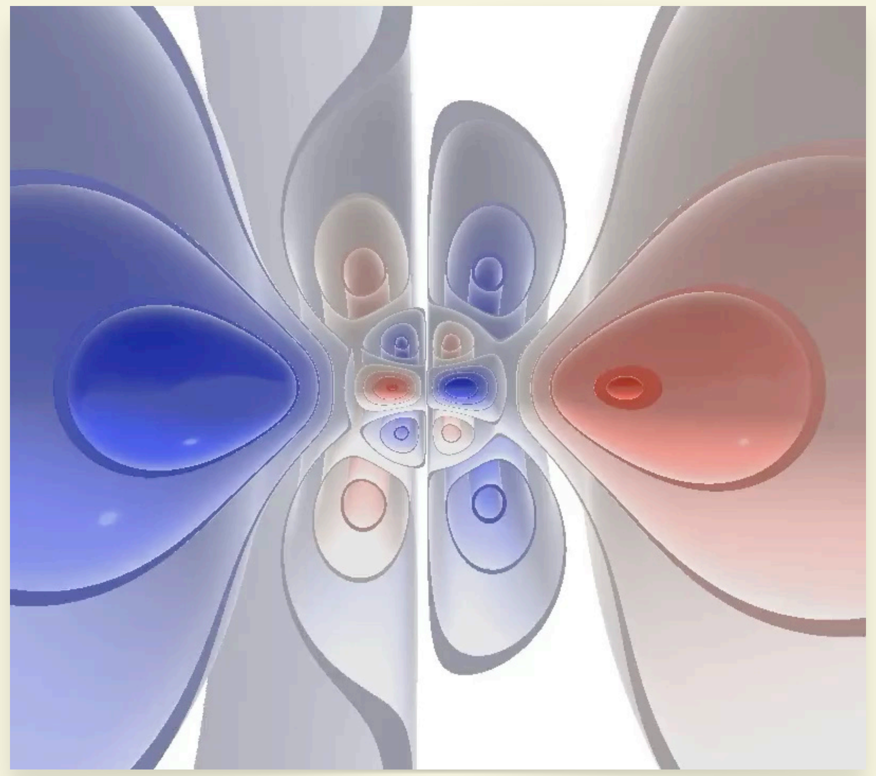
Poisson surface reconstruction
[Kazhdan *et al.* 2006]



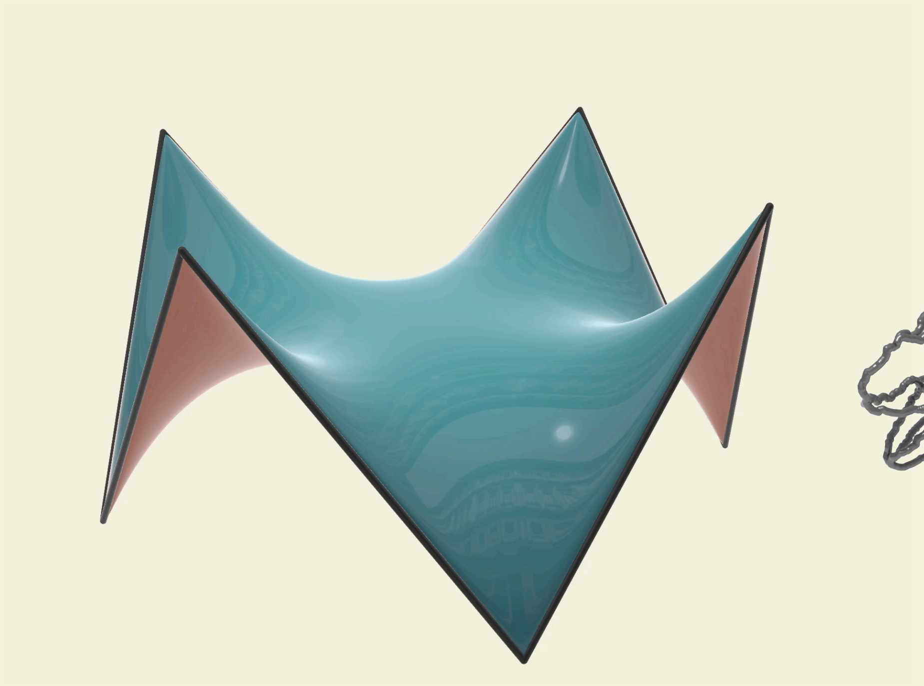
generalized winding numbers
[Jacobson *et al.* 2013]



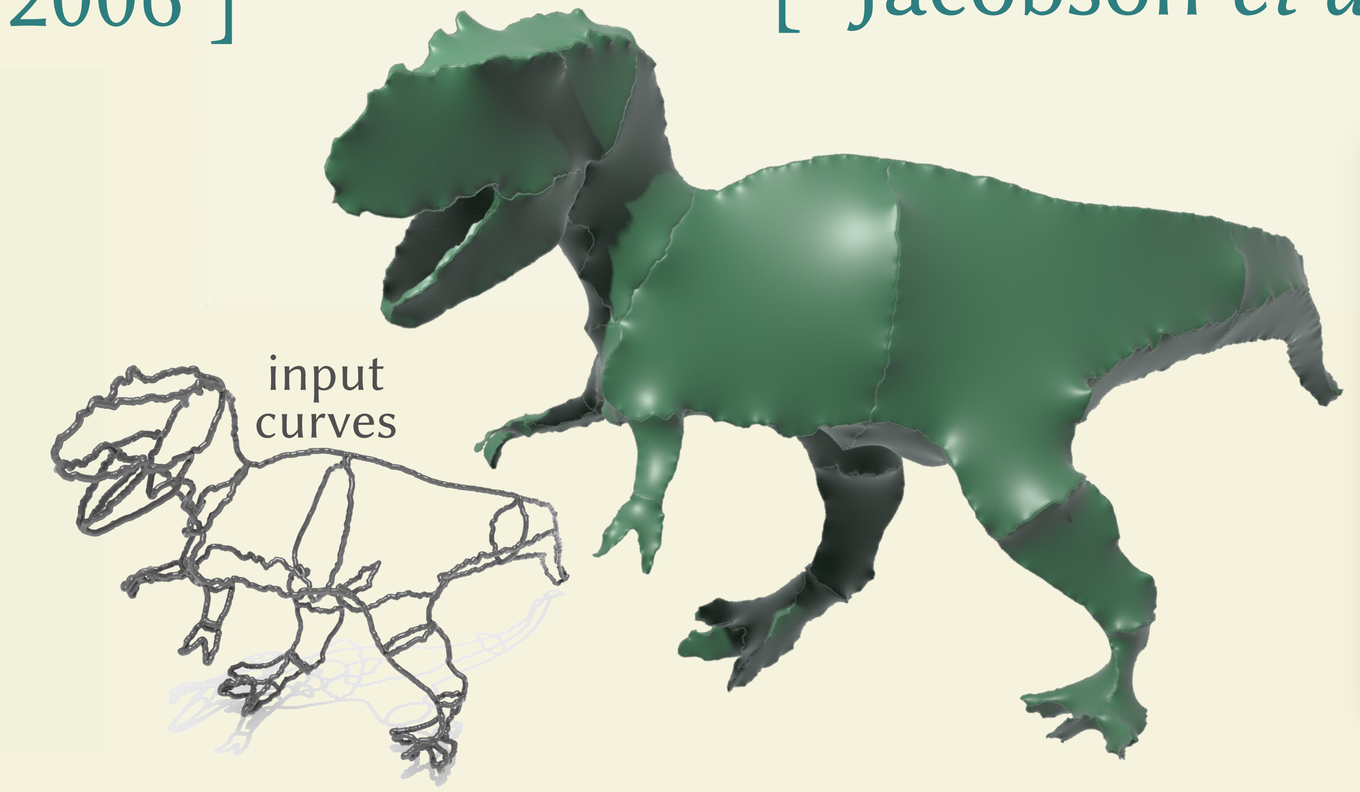
Riemann surfaces
[Riemann 1851]



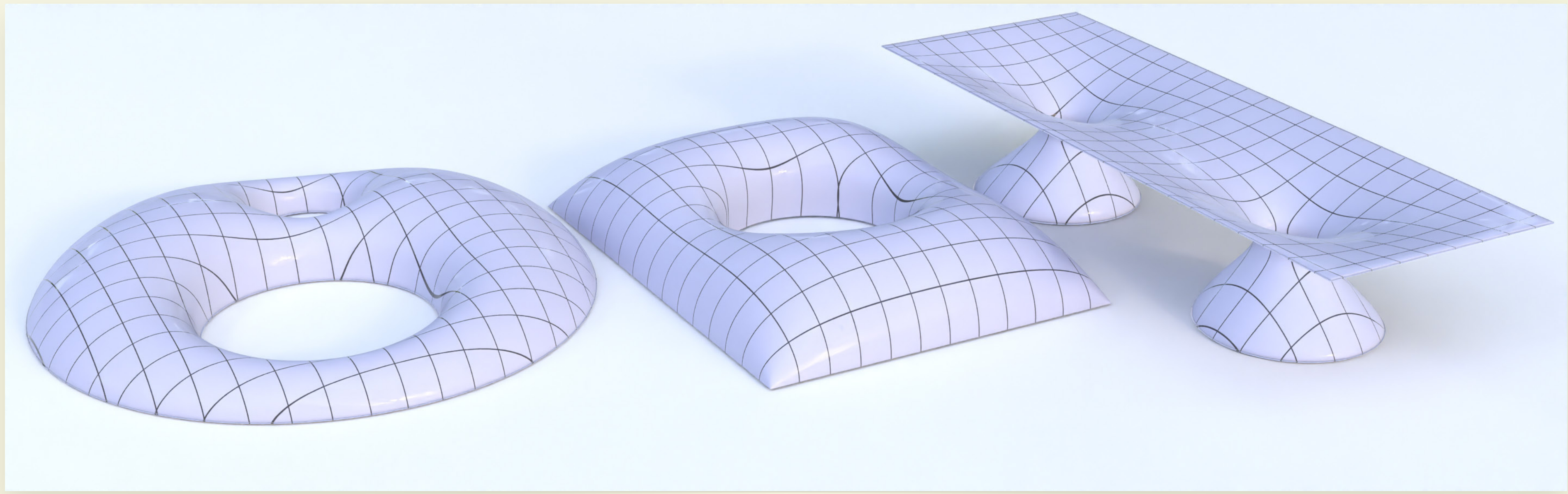
hyperspherical harmonics
[Fock 1935]



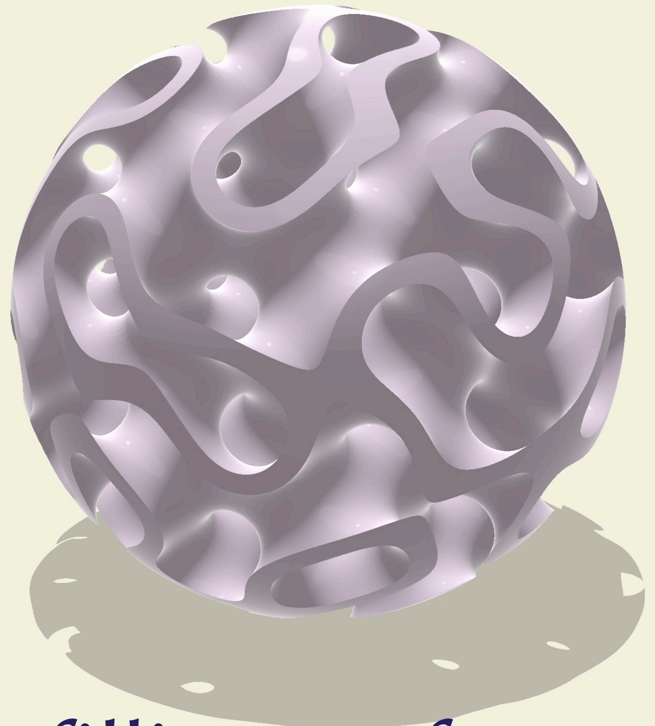
nonplanar polygons
[Maxwell 1873]



curve networks
[de Goes *et al.* 2011]



shell structures in architectural geometry [Adiels *et al.* 2022]



space-filling surfaces for digital fabrication

… but, they're hard to render with existing techniques

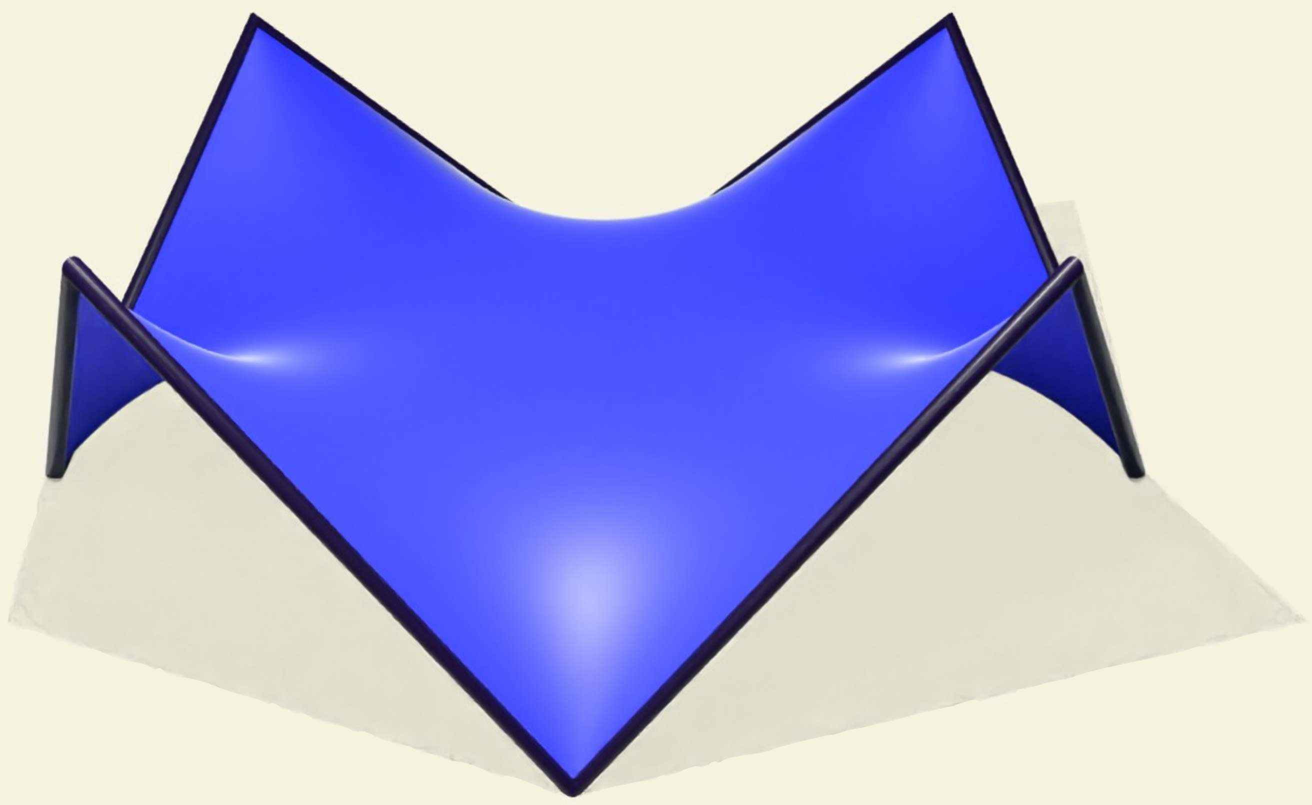
may have singularities

… but, they’re hard to render with existing techniques

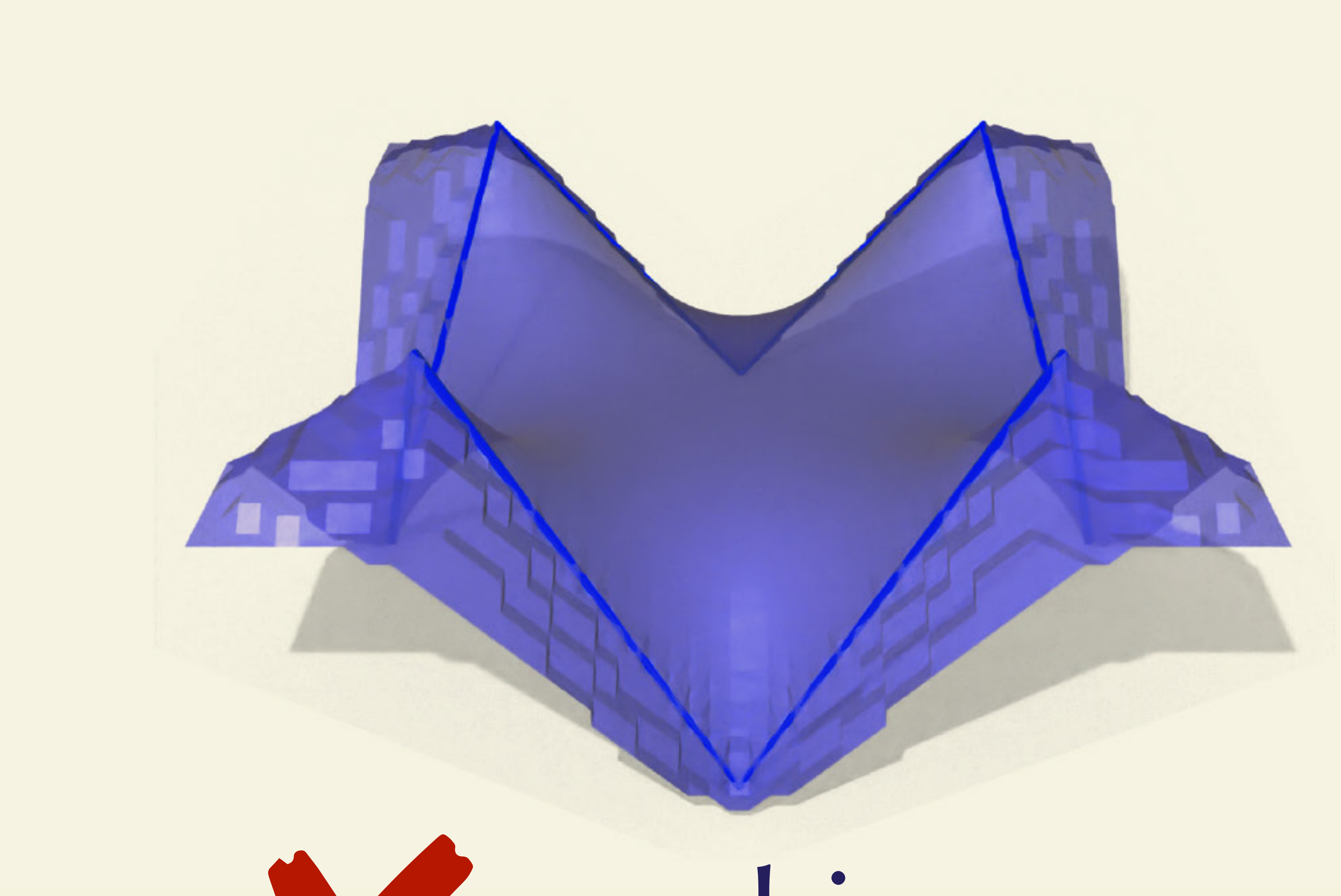
may have boundaries

… but, they're hard to render with existing techniques

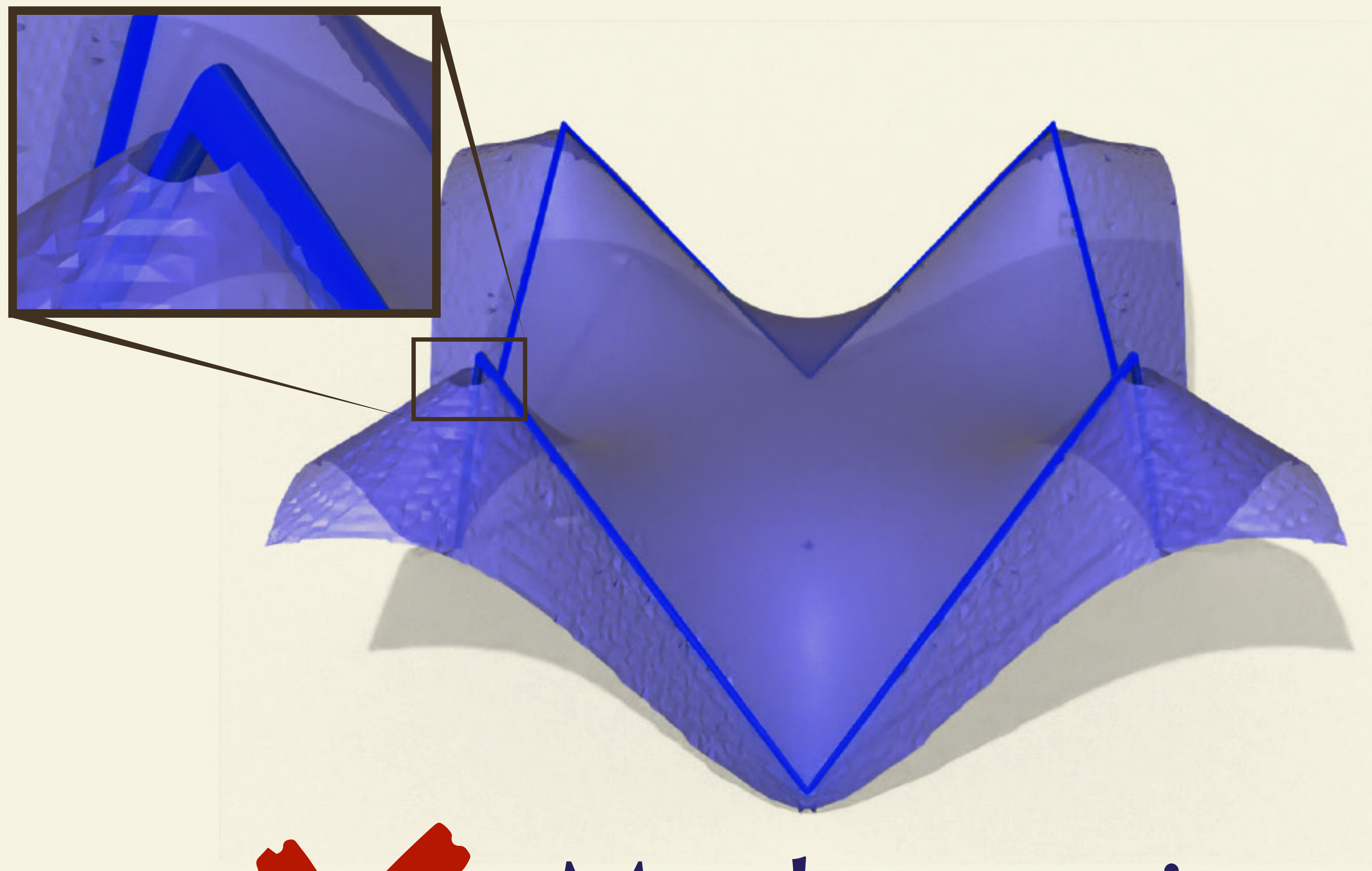
may have boundaries



ours



marching cubes



Mathematica (ContourPlot3D)

Sphere tracing

[Hart 1996]

compute intersections for *signed distance functions (SDFs)*

Sphere tracing

[Hart 1996]

compute intersections for *signed distance functions (SDFs)*

Sphere tracing: beyond SDFs

[Hart 1996]

- Easy to generalize to *Lipschitz* functions:

  (essentially, $|\nabla f| \leq L$)

- Important fact:

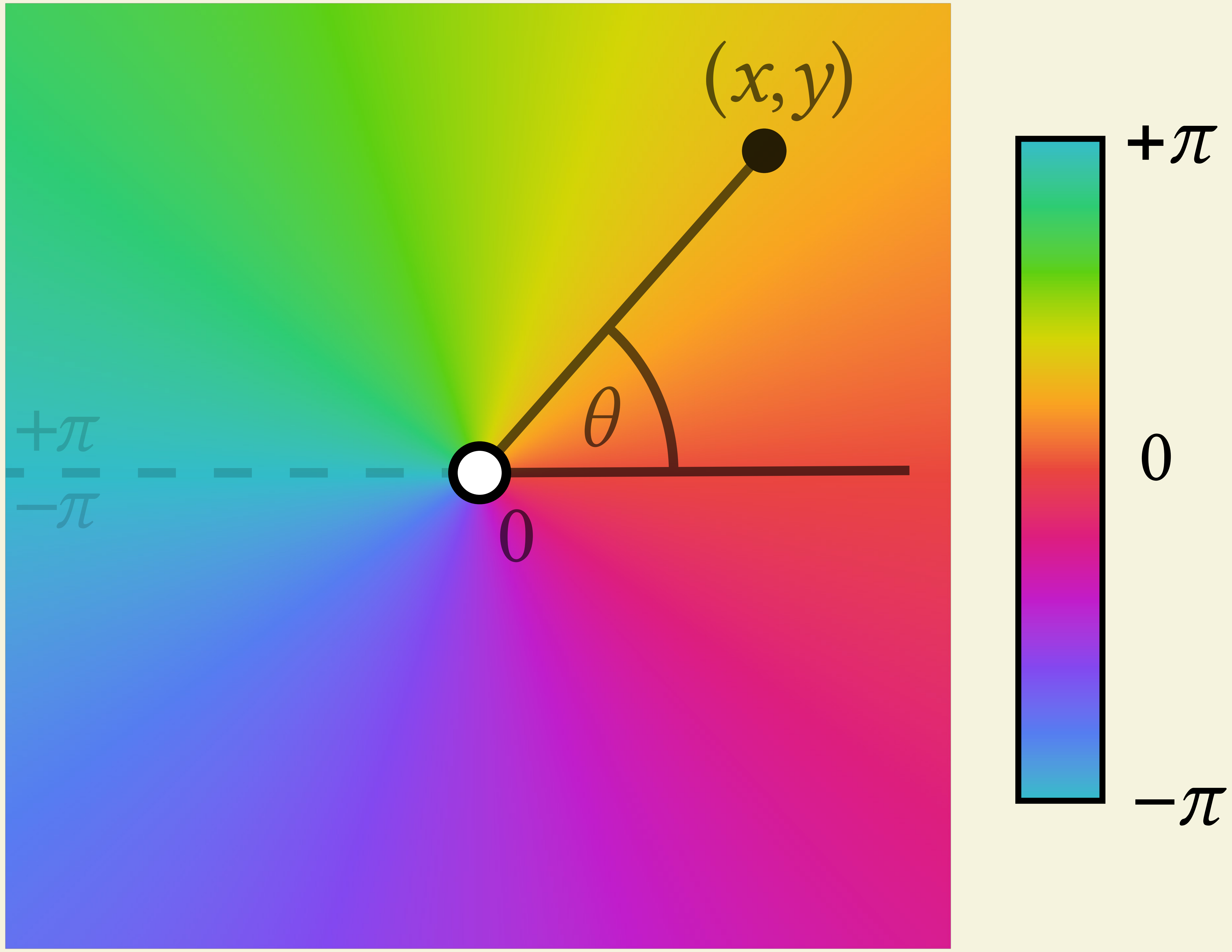
  $|f(x) - f(y)| \leq L|x - y|$

- provides a conservative bound on distance

[Inigo Quilez 2015]

Problem: many harmonic functions are not Lipschitz

$$\theta(x, y) = \text{atan2}(y, x)$$



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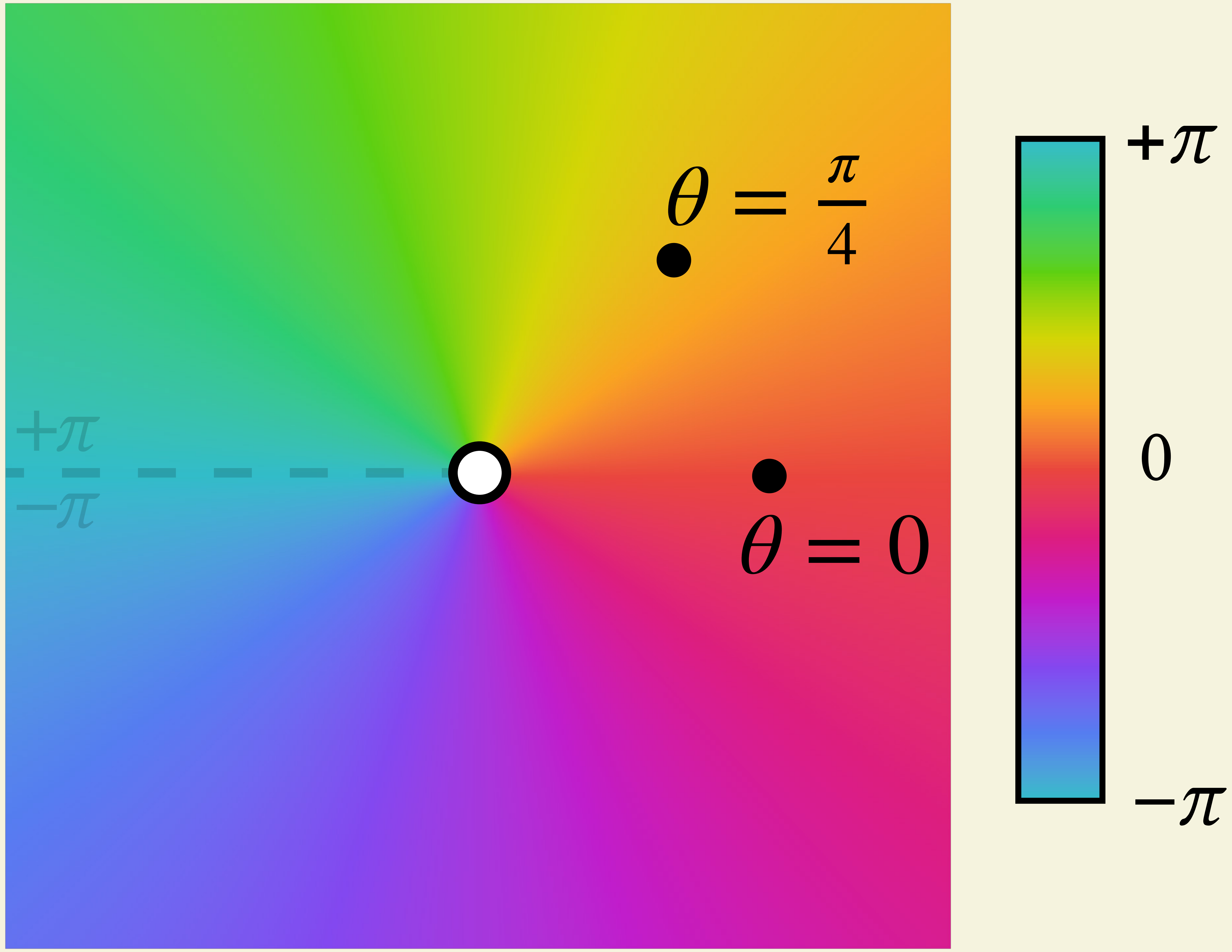

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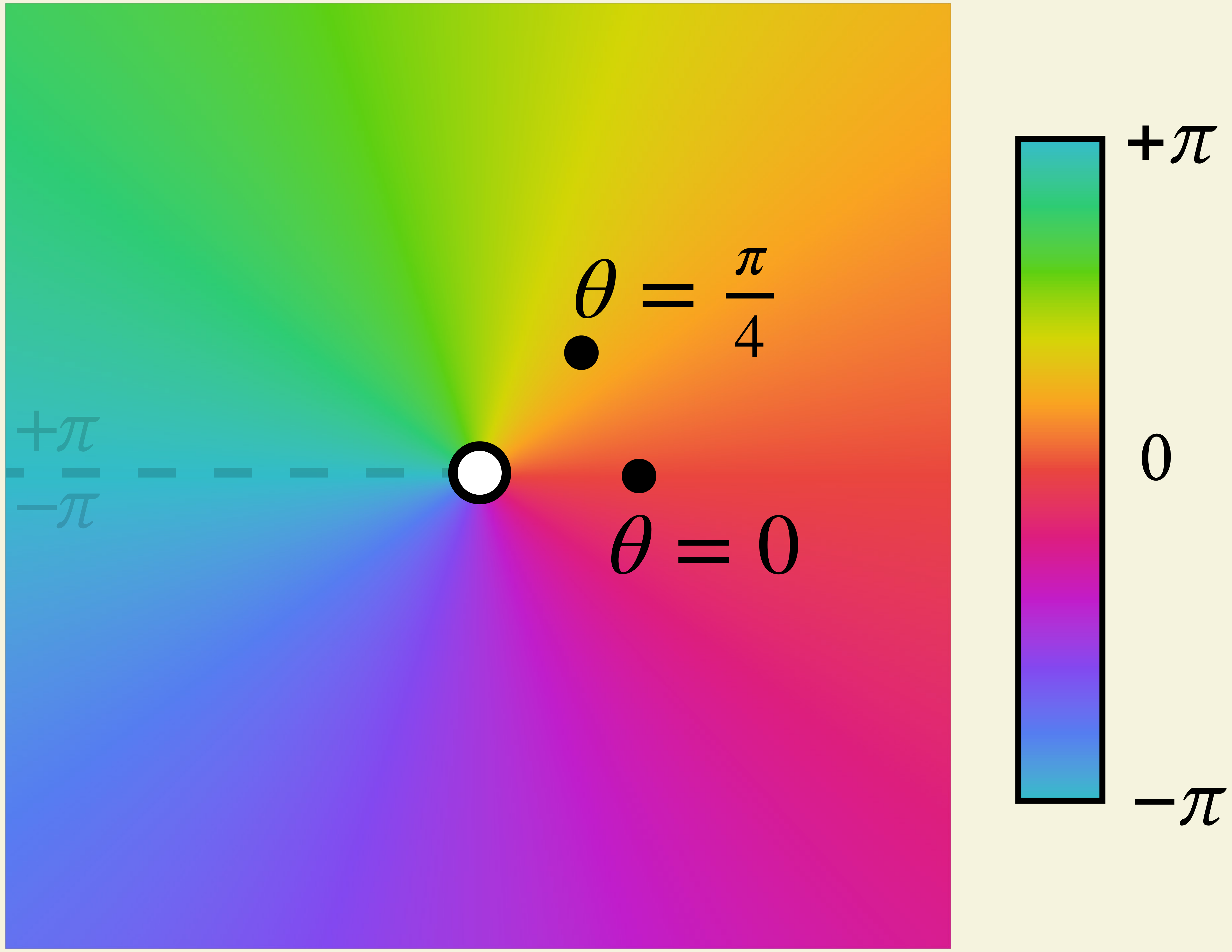
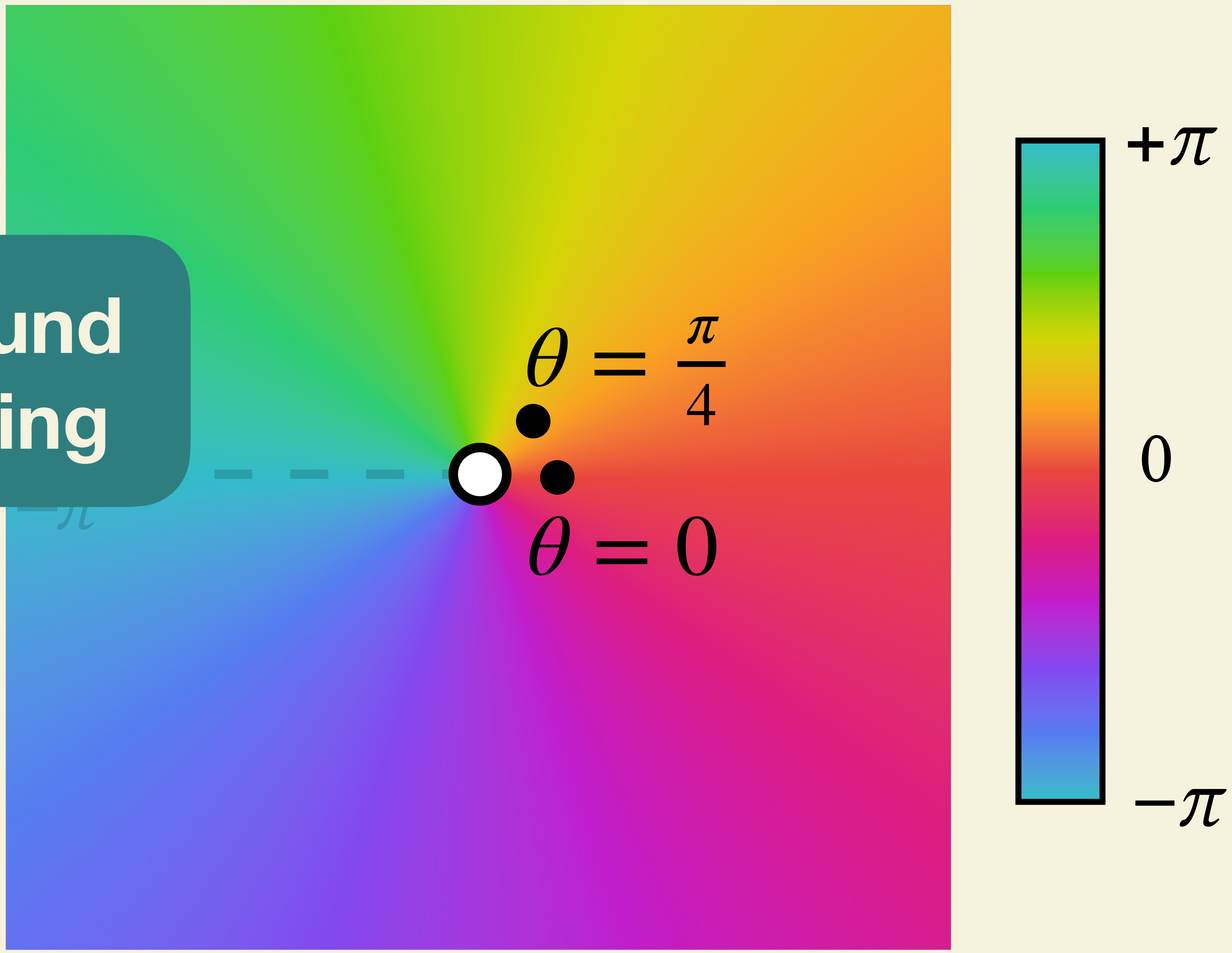


Problem: many harmonic functions are not Lipschitz

$$\theta(x, y) = \text{atan2}(y, x)$$

No matter how close points get, function values never get closer

no distance bound for sphere tracing



Main idea: get distance bounds from Harnack's inequality

Let f be a positive harmonic function on a ball:

$$\frac{1 - r/R}{(1 + r/R)^2} f(x) \leq f(y) \leq \frac{1 + r/R}{(1 - r/R)^2} f(x)$$

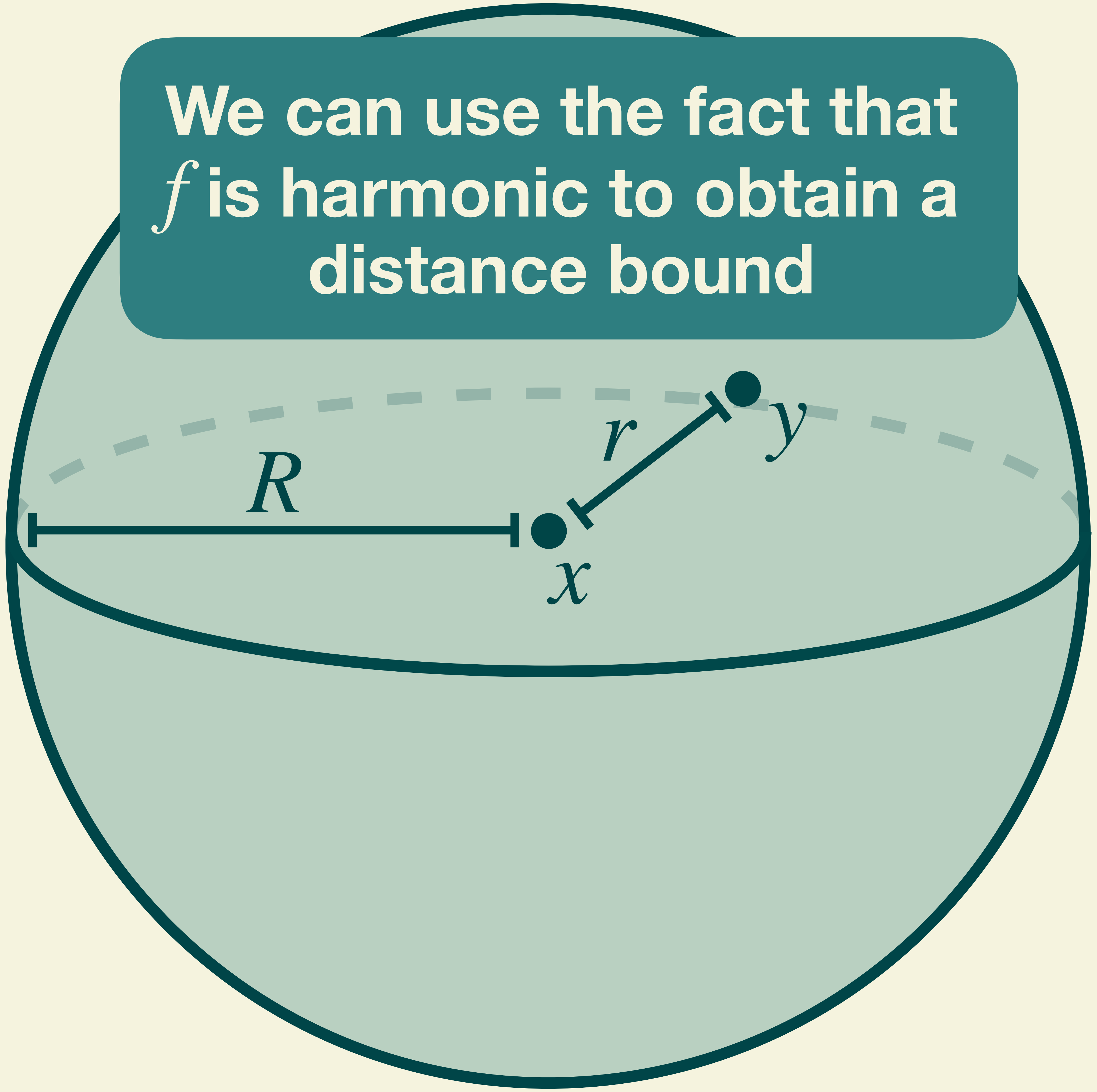
lower bound — upper bound

always safe to take step of size

$$\frac{R}{2}\left| a + 2 - \sqrt{a^2 + 8a} \right|,$$

where $a = \dfrac{f(x)}{f^*}$

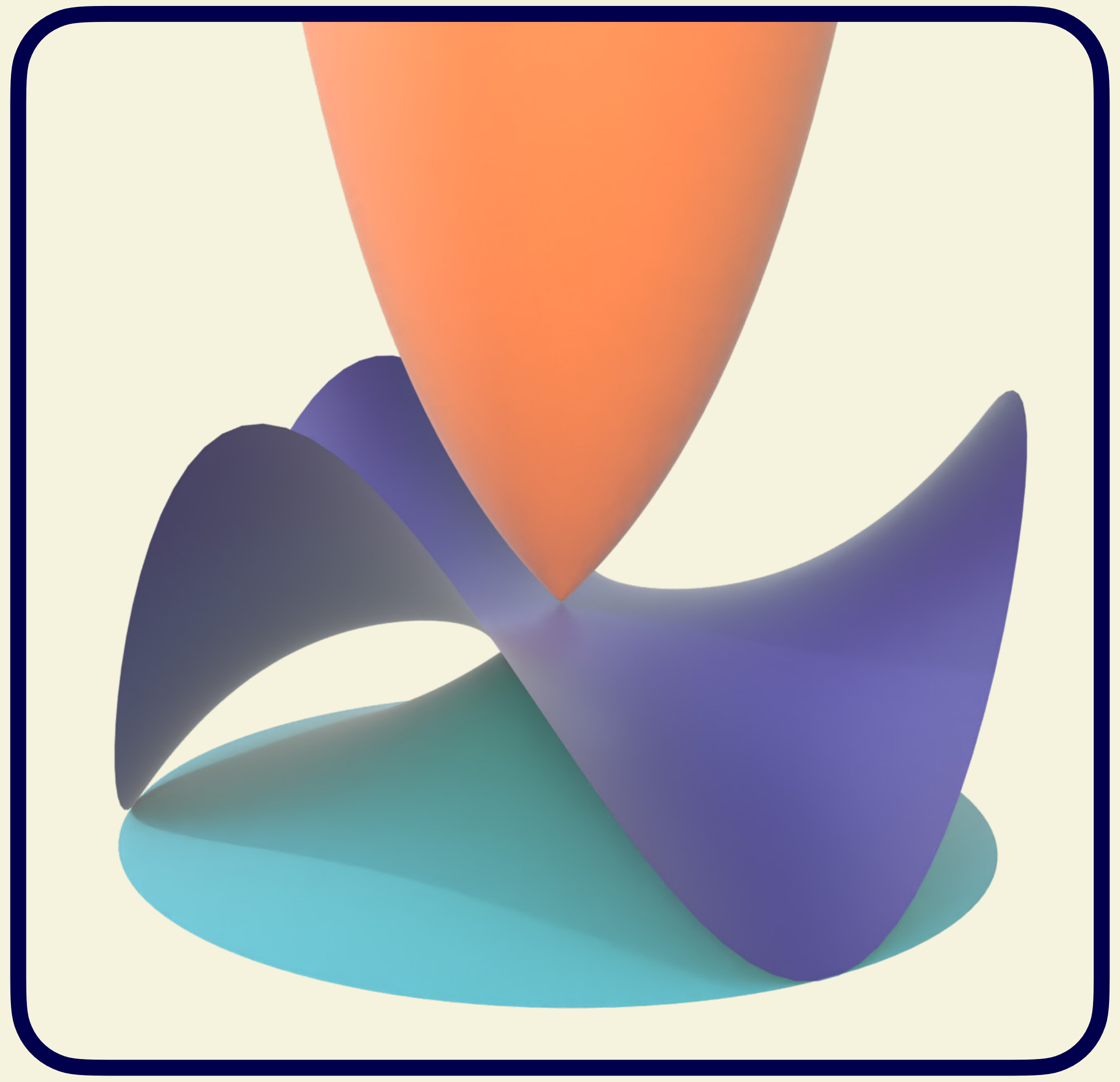
We can use the fact that f is harmonic to obtain a distance bound



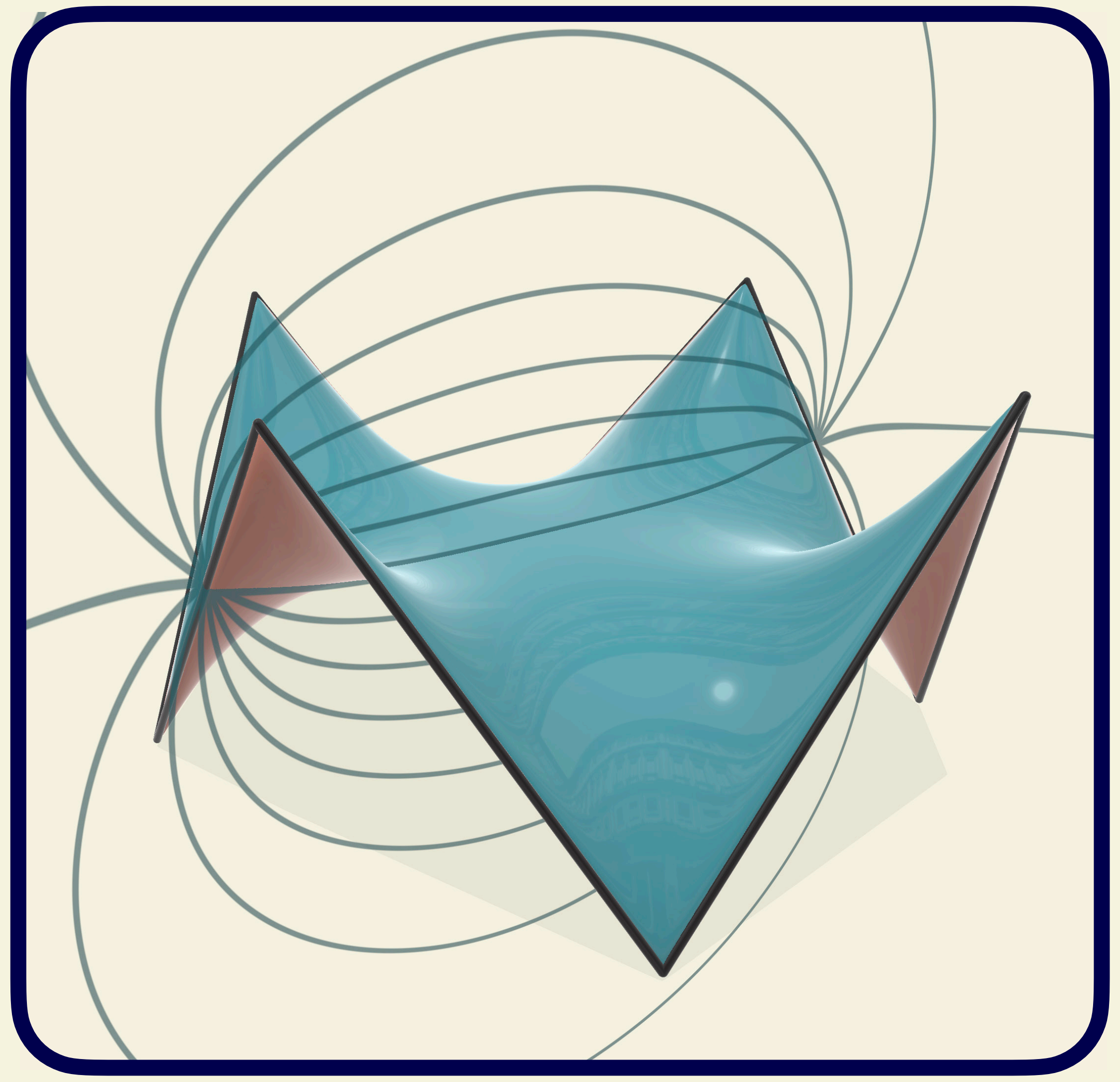
2D example

Outline

I. Harnack's Inequality



II. Harnack Tracing



III. Examples

IV. Future Work

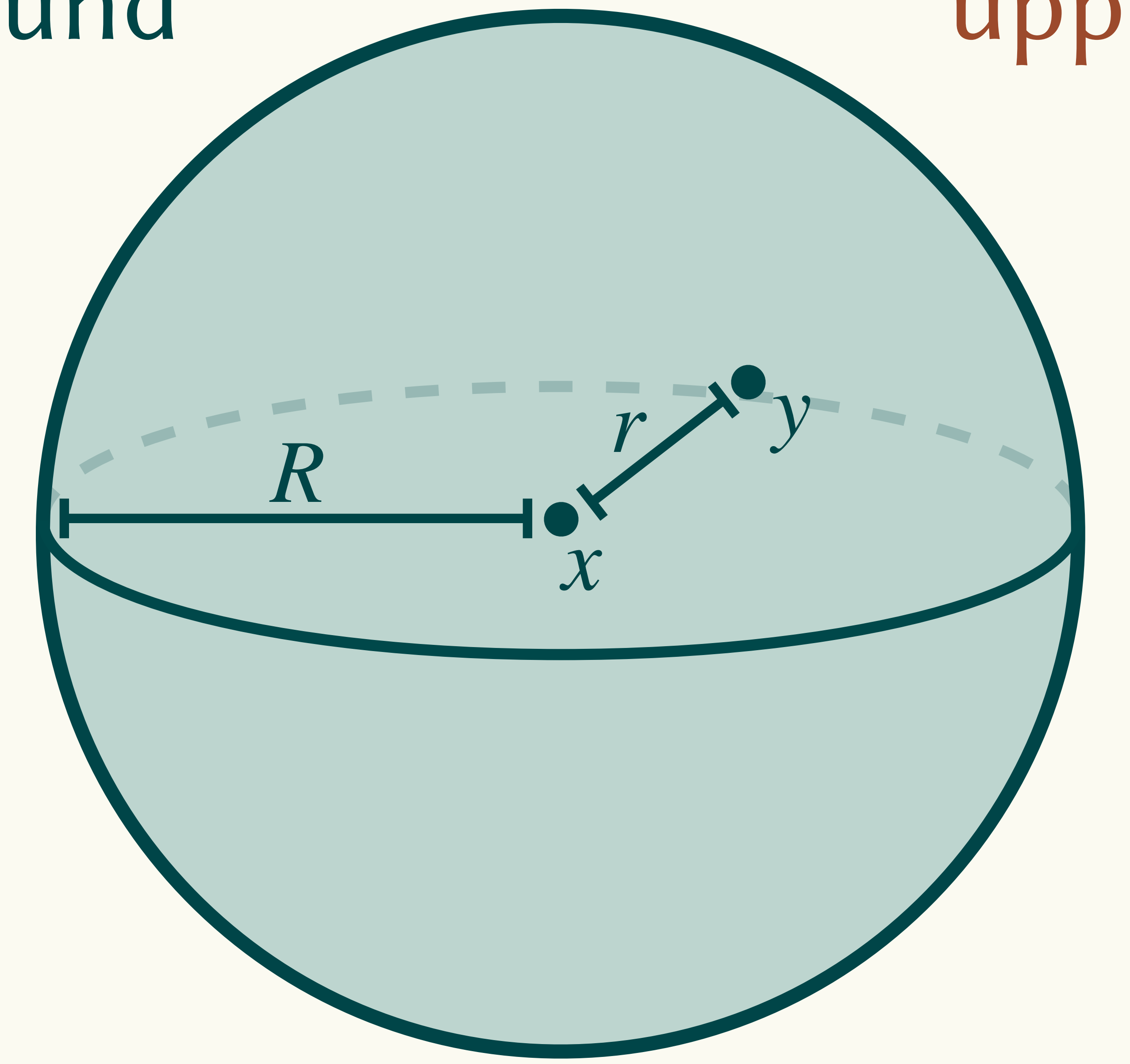
I. Harnack’s Inequality

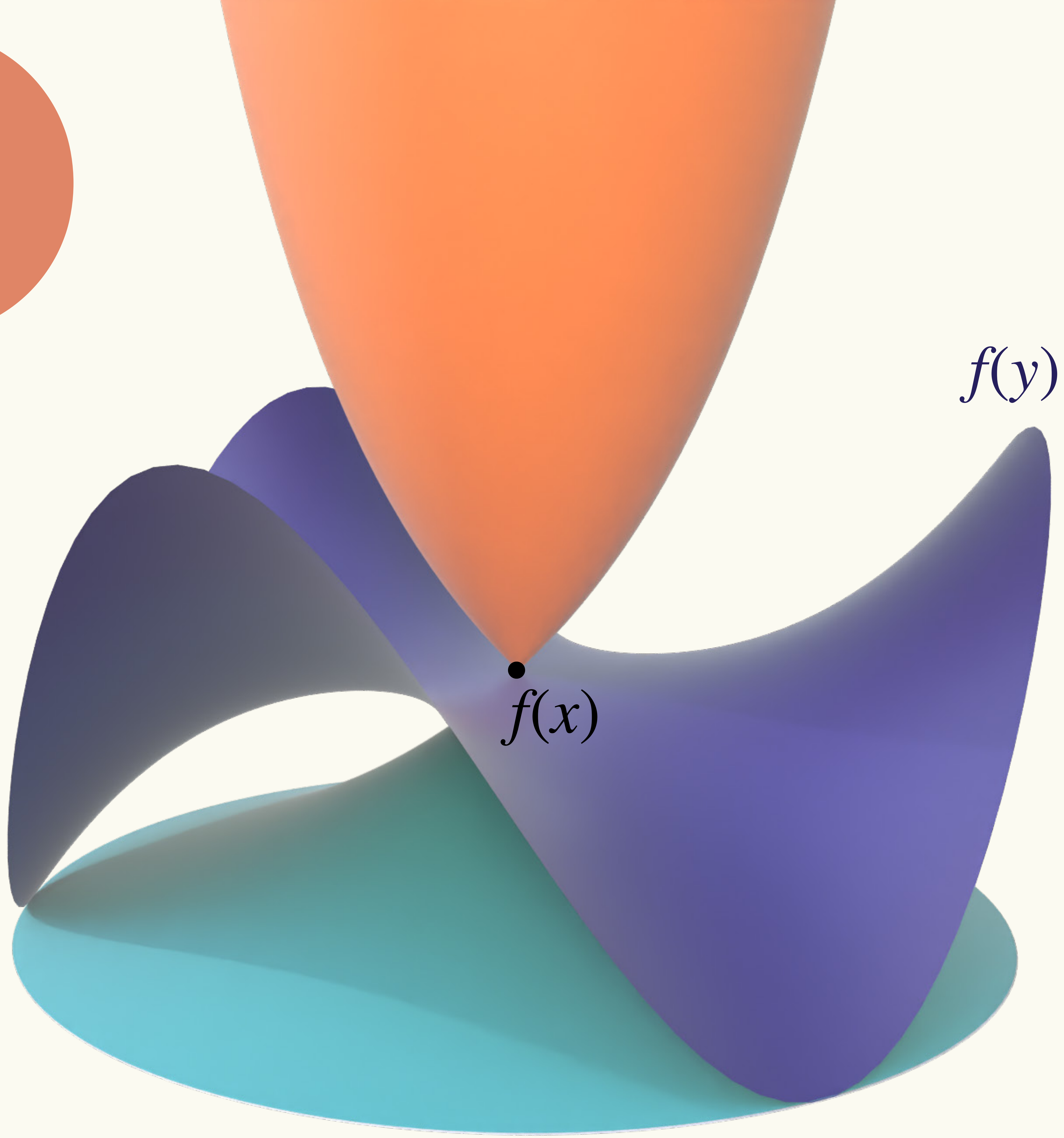
Harnack's Inequality

$\left(\text{in } \mathbb{R}^d\right)$

$$\frac{1 - r/R}{(1 + r/R)^{d-1}} f(x) \leq f(y) \leq \frac{1 + r/R}{(1 - r/R)^{d-1}} f(x)$$

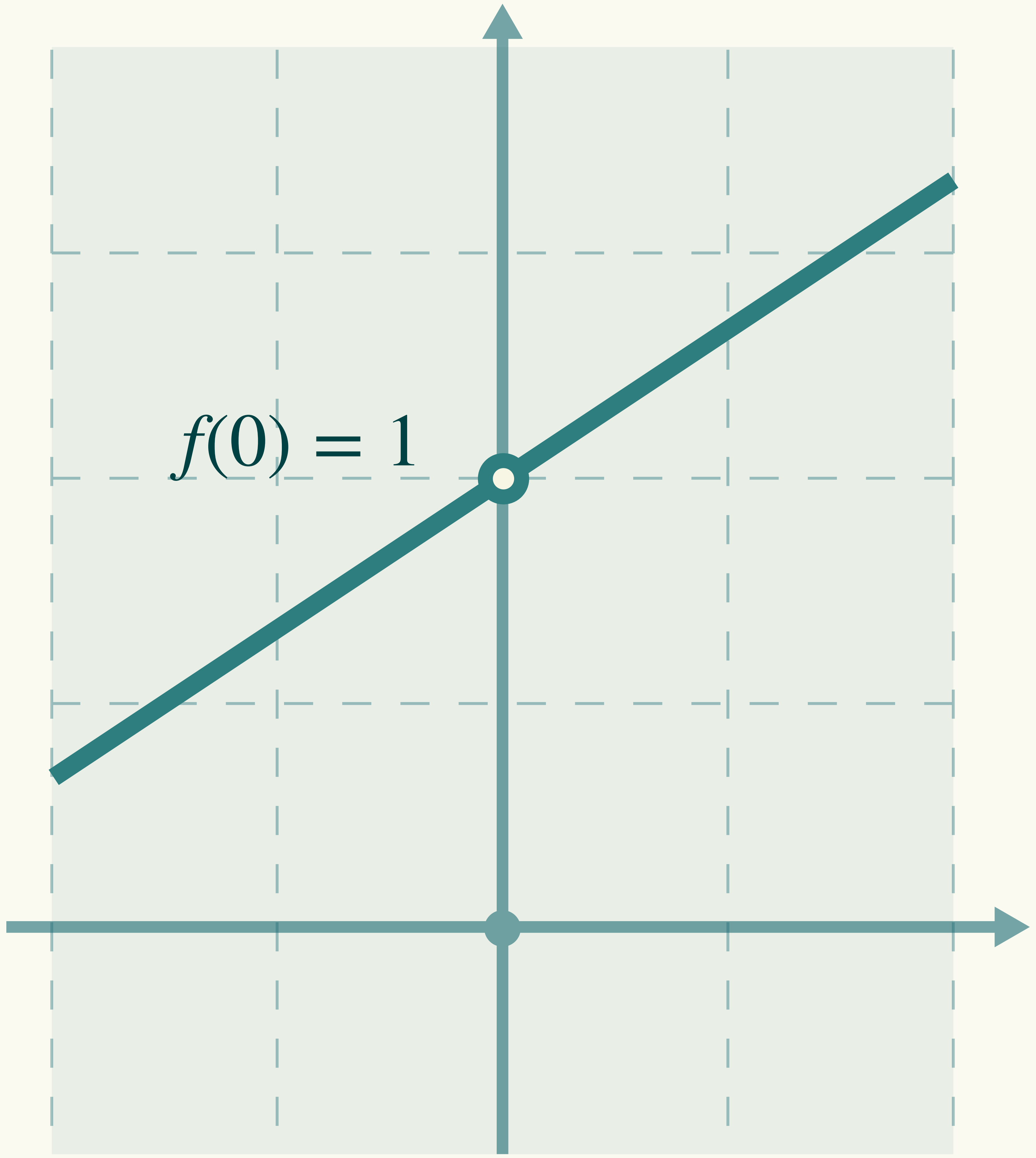
lower bound | upper bound





Prelude: Bounding Positive Linear Functions

a linear function can
change arbitrarily fast



*technically speaking, positive affine functions

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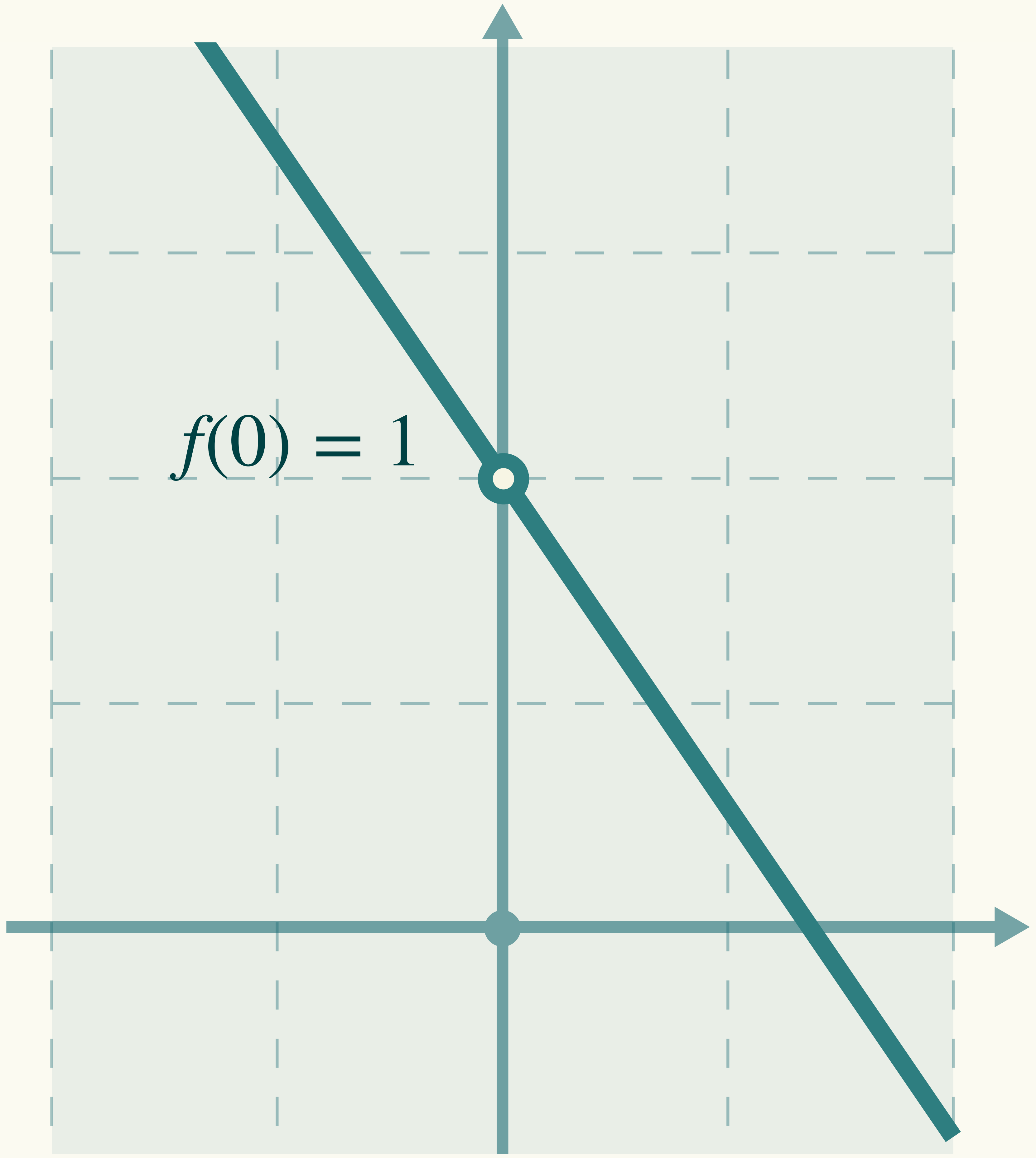

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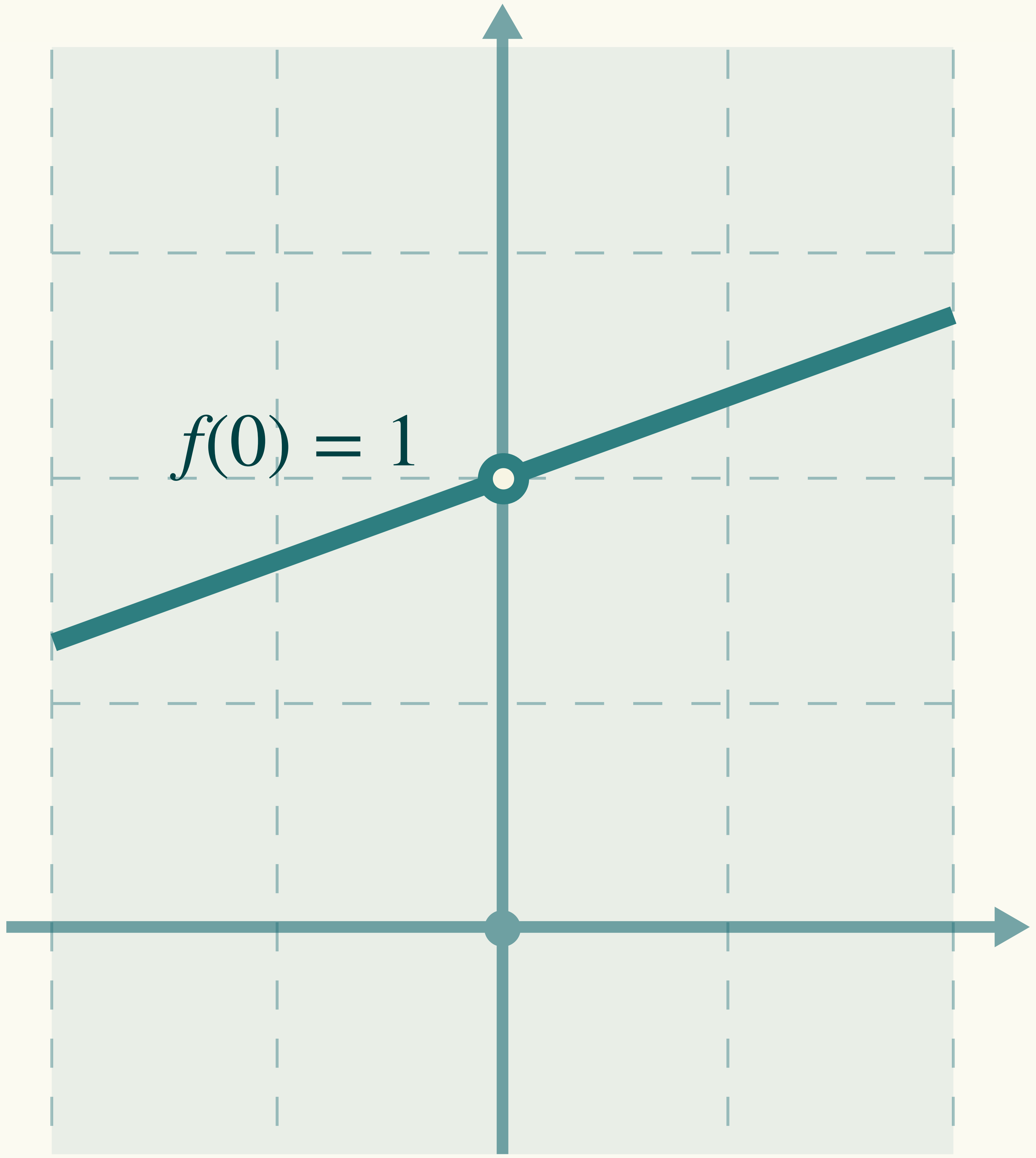

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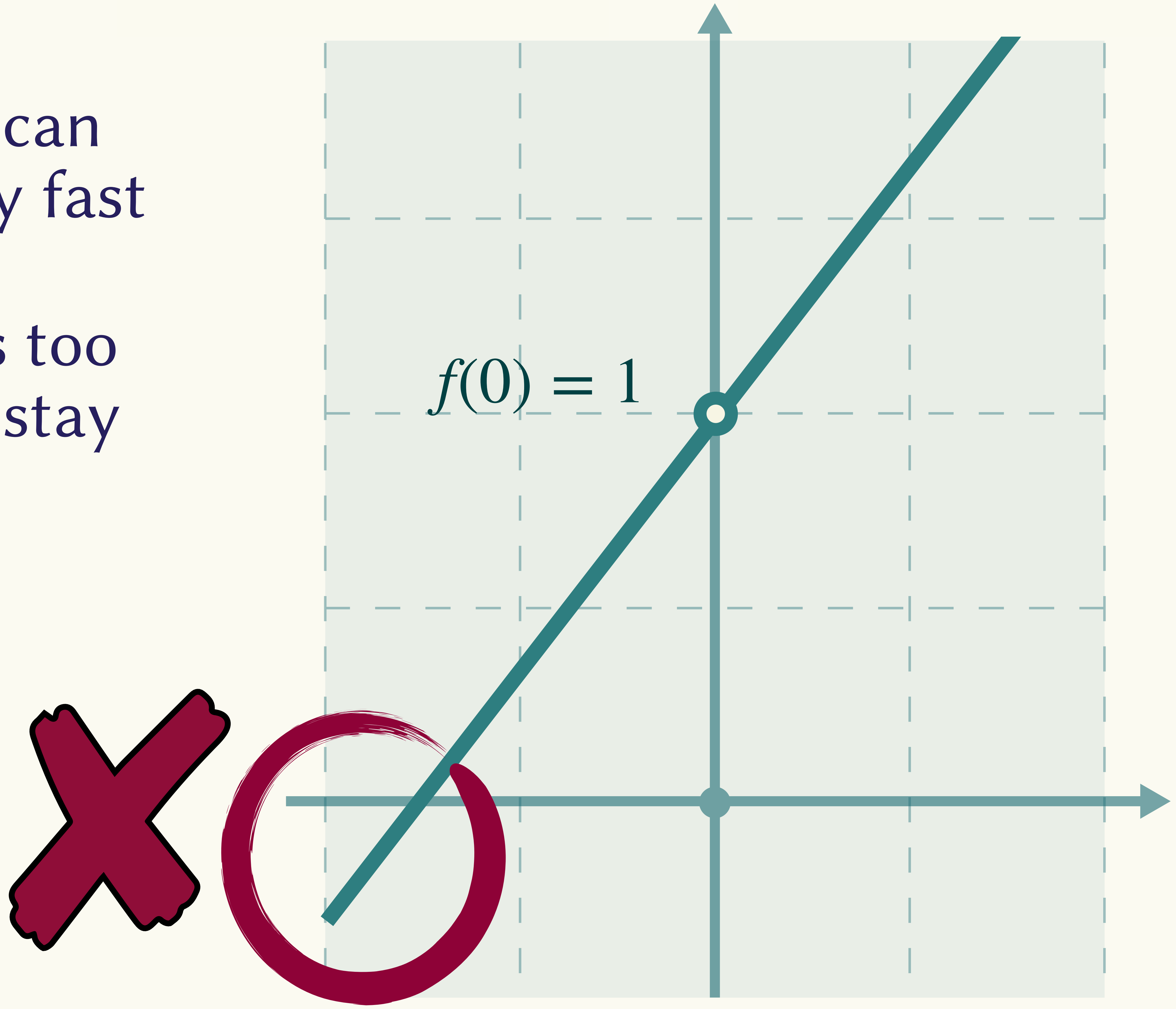
Prelude: Bounding Positive Linear Functions

a linear function can change arbitrarily fast

but if it changes too fast, it does not stay *positive*



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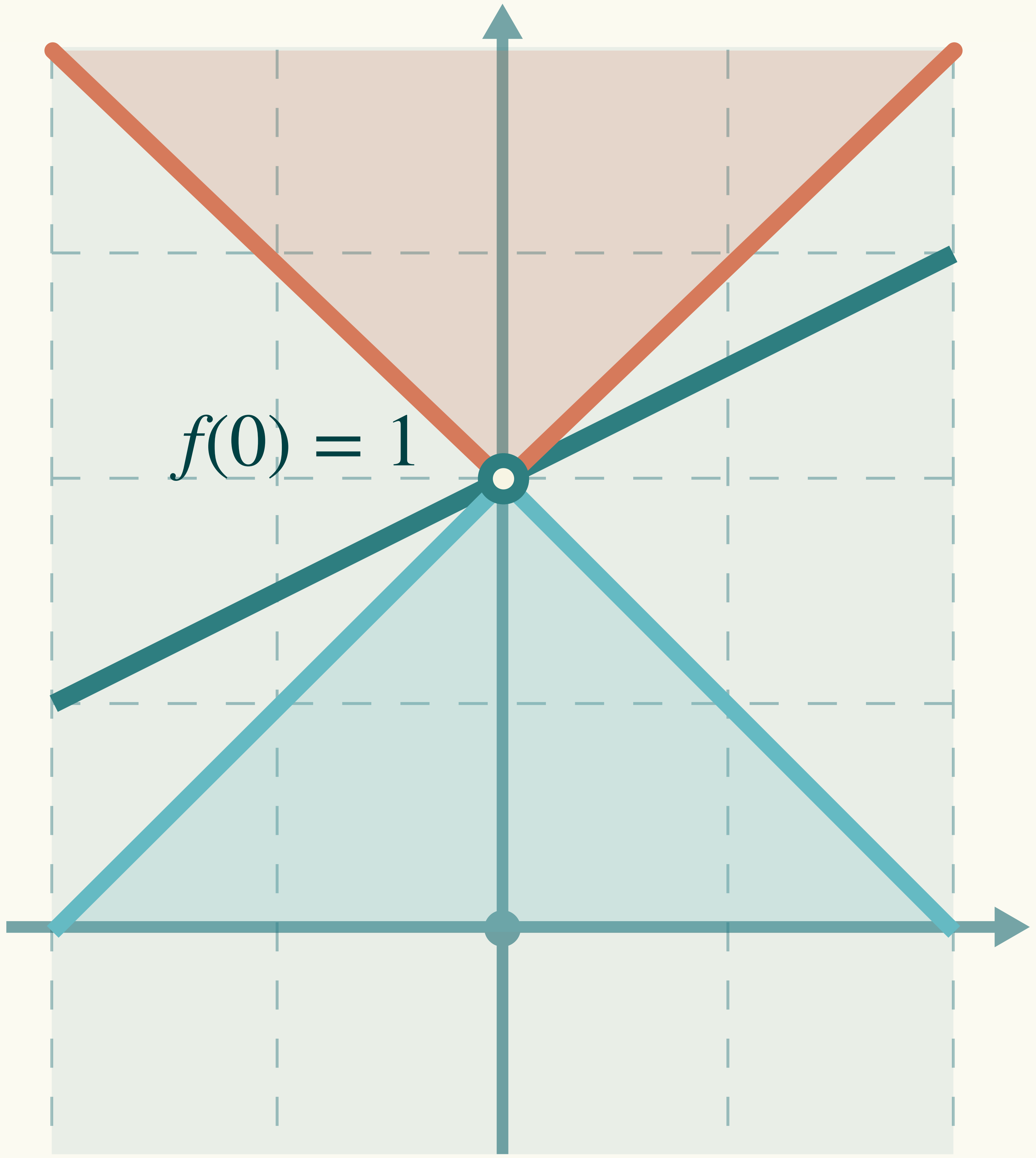
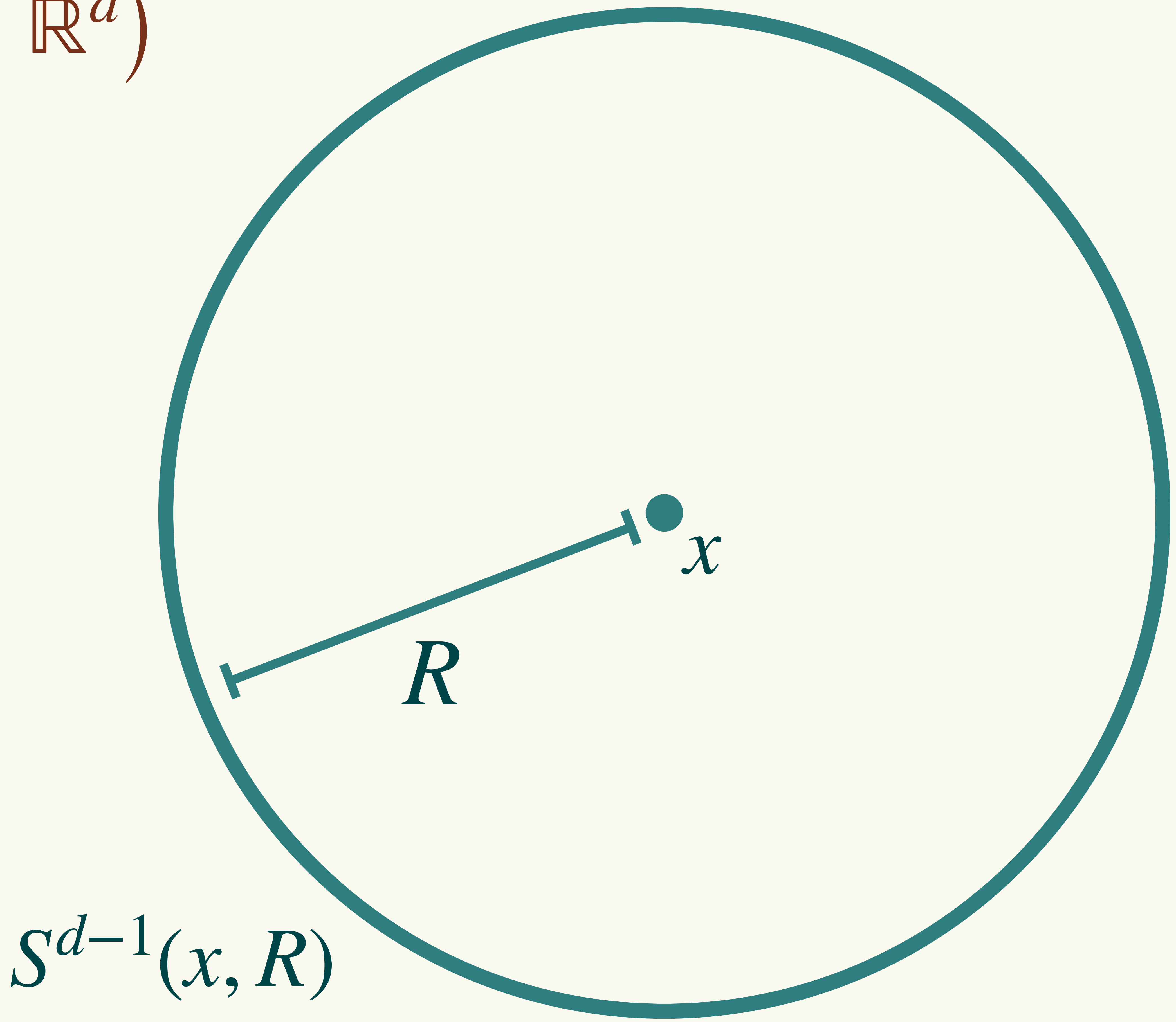


positive linear functions must stay between the upper and lower bounds

*technically speaking, positive affine functions

The Mean Value Property

(in $\mathbb{R}^d$)



The Mean Value Property

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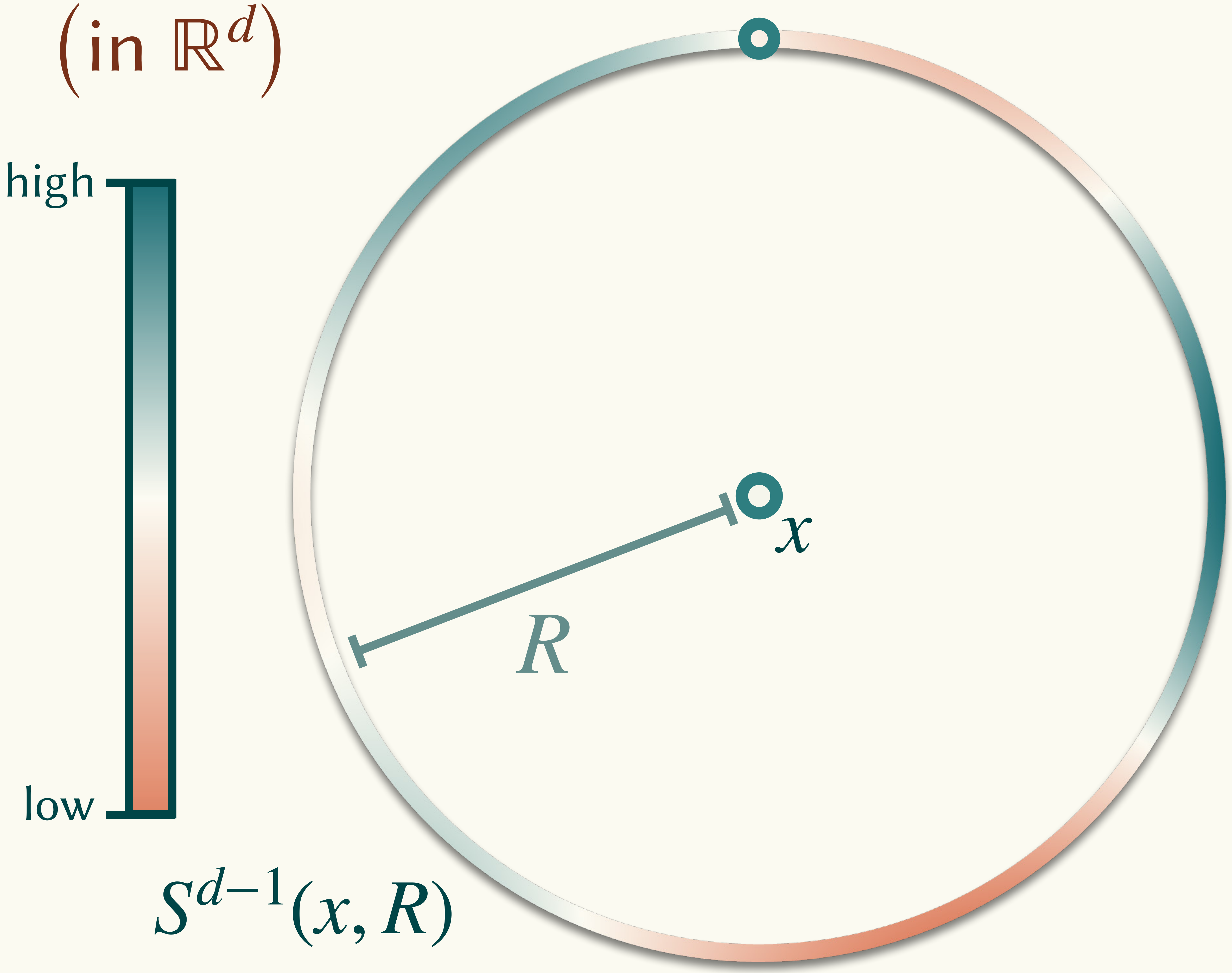
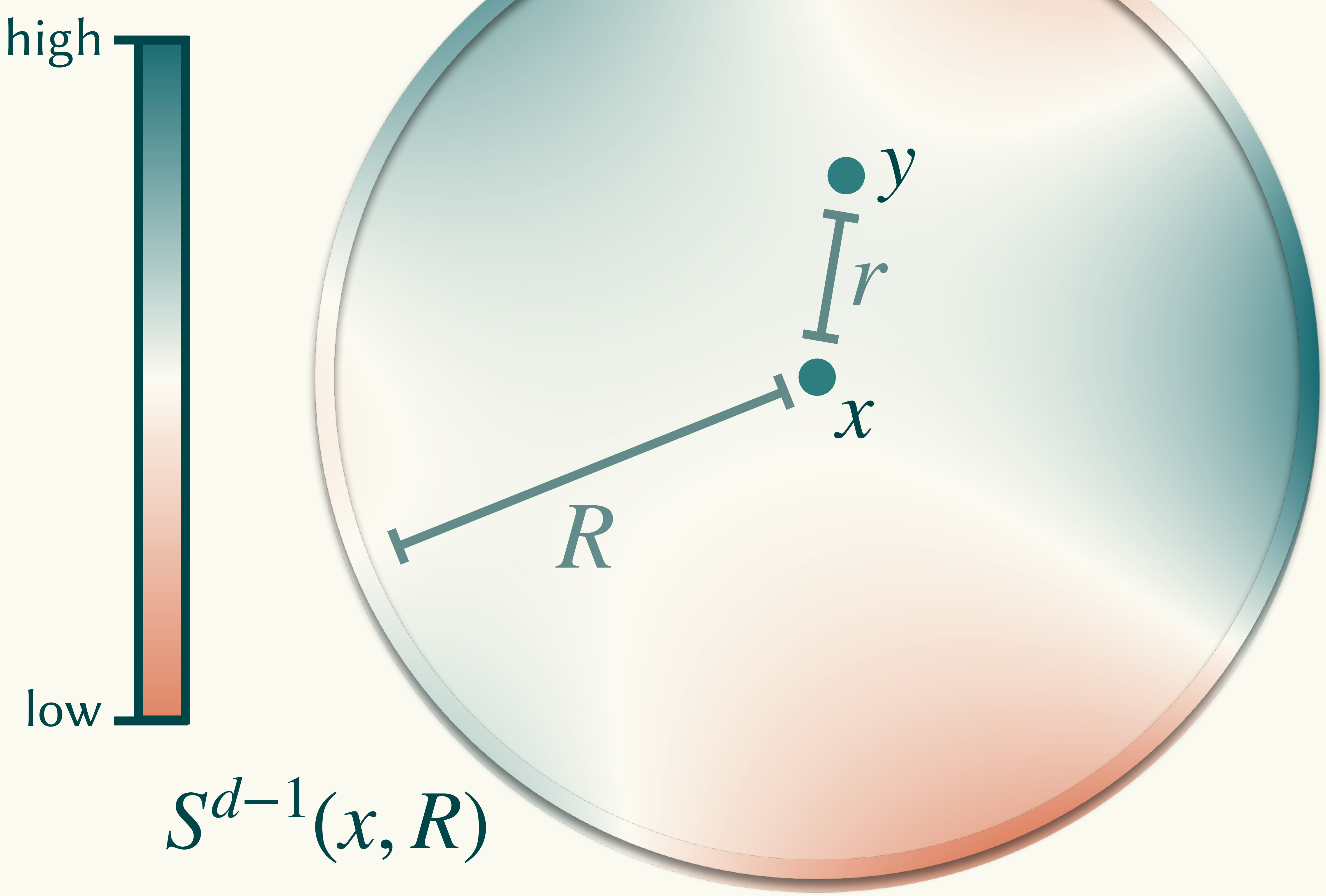


mean value property

$$f(x) = \frac{1}{\mathrm{vol}(S)} \int_S f(z)\, dz$$

The Poisson Kernel

(in $\mathbb{R}^d$)



$S^{d-1}(x, R)$

mean value property

$$f(x) = \frac{1}{\mathrm{vol}(S)} \int_S f(z)\, dz$$

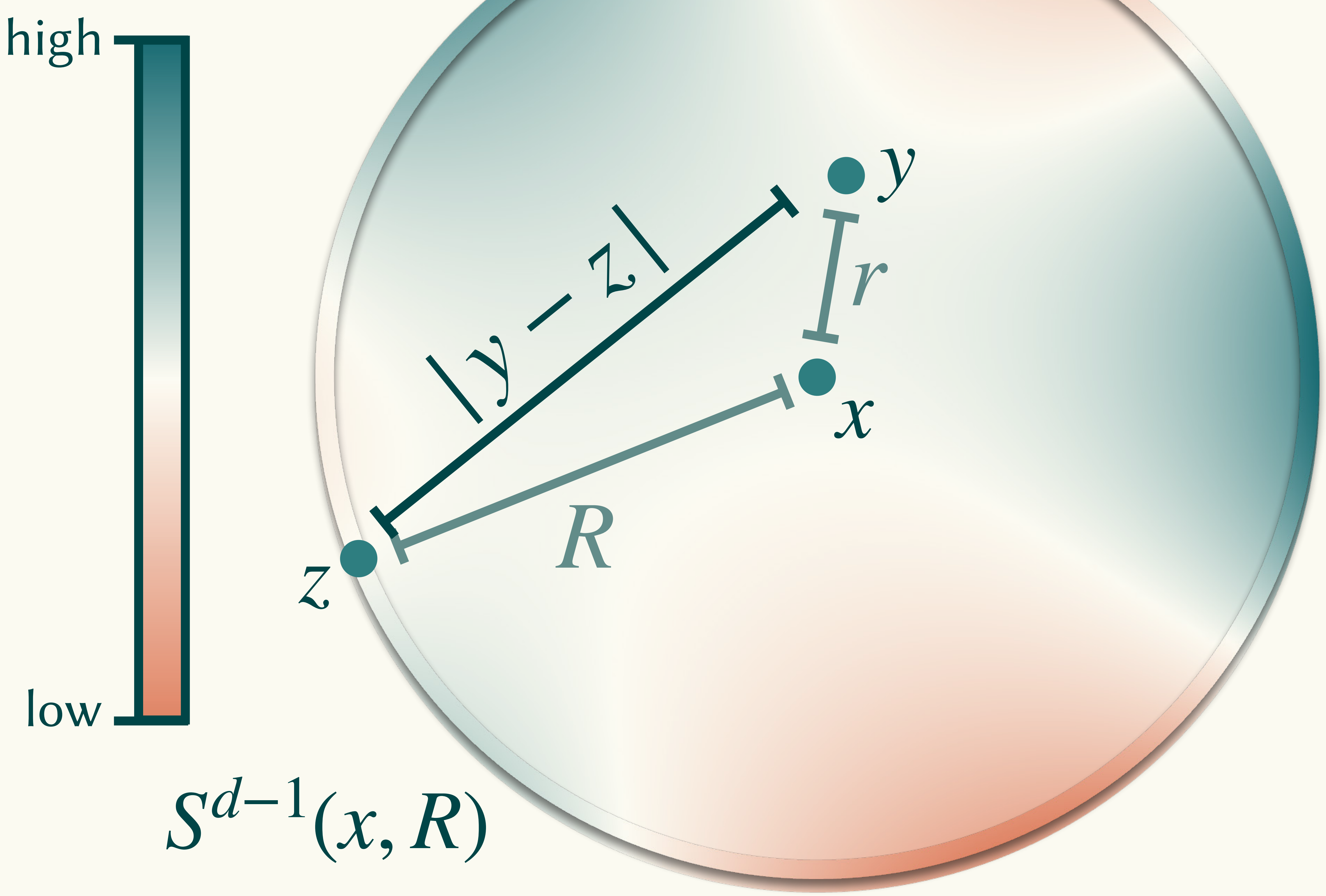
Poisson kernel

$$f(y) = \frac{1}{\mathrm{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d} |y - z|^d} f(z)\, dz$$

weighted average

The Poisson Kernel

(in $\mathbb{R}^d$)



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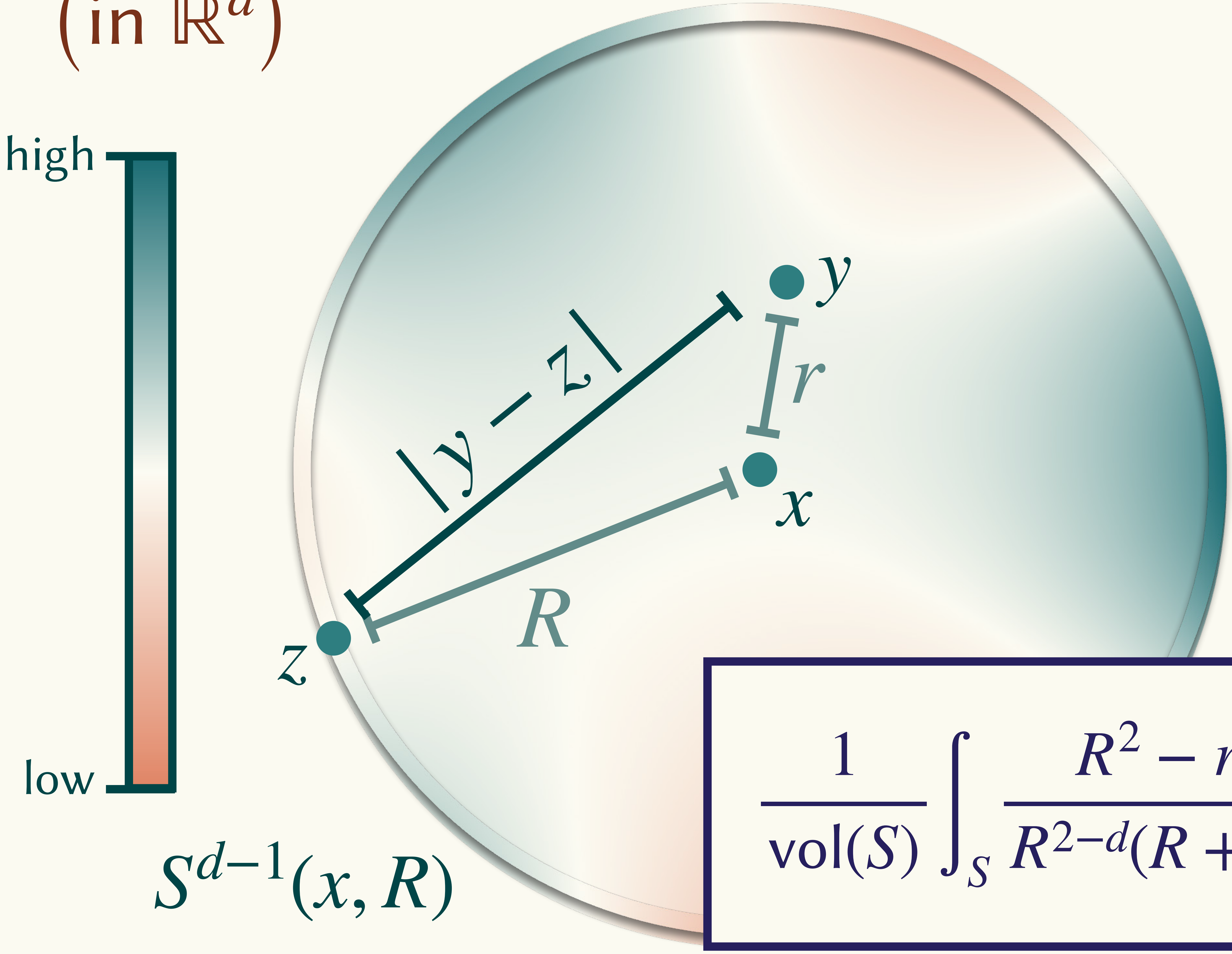
$$R - r \leq |y - z| \leq R + r$$

The Poisson Kernel

(in $\mathbb{R}^d$)



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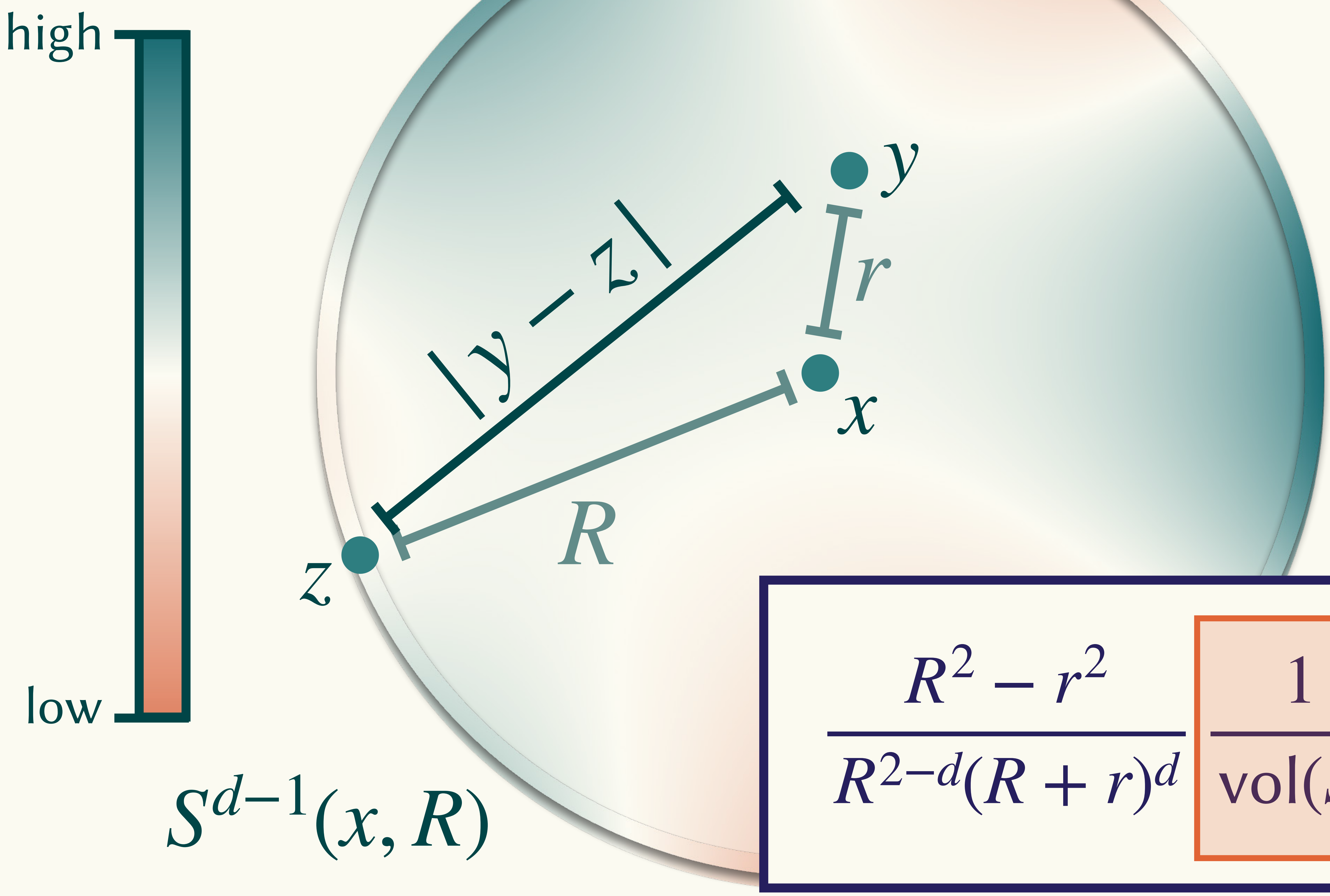
The Poisson Kernel

$\left(\text{in } \mathbb{R}^d\right)$



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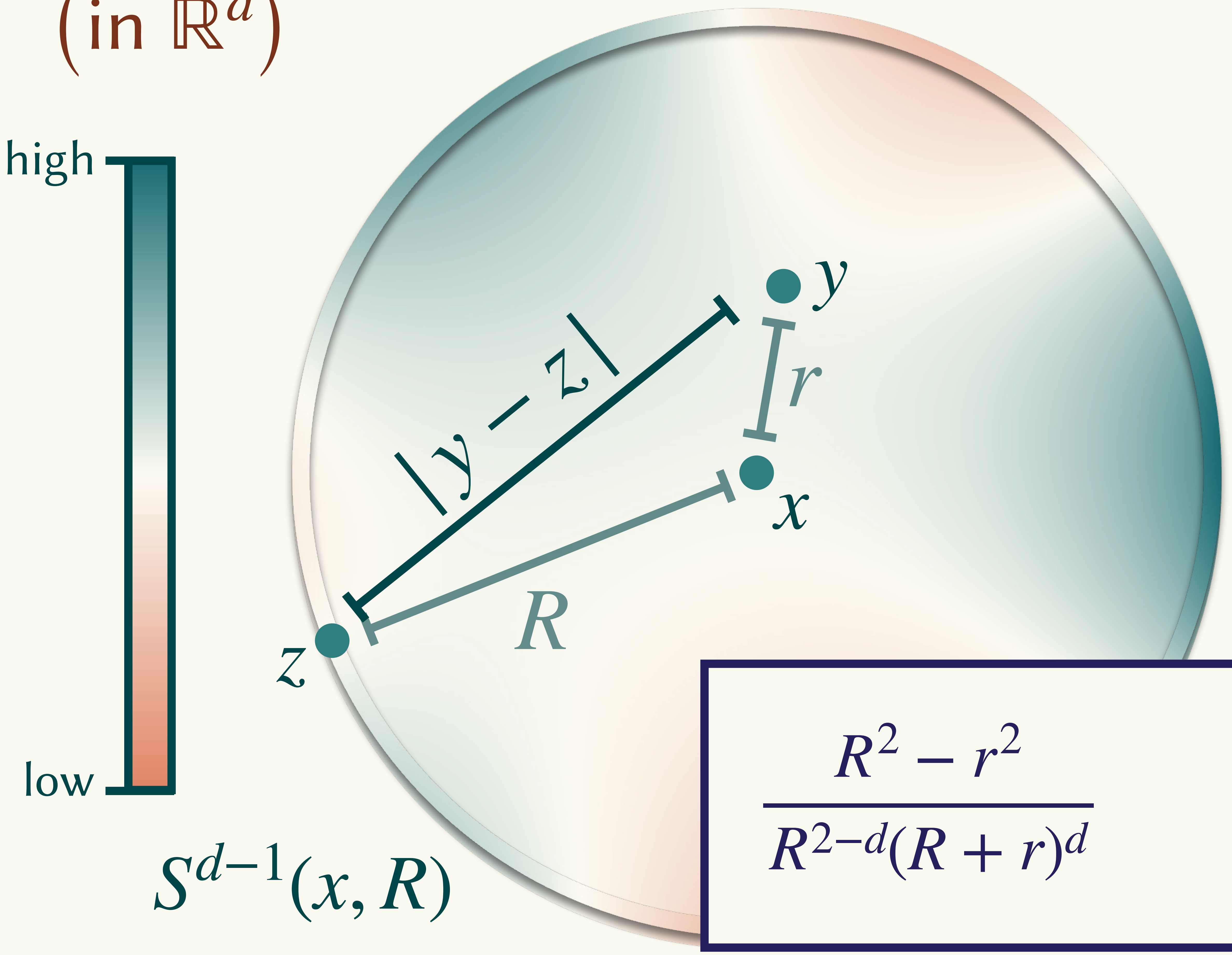

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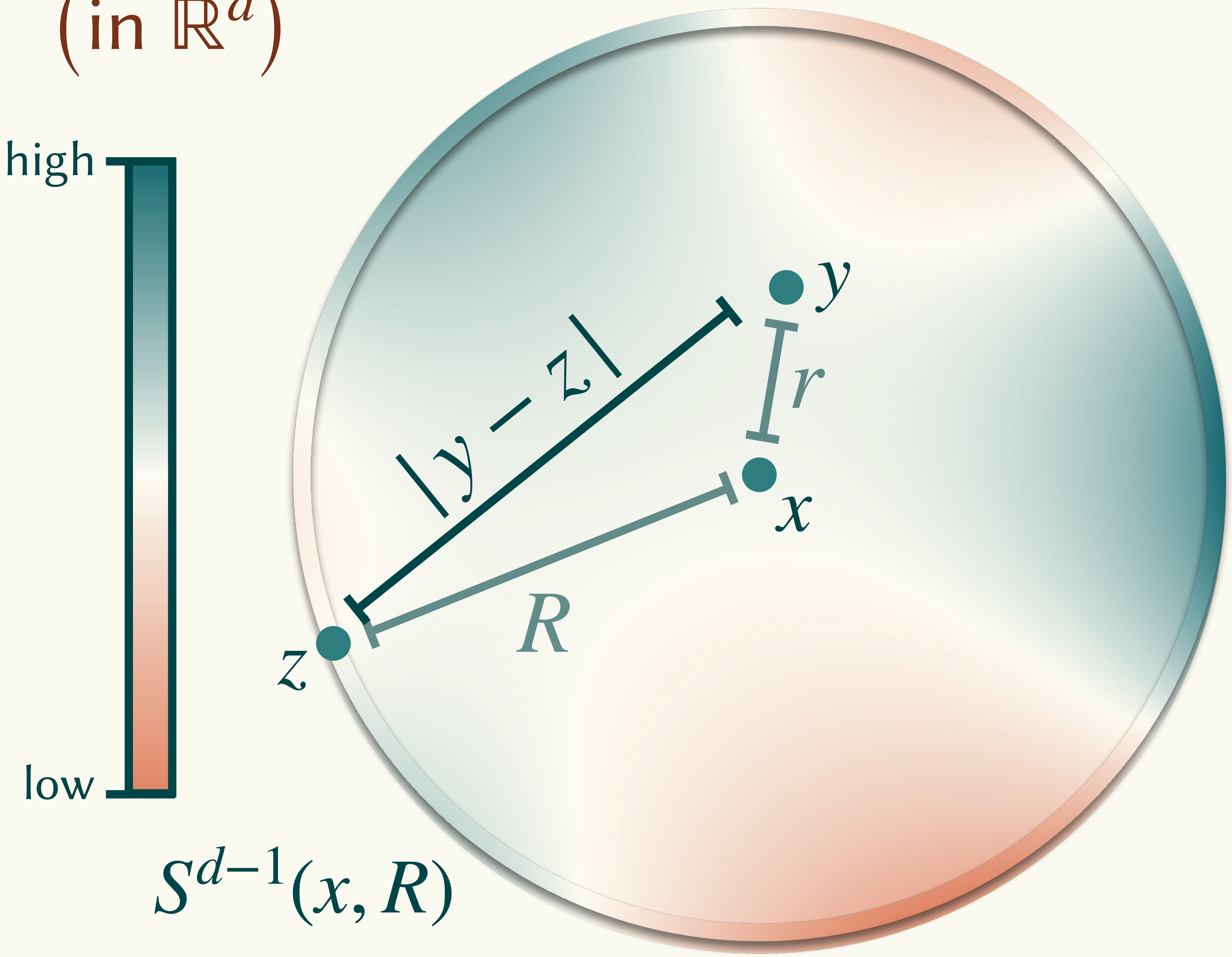
$$\frac{R^2 - r^2}{R^{2-d}(R + r)^d} \quad f(x) \quad \leq f(y) \leq \frac{R^2 - r^2}{R^{2-d}(R - r)^d} \quad f(x)$$

Harnack's Inequality

(in $\mathbb{R}^d$)



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Harnack's inequality

$$\frac{R^2 - r^2}{R^{2-d}(R + r)^d} f(x) \leq f(y) \leq \frac{R^2 - r^2}{R^{2-d}(R - r)^d} f(x)$$

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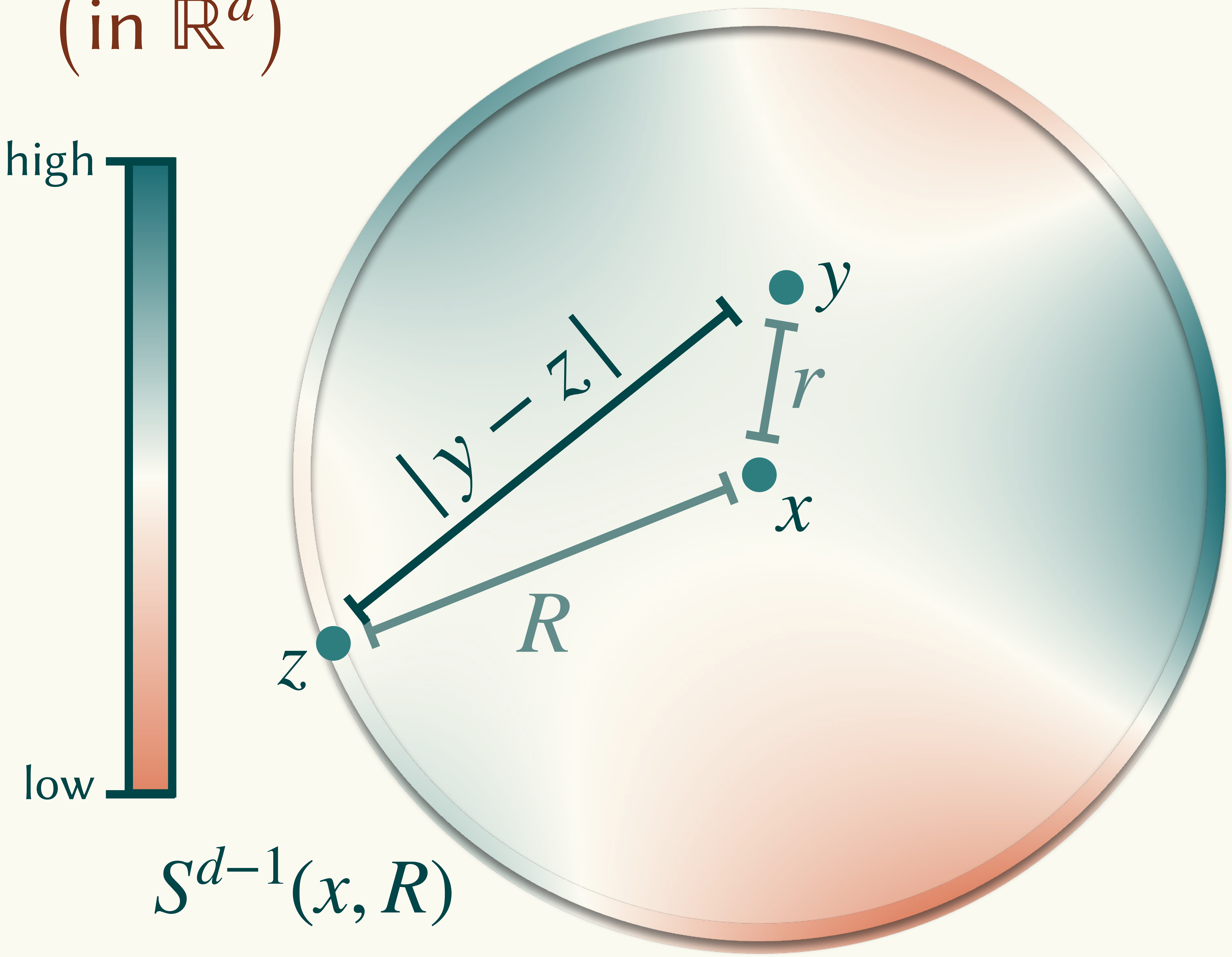

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Harnack's inequality

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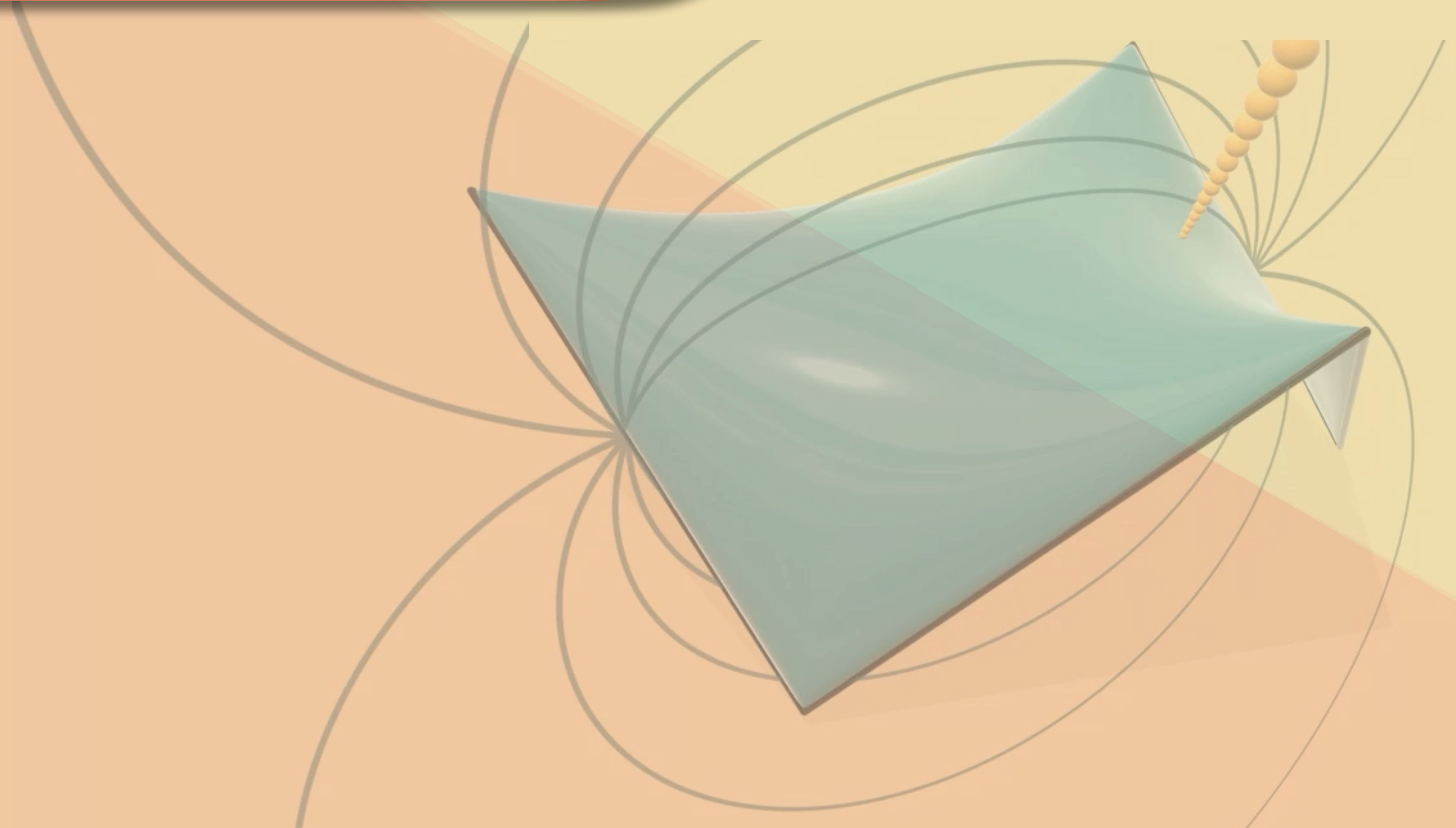
II. Harnack Tracing



Distance bounds from Harnack's inequality

Let f be a positive harmonic function on a ball:

$$\frac{1 - r/R}{(1 + r/R)^2} f(x) \leq f(y) \leq \frac{1 + r/R}{(1 - r/R)^2} f(x)$$

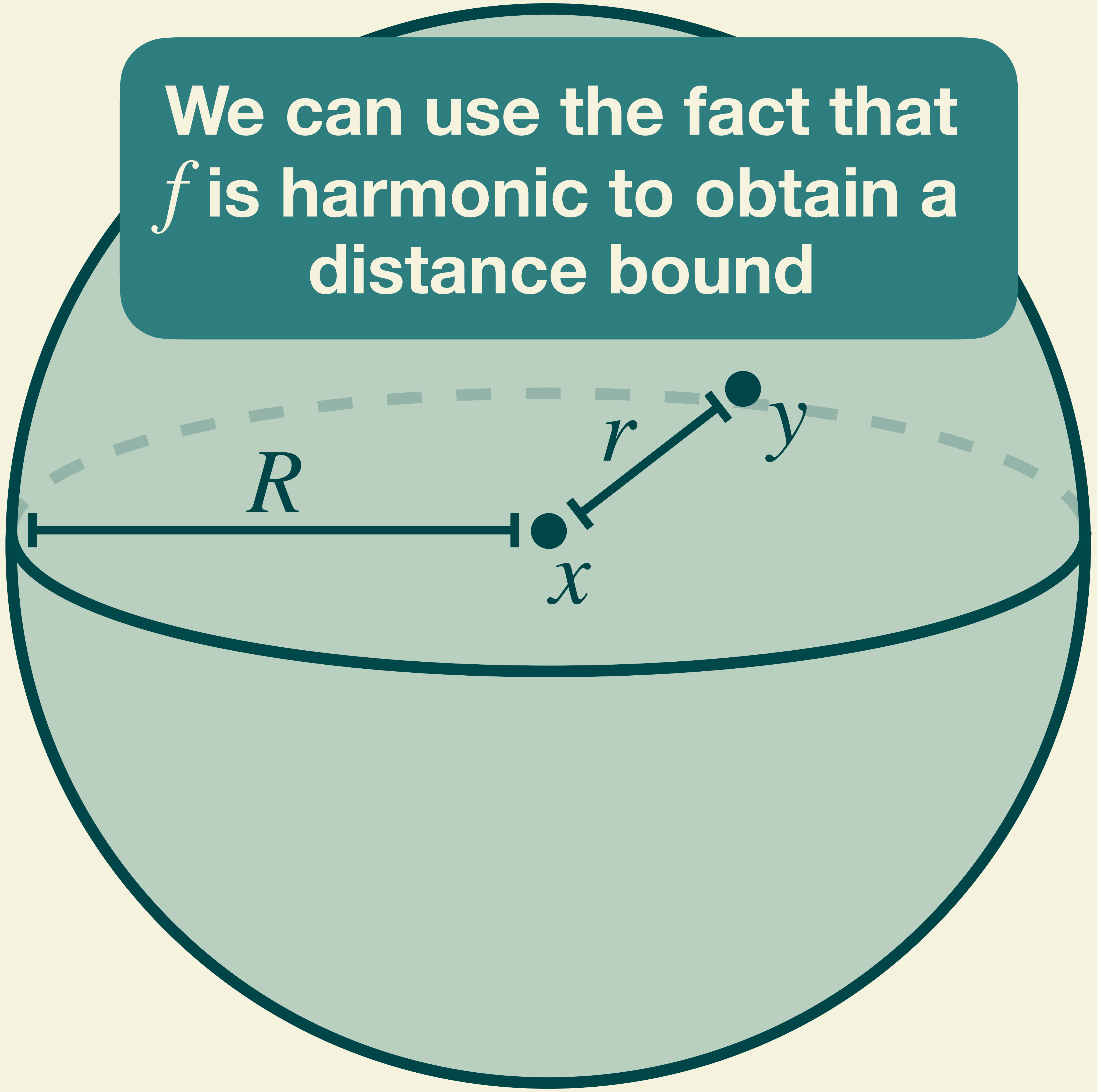
lower bound — upper bound

always safe to take step of size

$$\frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$$

where $a = \frac{f(x)}{f^*}$

We can use the fact that f is harmonic to obtain a distance bound



Distance bounds from Harnack's inequality

Let f be a positive harmonic function on a ball:

What if f is not positive?

Just add a constant to make it positive on the ball

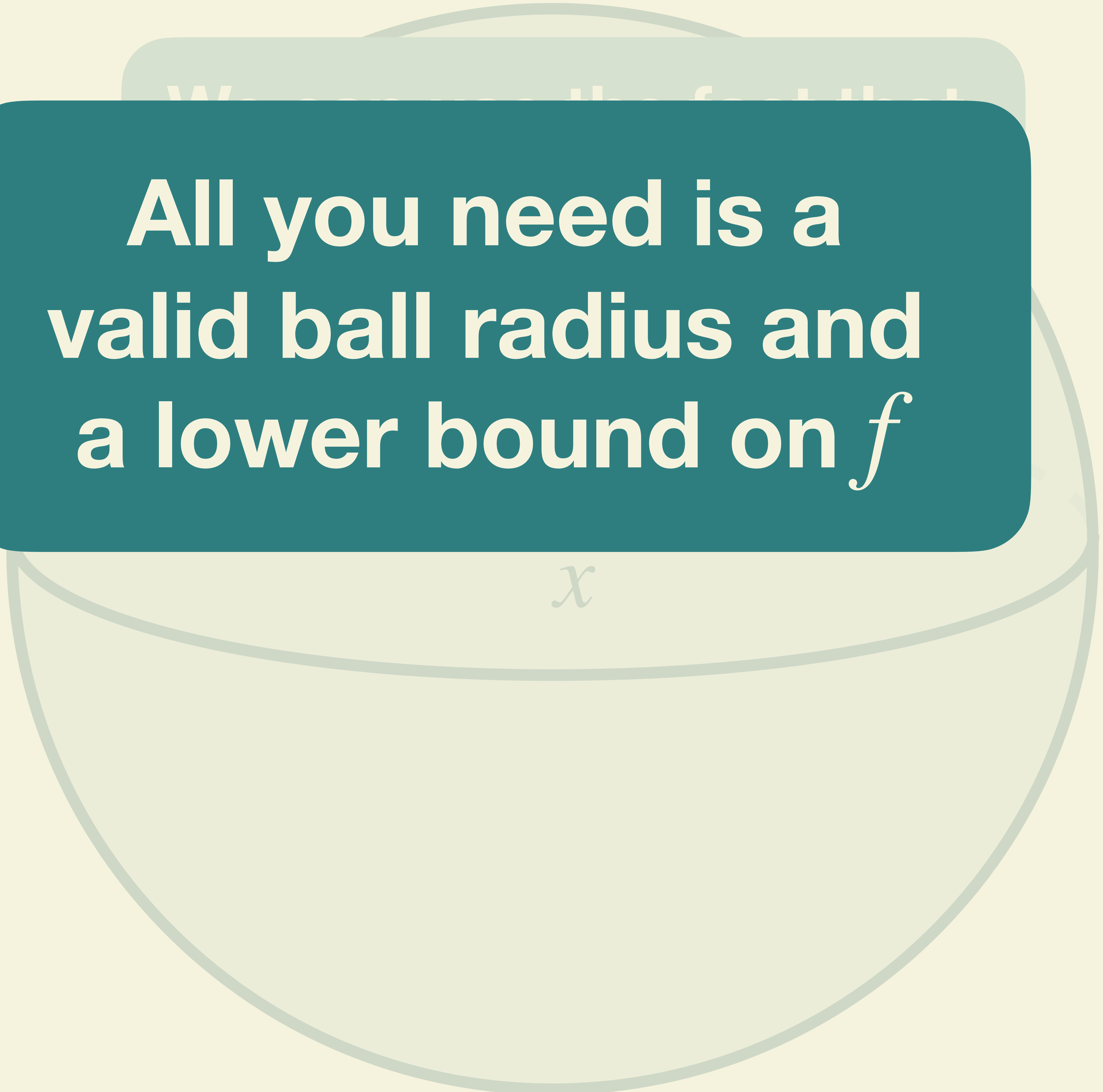
lower bound

upper bound

always safe to take step of size

$$r = \frac{R}{2}\left|a + 2 - \sqrt{a^2 + 8a}\right|,$$

where $a = \frac{f(x)}{f^*}$



All you need is a valid ball radius and a lower bound on f

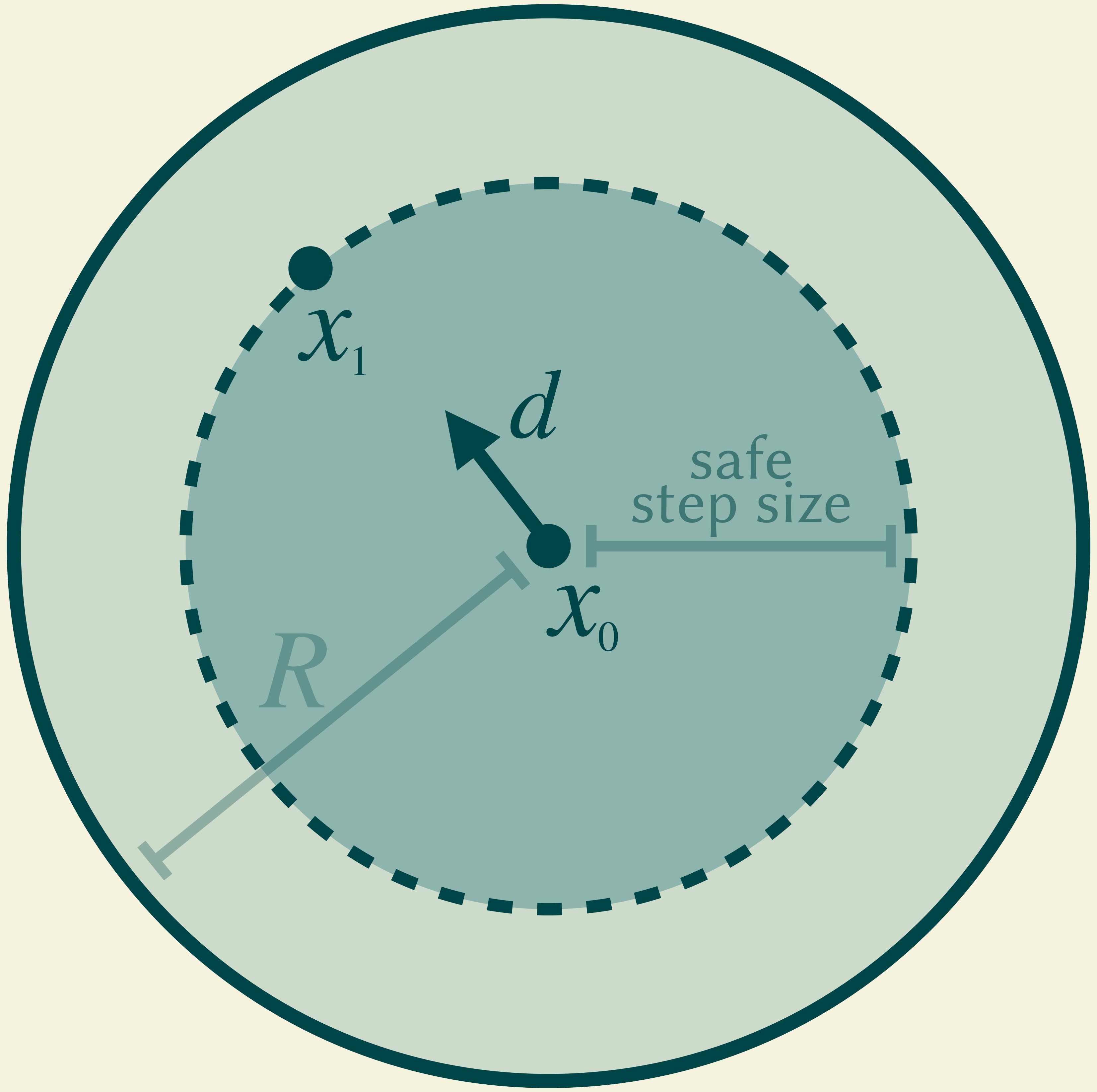
Algorithm sketch

Harnack Tracing

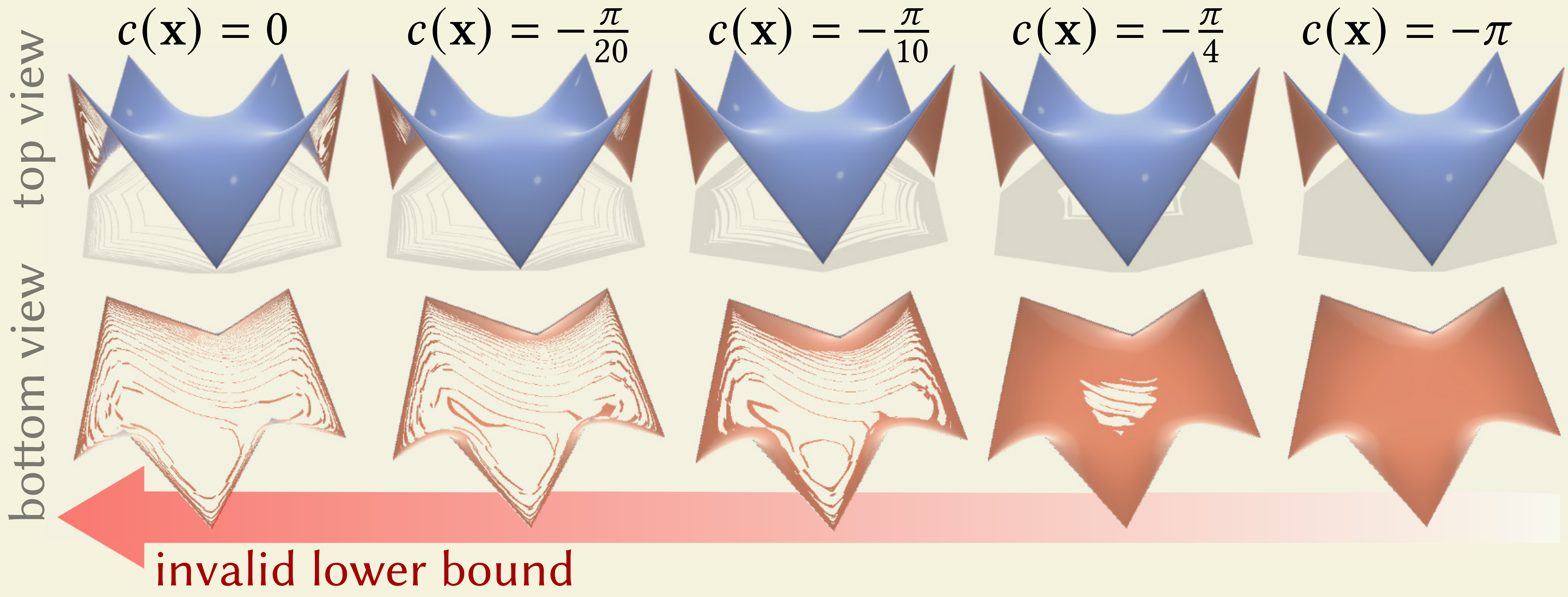
Starting from point x_0 in direction d:

- Pick ball radius
- Shift f to be positive on ball
- Calculate safe step size
- Take safe step in ray direction

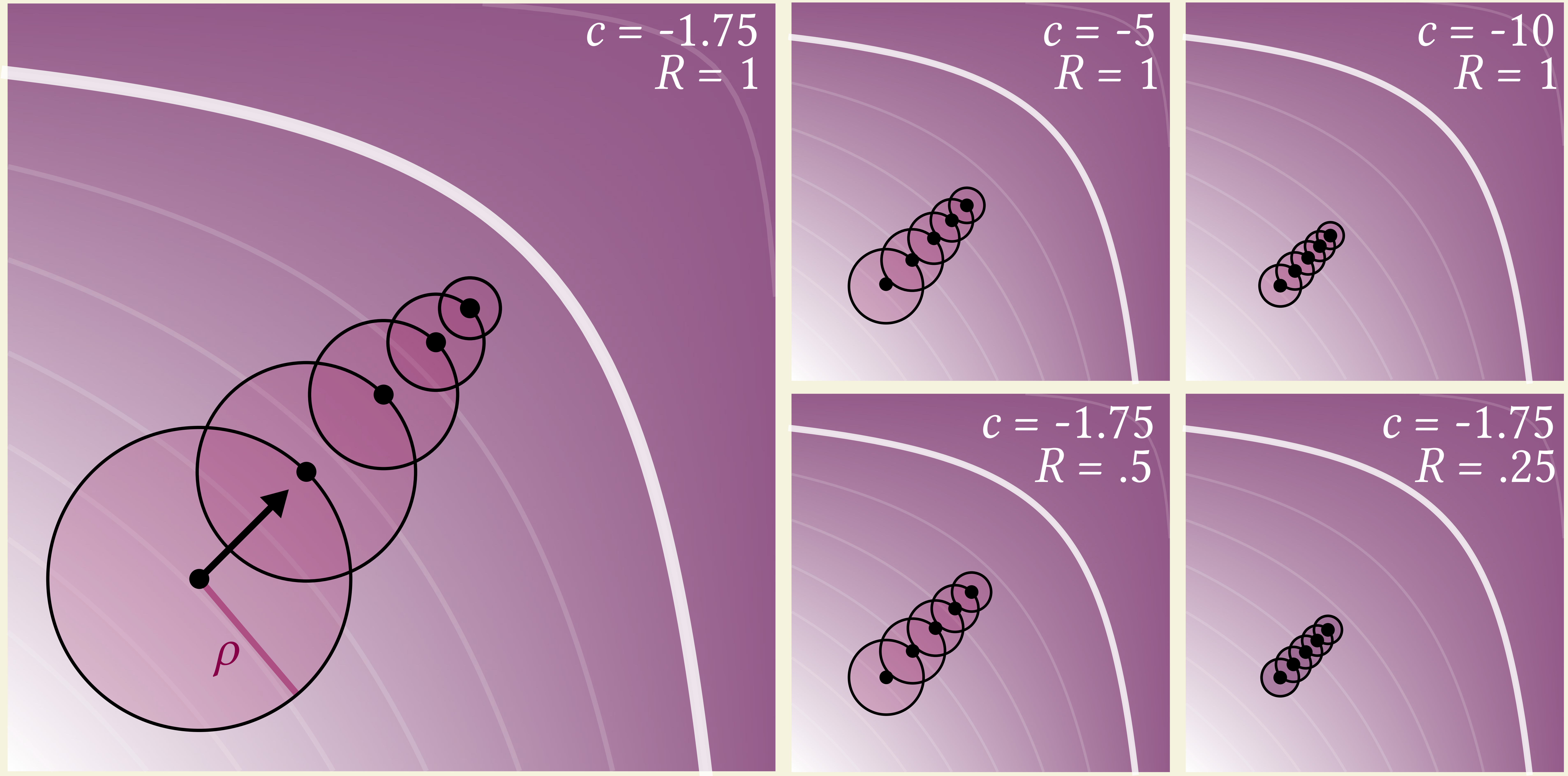
Repeat until f is sufficiently close to f^*



Invalid lower bounds



Balancing the radius and shift



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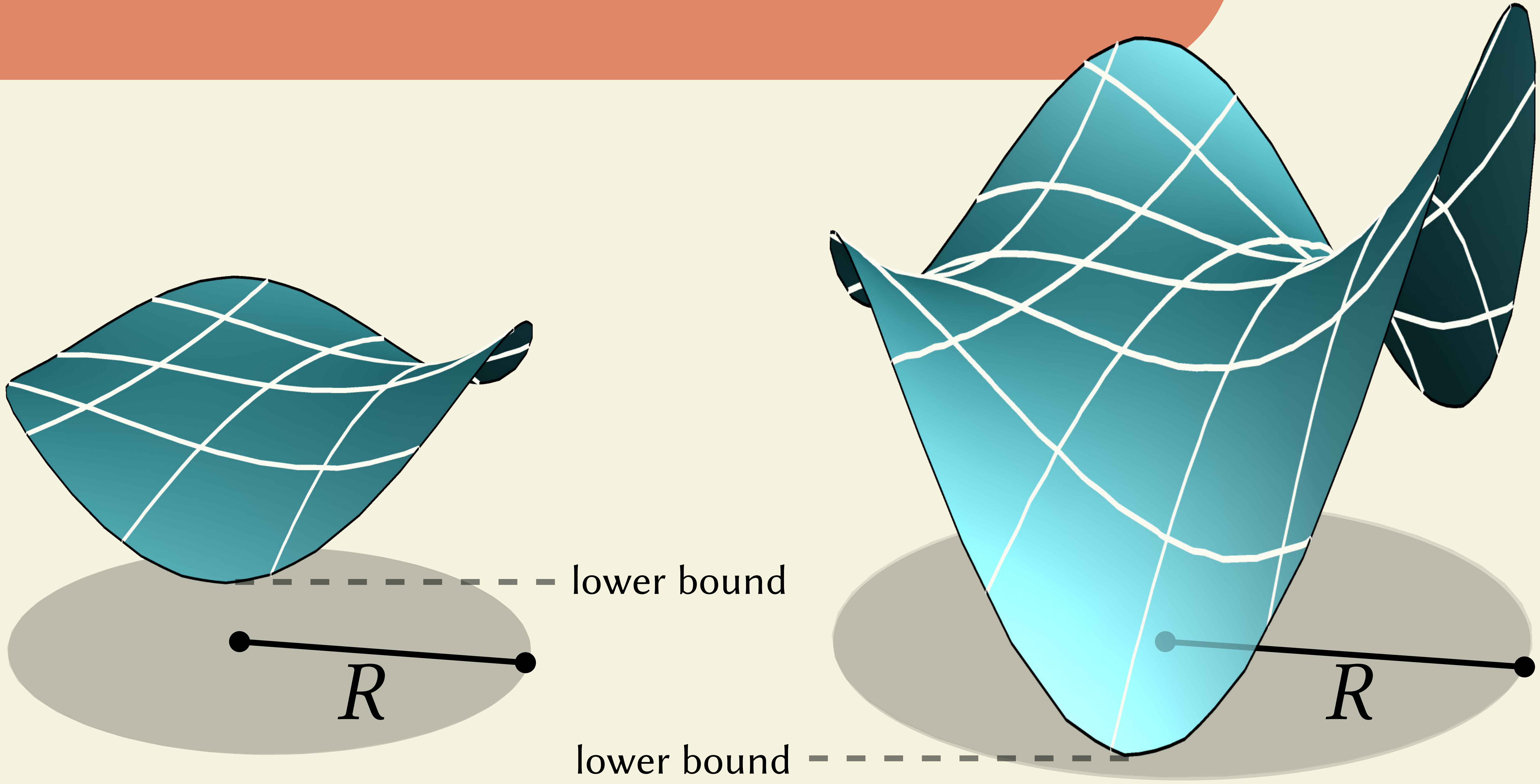


smaller radius, larger shift

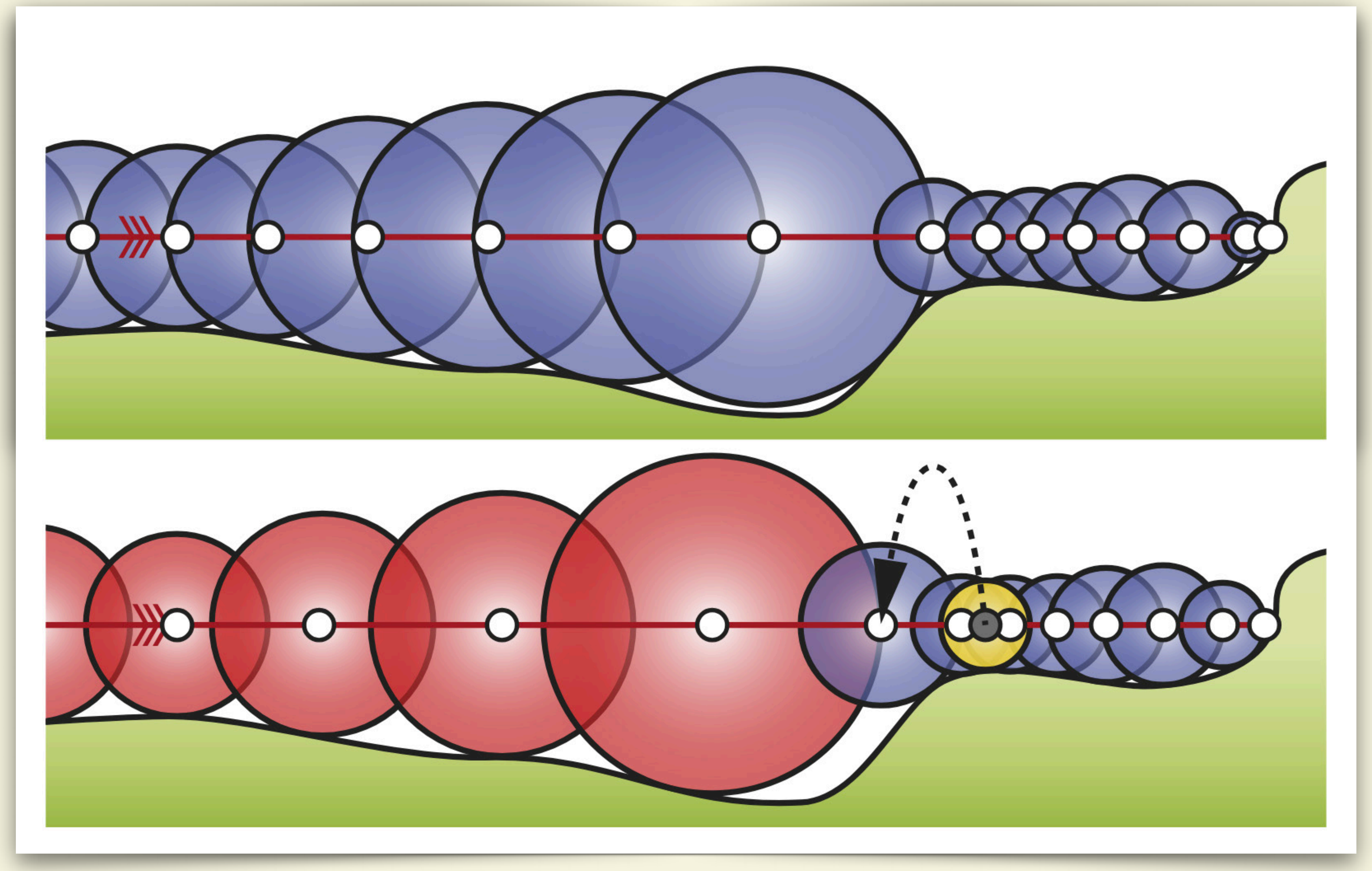
larger radius, smaller shift

Sphere tracing acceleration

[Keinert *et al.* 2014]: “over-stepping”

conservative steps

valid oversteps



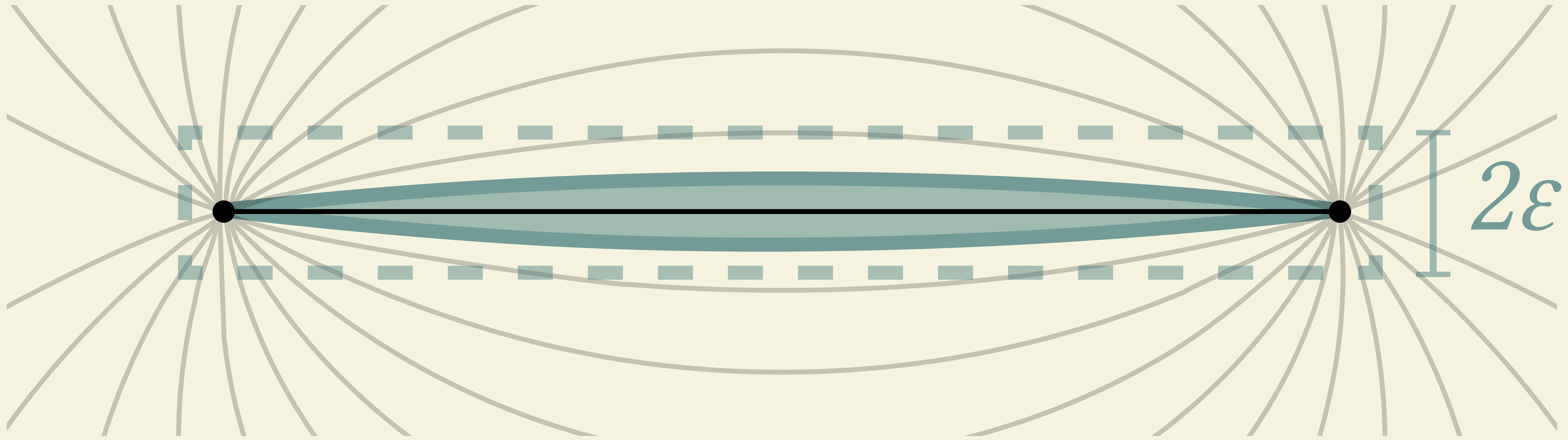
Acceleration: gradient termination

How do you decide when you have “hit” the surface?

$$|f(\mathbf{x}) - f^*| < \varepsilon$$

Acceleration: gradient termination

How do you decide when you have "hit" the surface?



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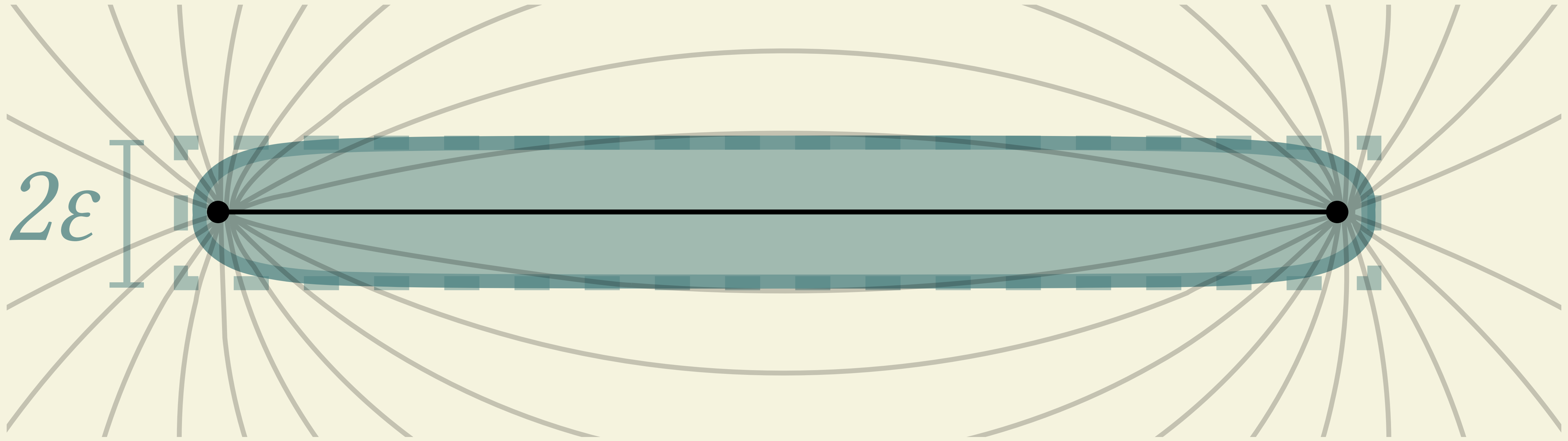
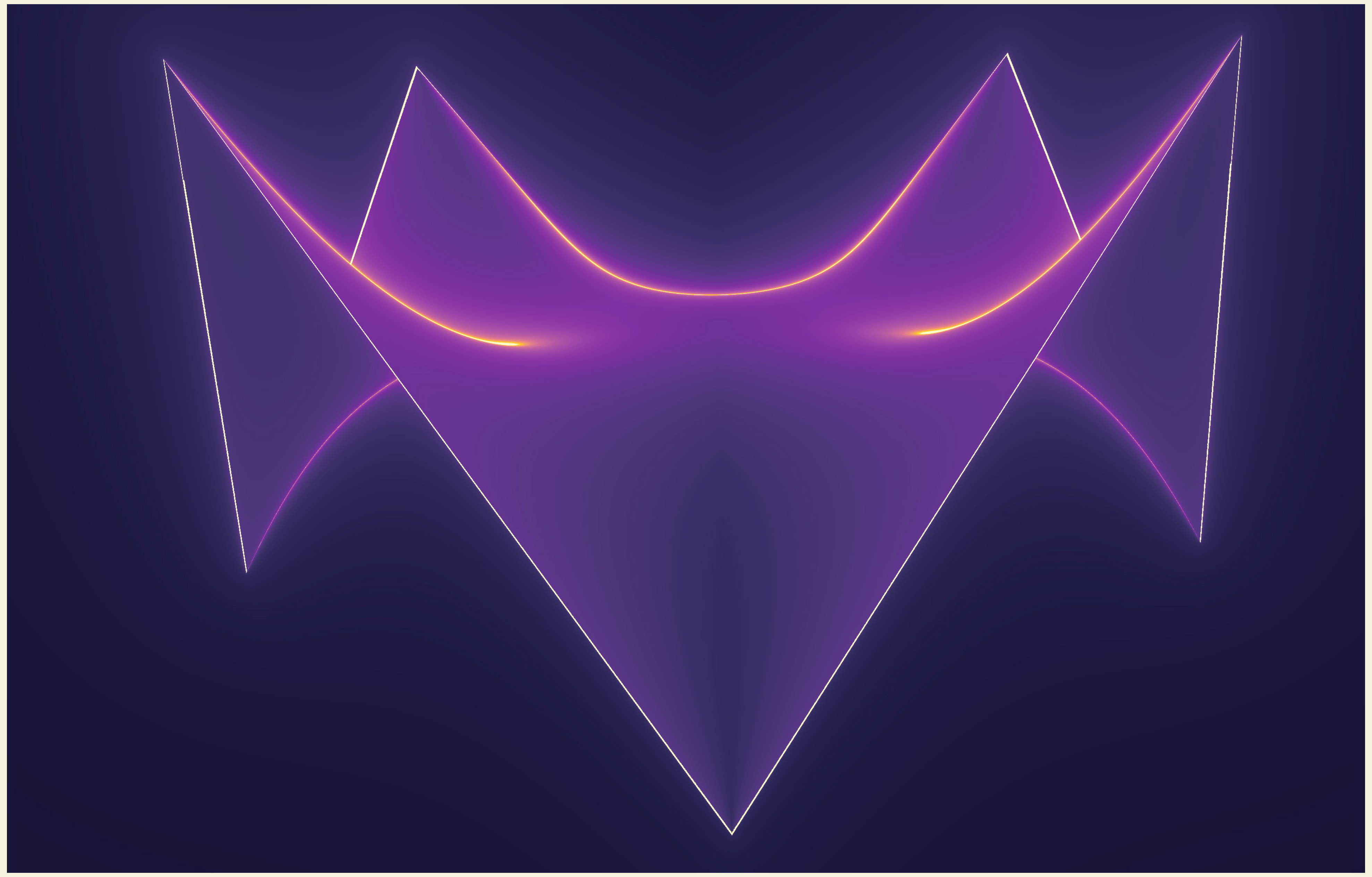


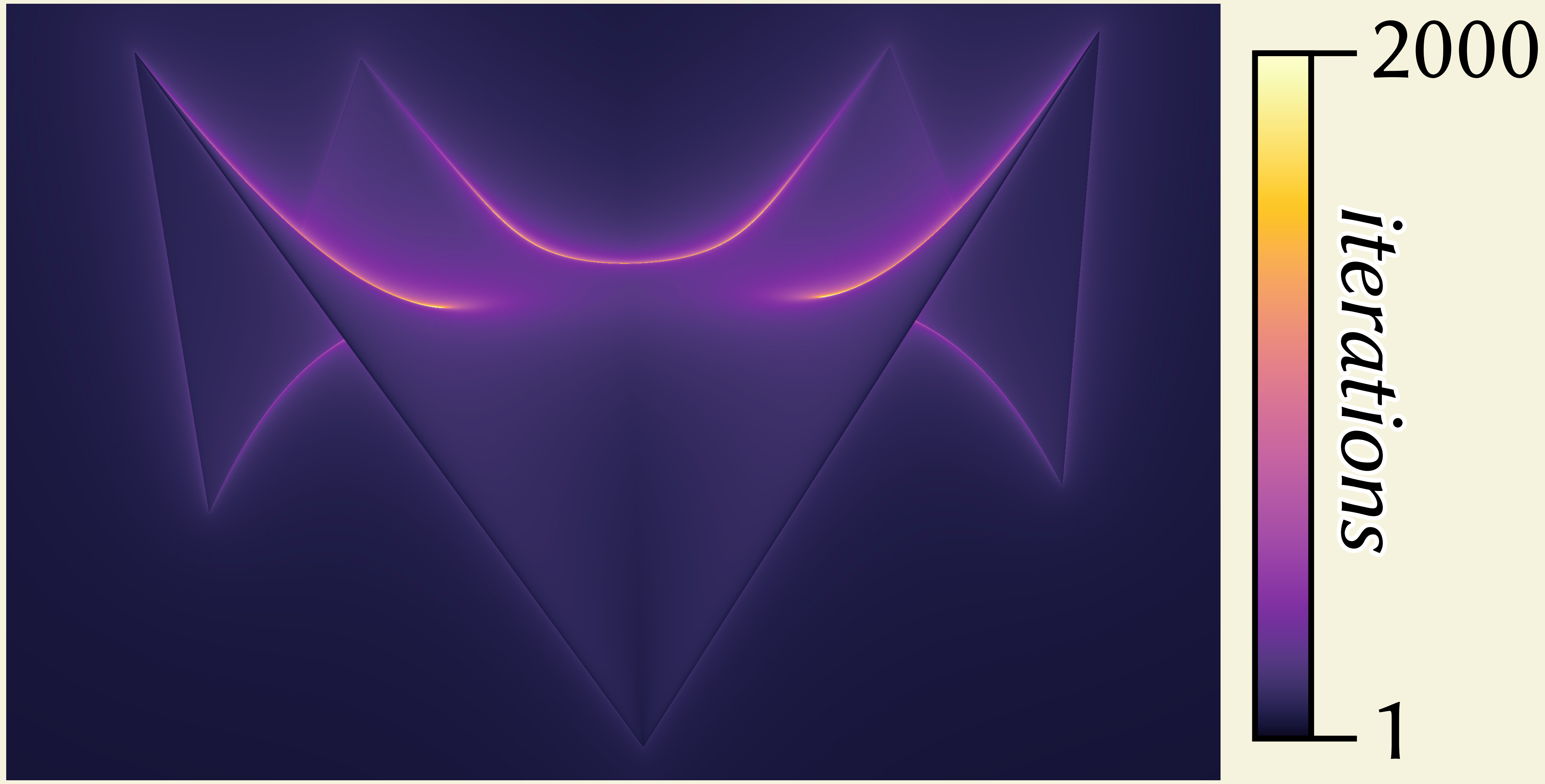
$$\frac{|f(\mathbf{x}) - f^*|}{|\nabla f(\mathbf{x})|} < \varepsilon$$

Acceleration: gradient termination

How do you decide when you have “hit” the surface?



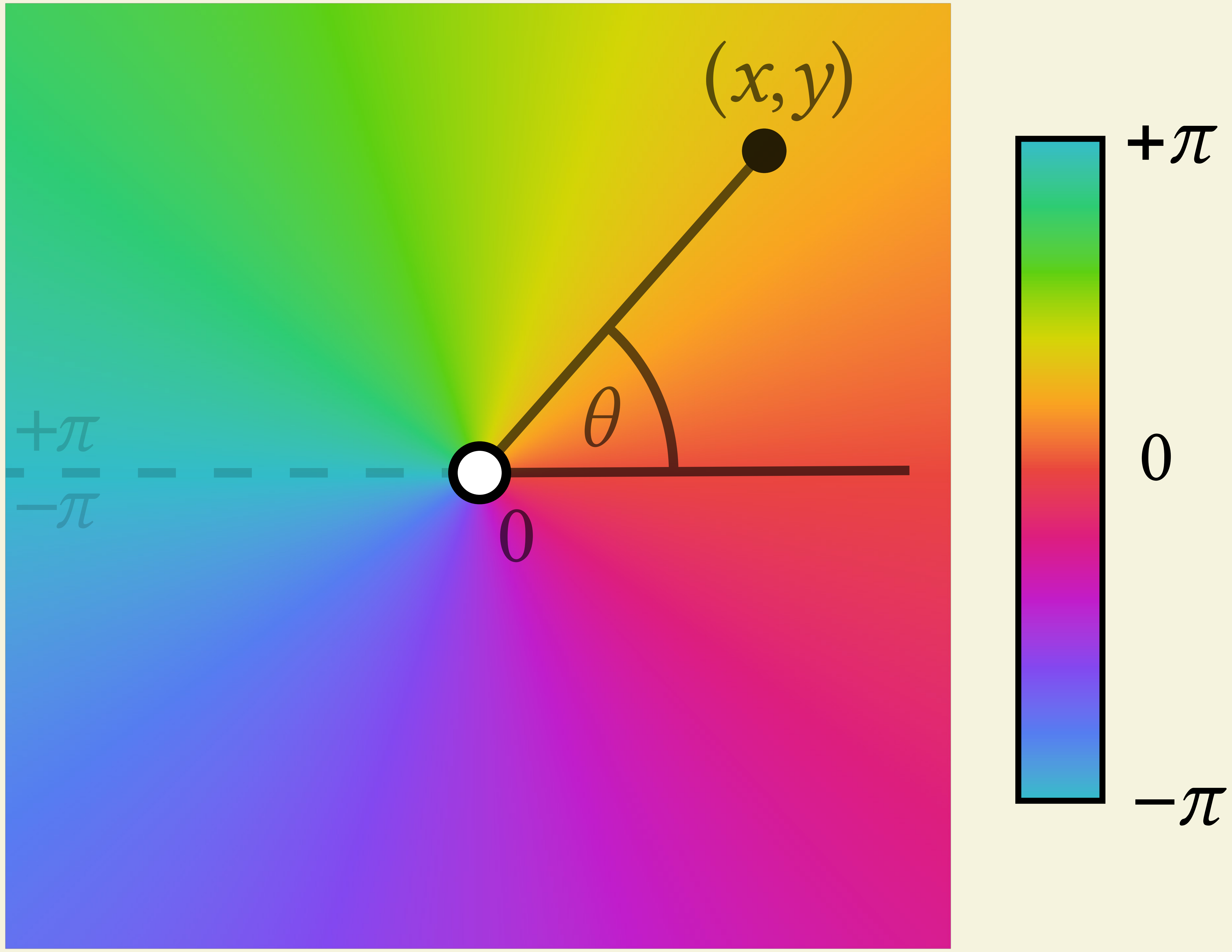
$$|f(\mathbf{x}) - f^*| < \varepsilon$$



$$\frac{|f(\mathbf{x}) - f^*|}{|\nabla f(\mathbf{x})|} < \varepsilon$$

Angle-valued functions

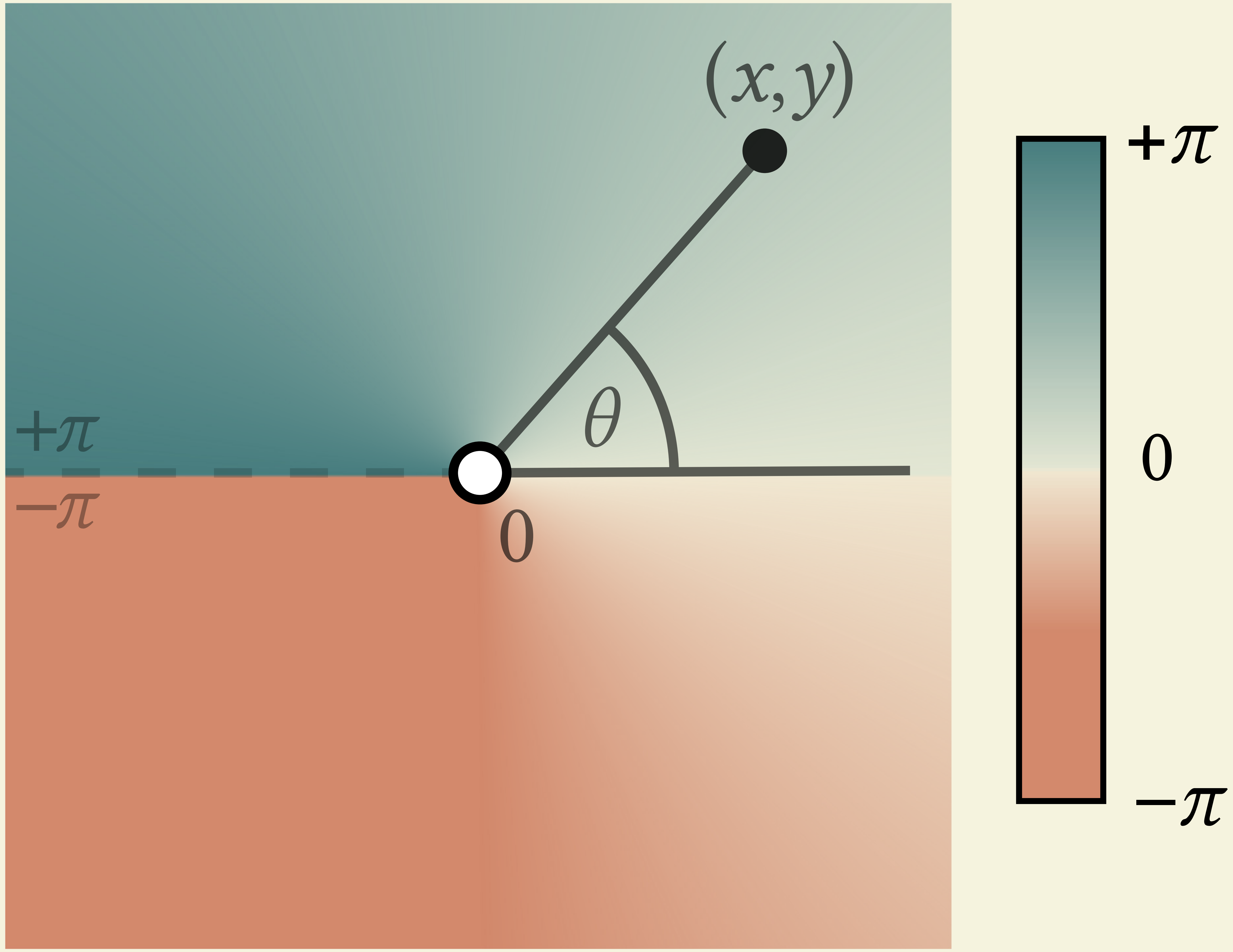
$$\theta(x, y) = \mathrm{atan2}(y, x)$$



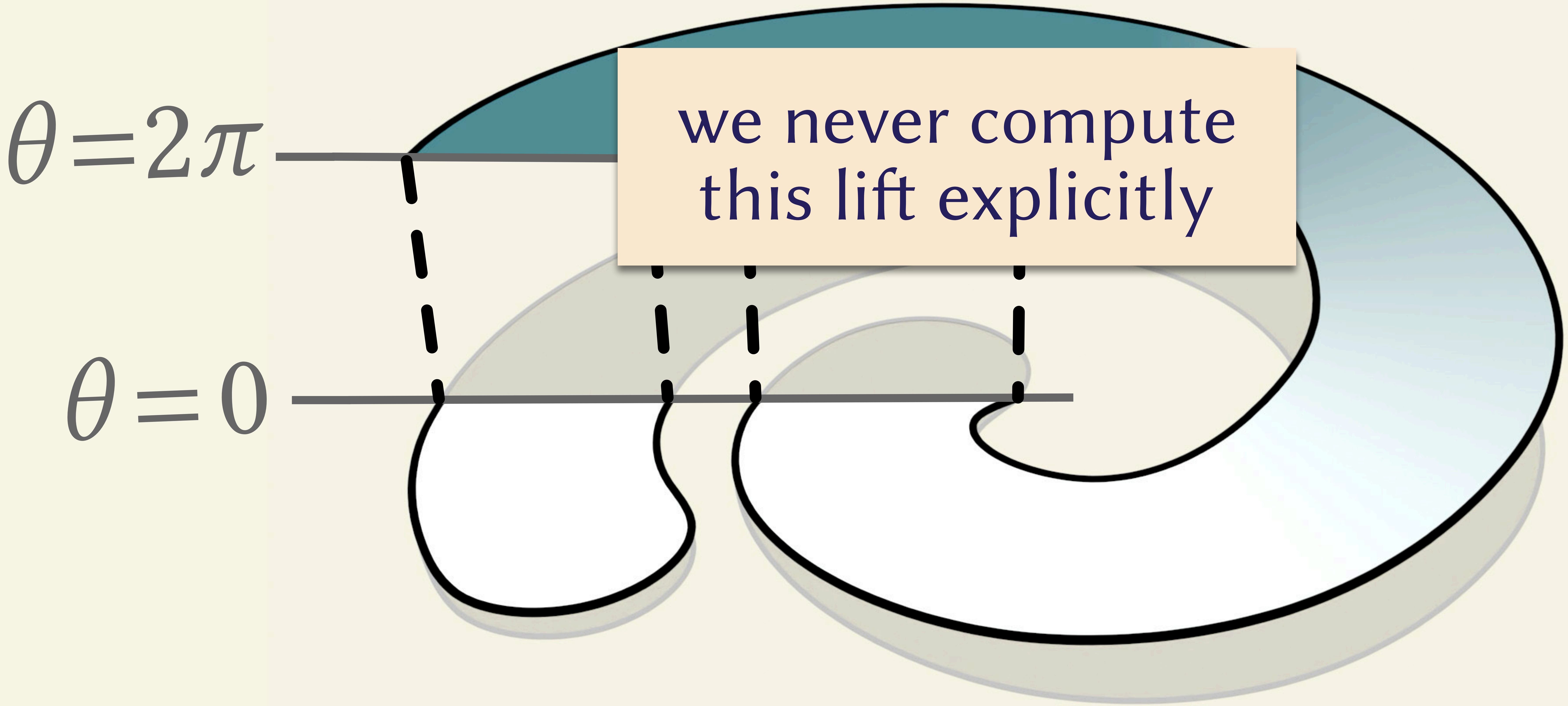
Angle-valued functions

$$\theta(x, y) = \mathrm{atan2}(y, x)$$

continuous when viewed modulo 2π



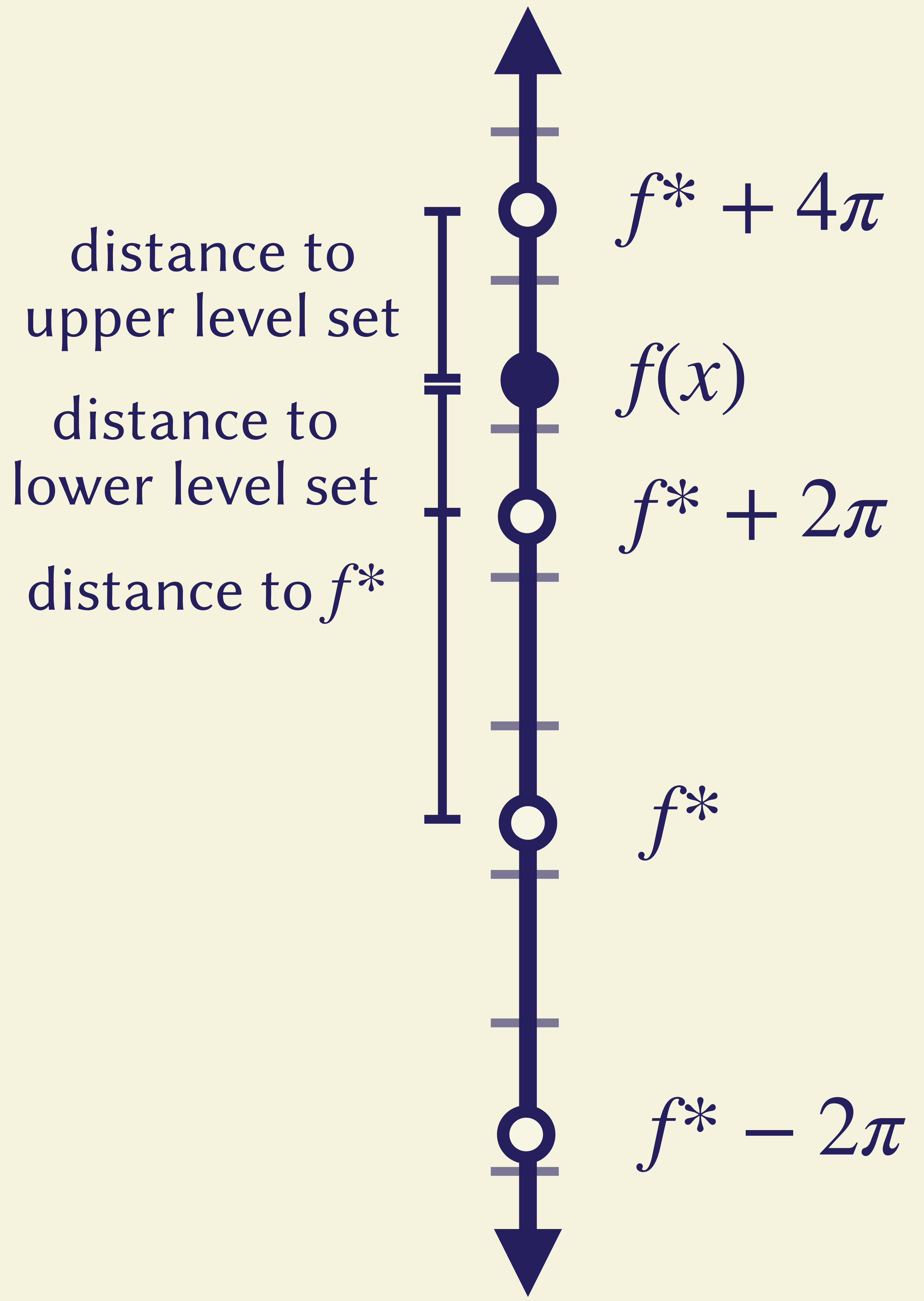
Angle-valued functions → continuous functions



Discontinuous Function

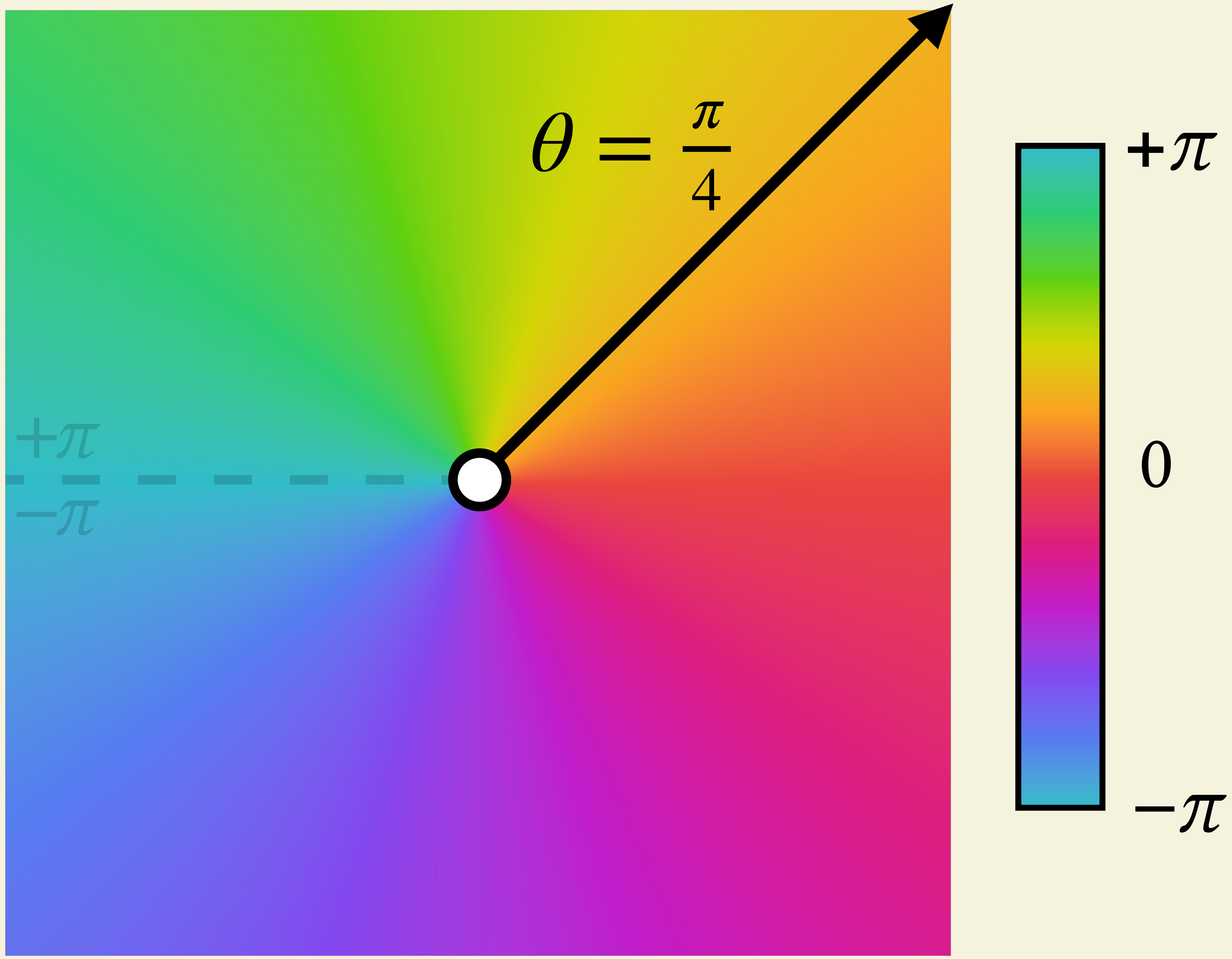
Continuous Function

In practice: look for level sets above and below

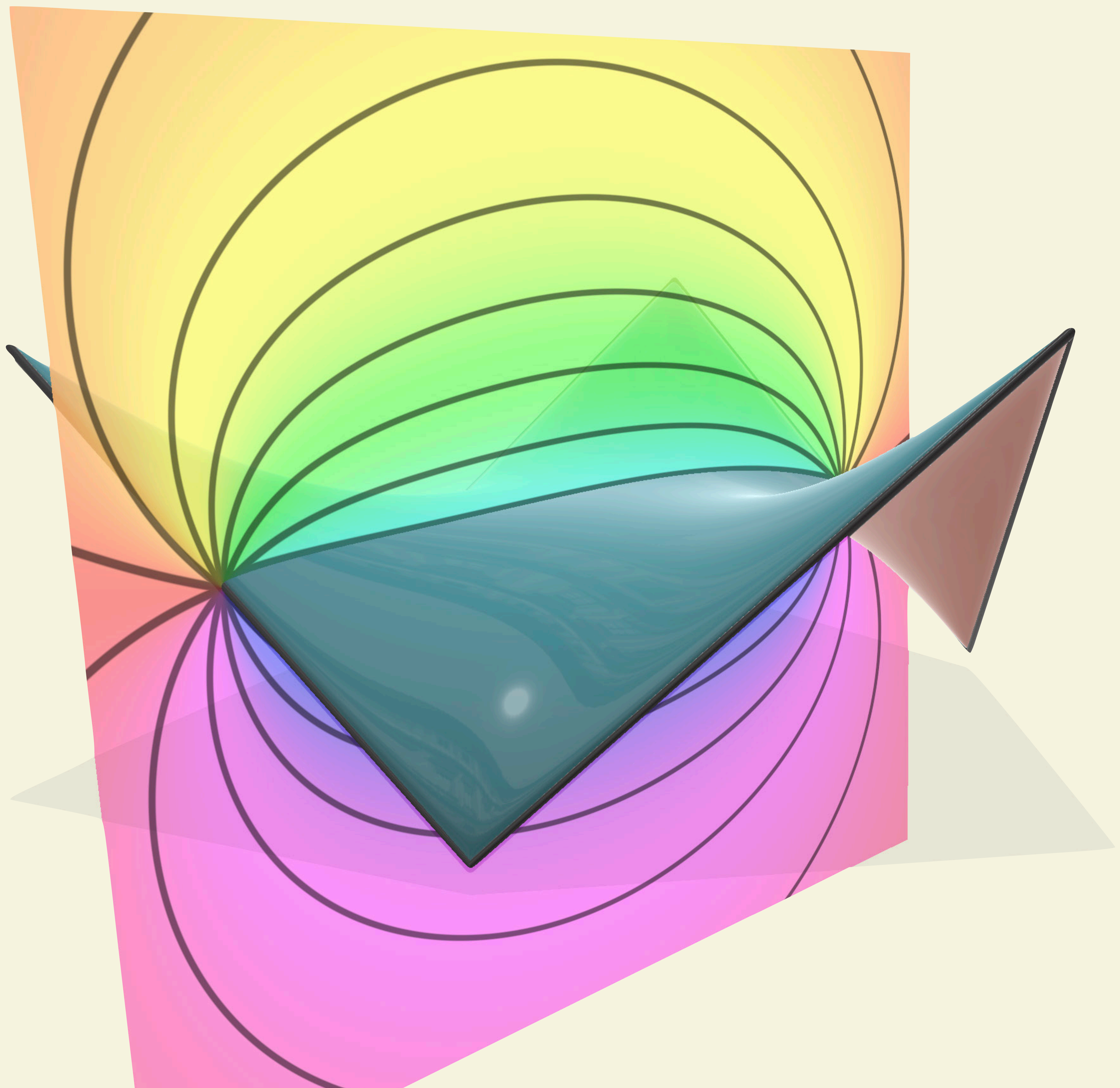


Angle-valued functions allow for boundaries

$$\theta(x, y) = \mathrm{atan2}(y, x)$$

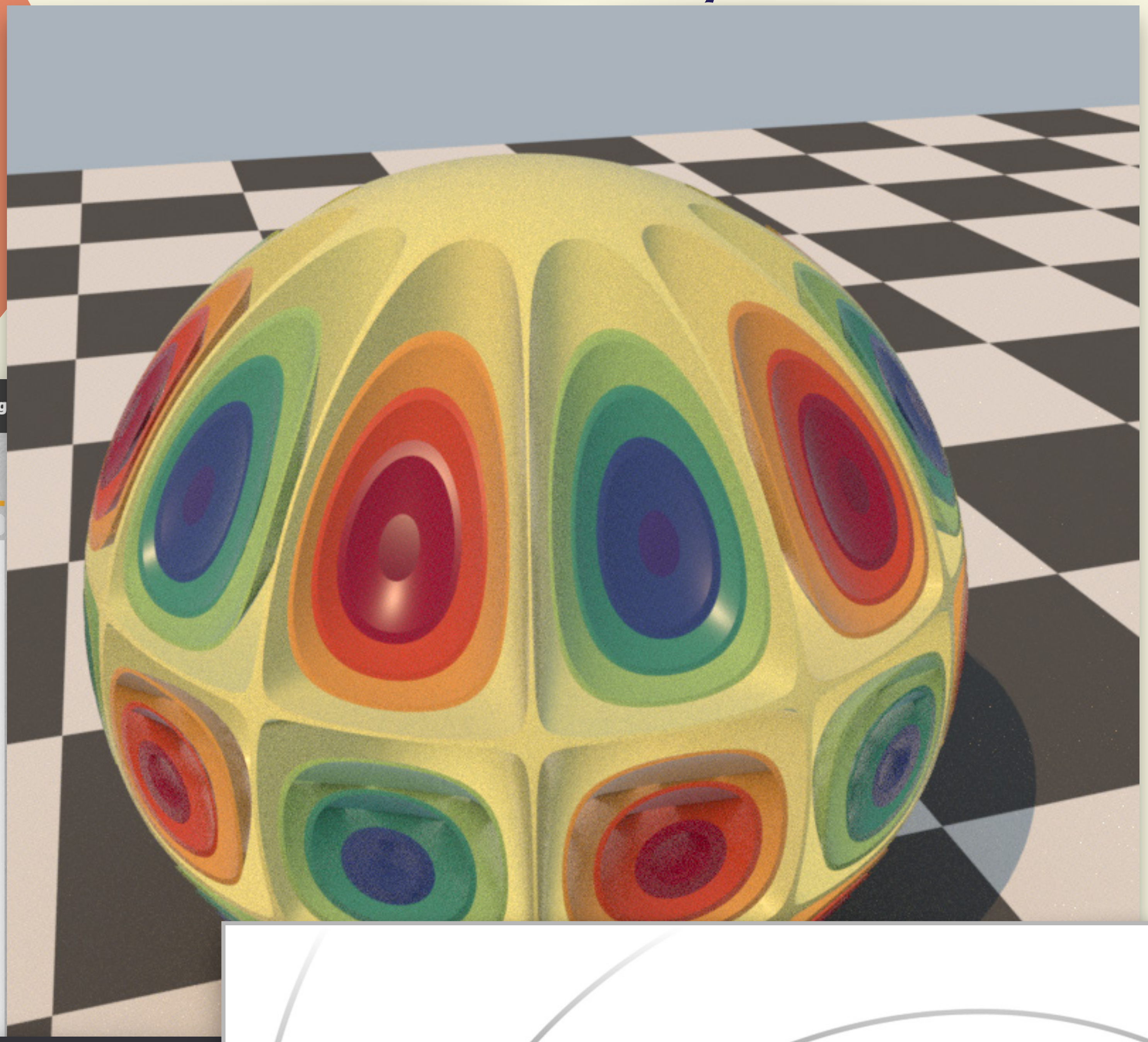


Angle-valued functions allow for boundaries

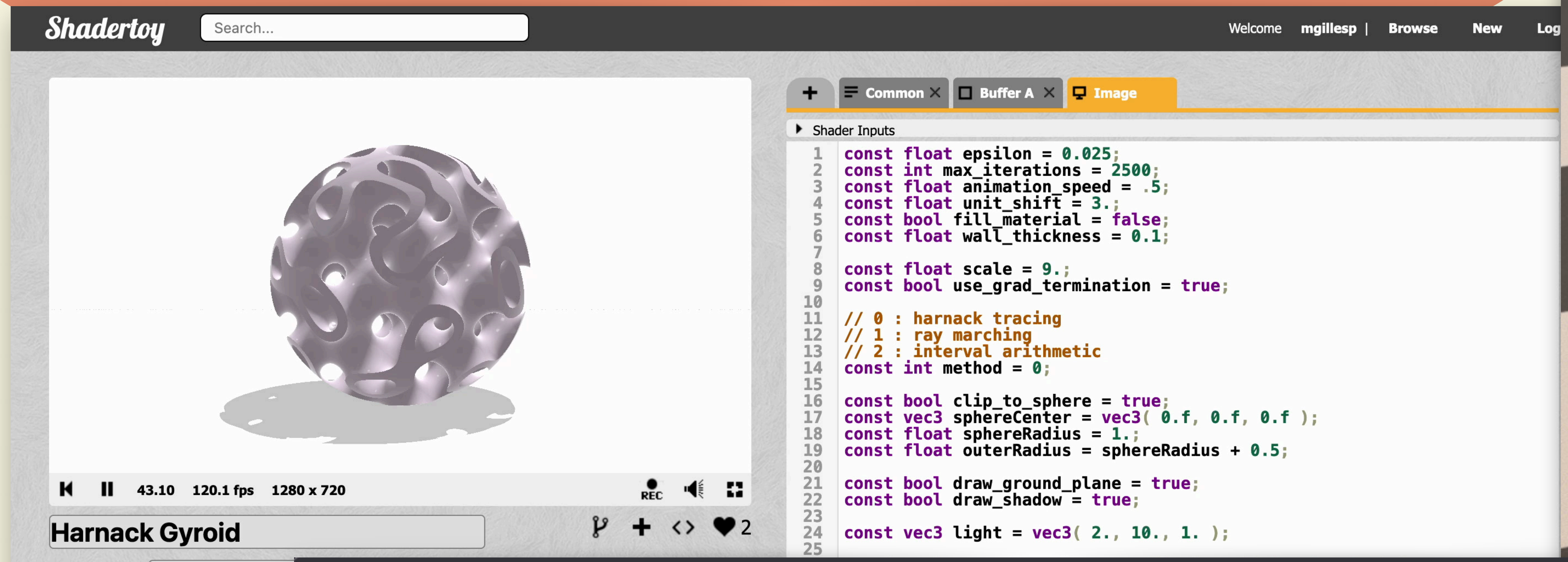


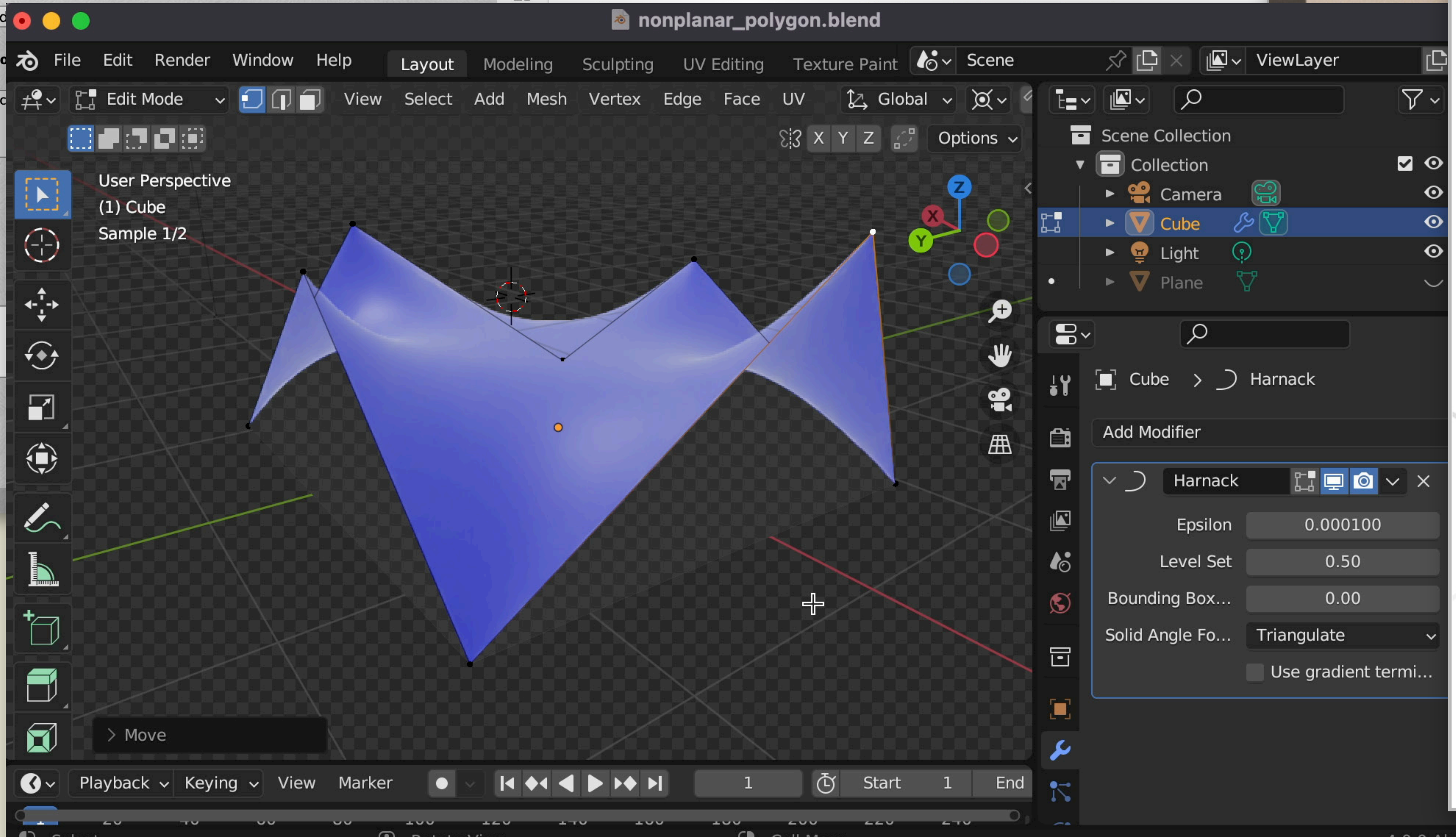
Simple to implement

PBRT (CPU ray tracer)

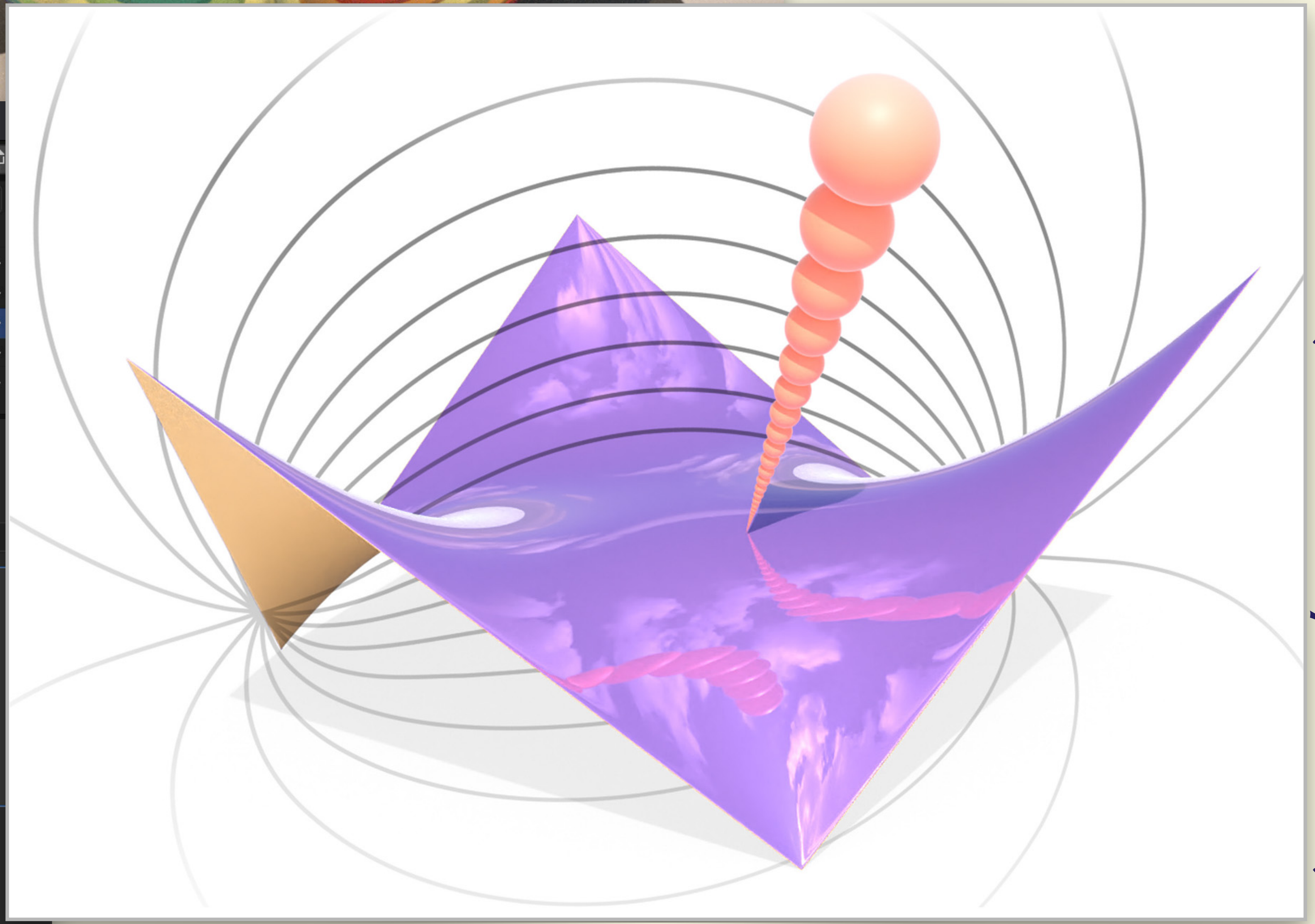


ShaderToy (WebGL shaders)



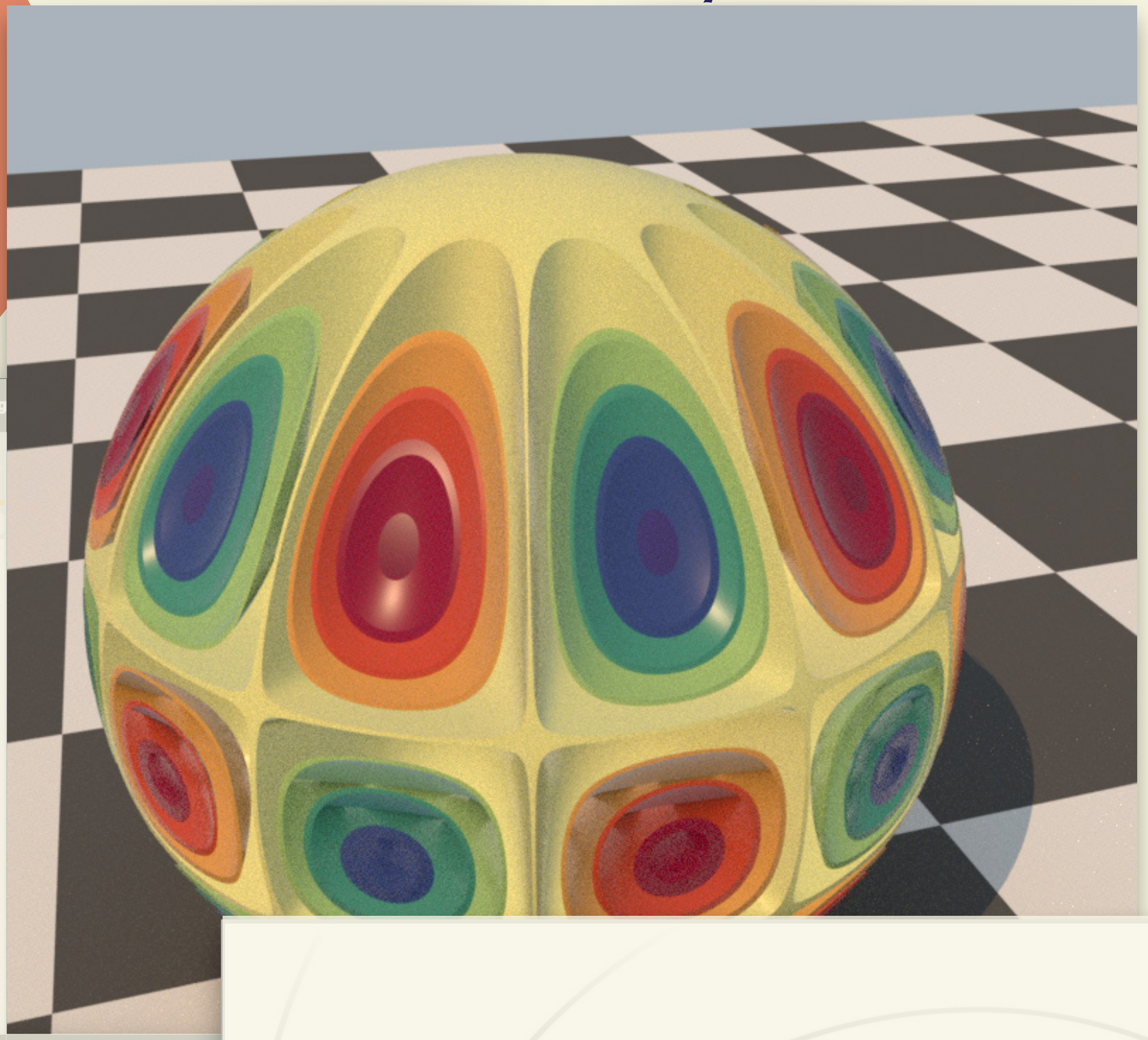


Blender (CPU ray tracer)

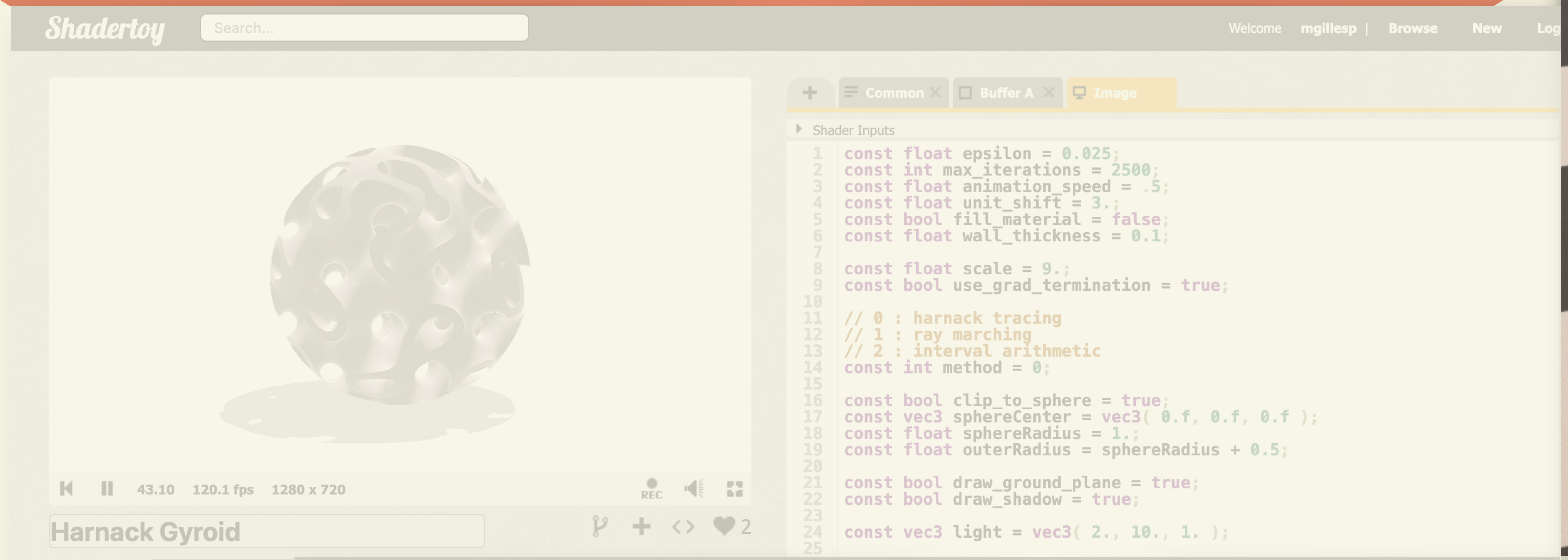


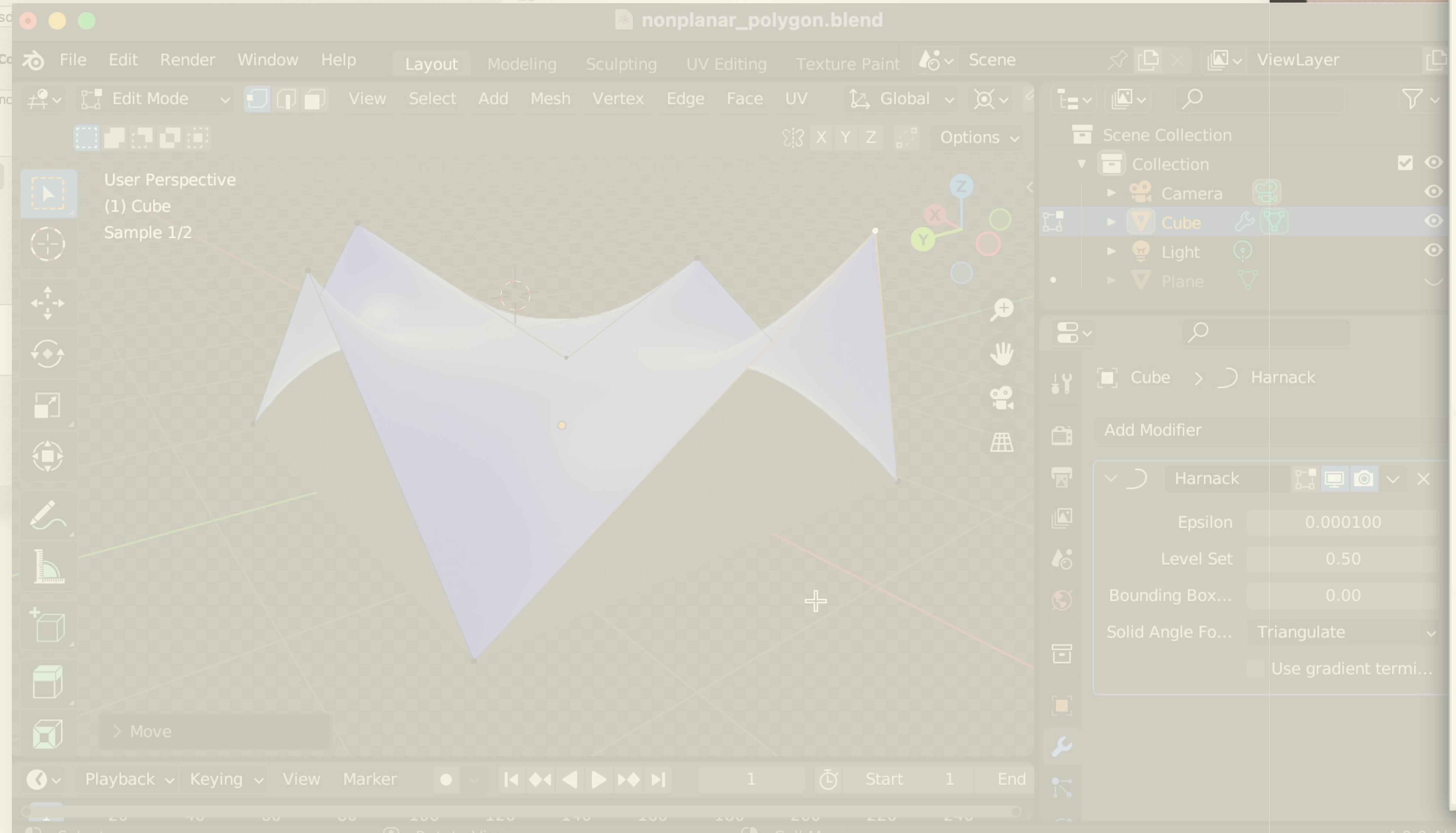
Simple to implement

PBRT (CPU ray tracer)



ShaderToy (WebGL shaders)

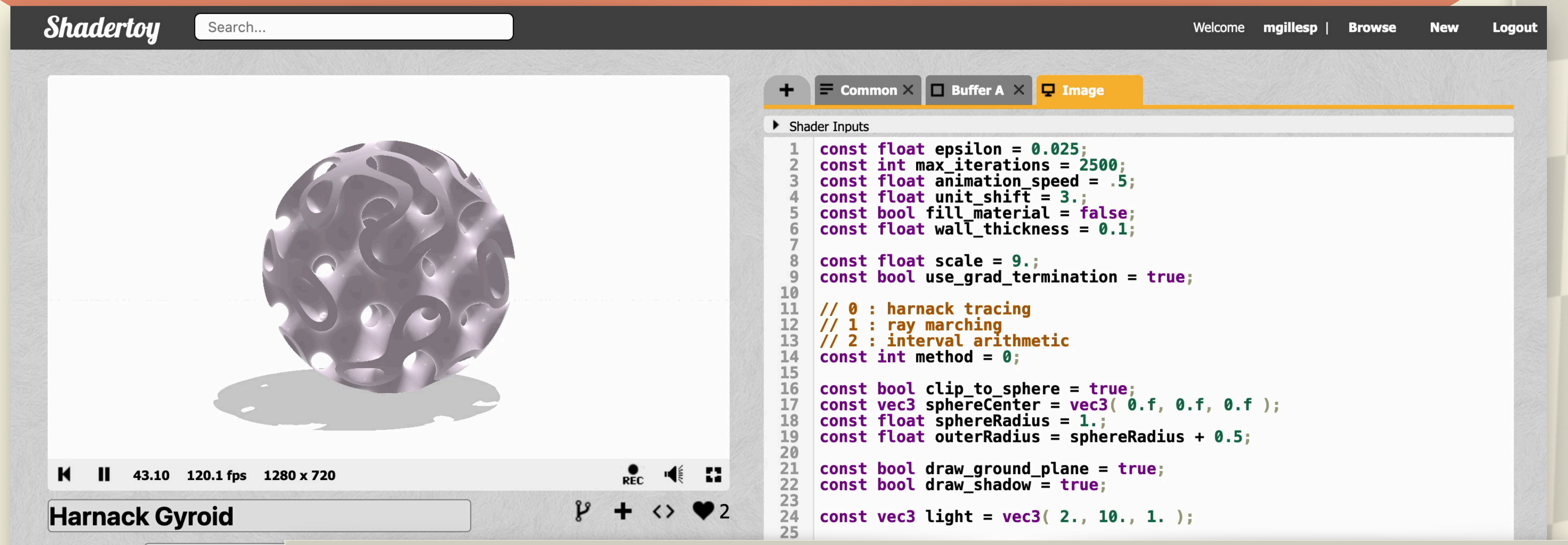




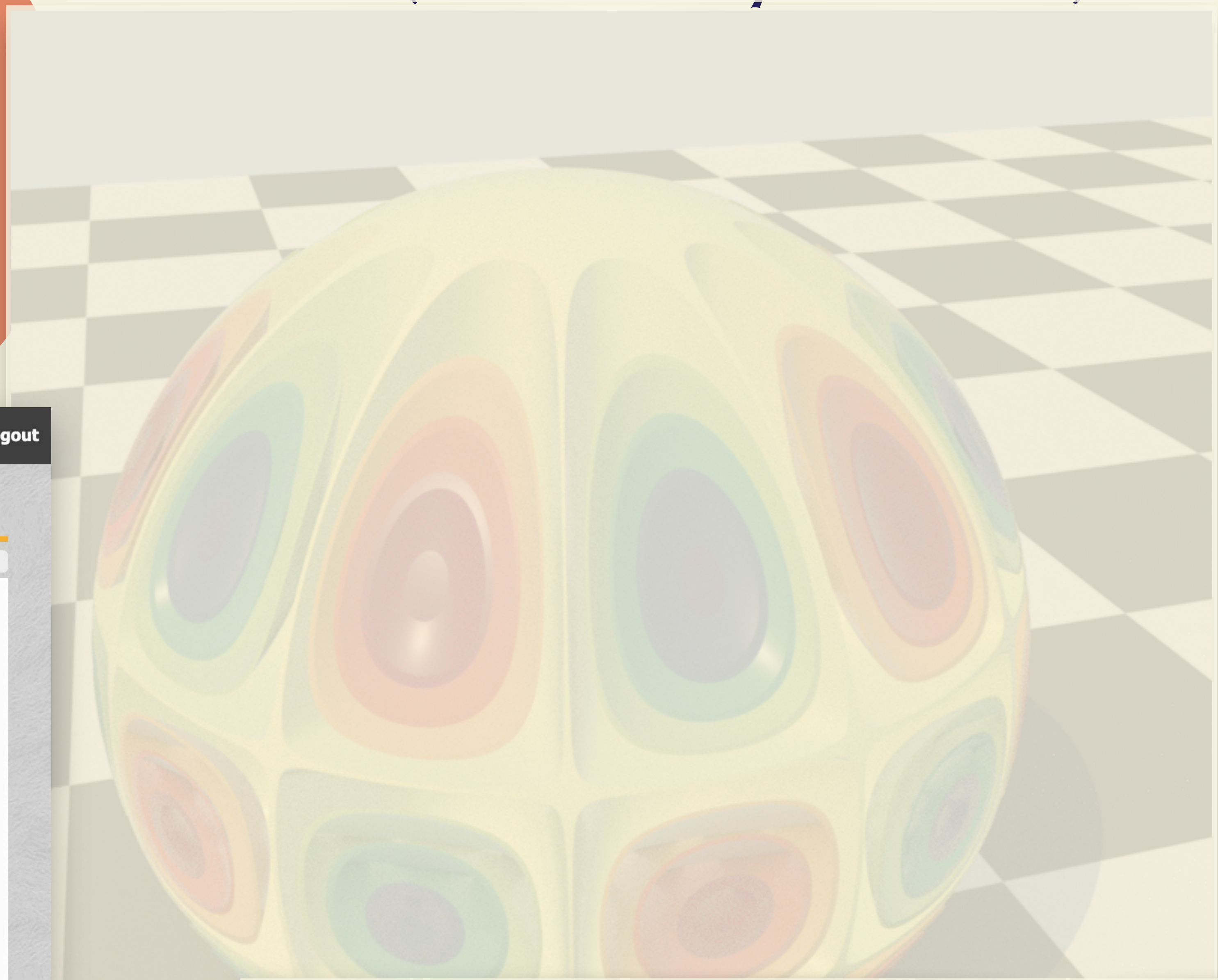
Blender (CPU ray tracer)

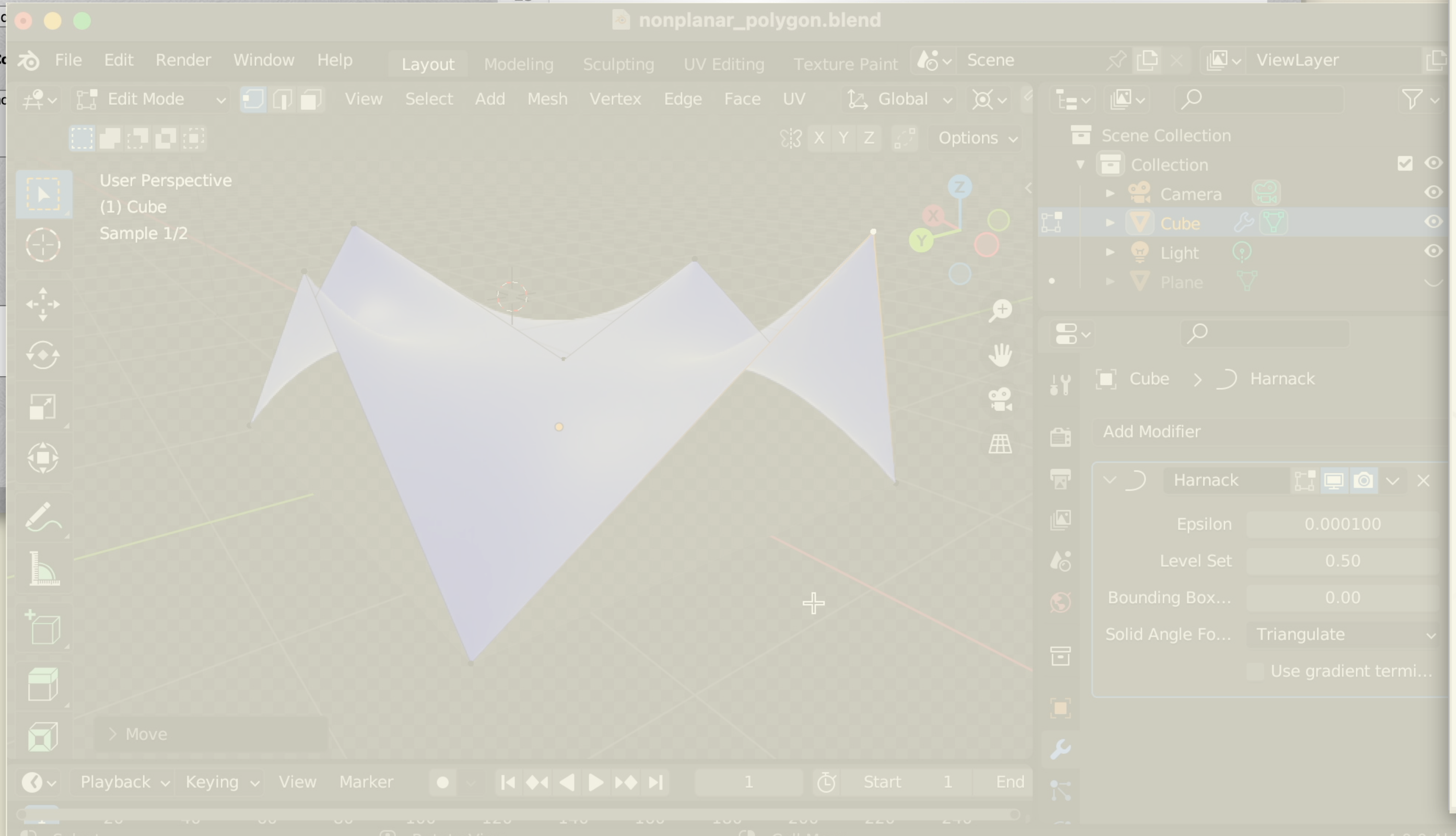
Simple to implement

ShaderToy (WebGL shaders)



PBRT (CPU ray tracer)

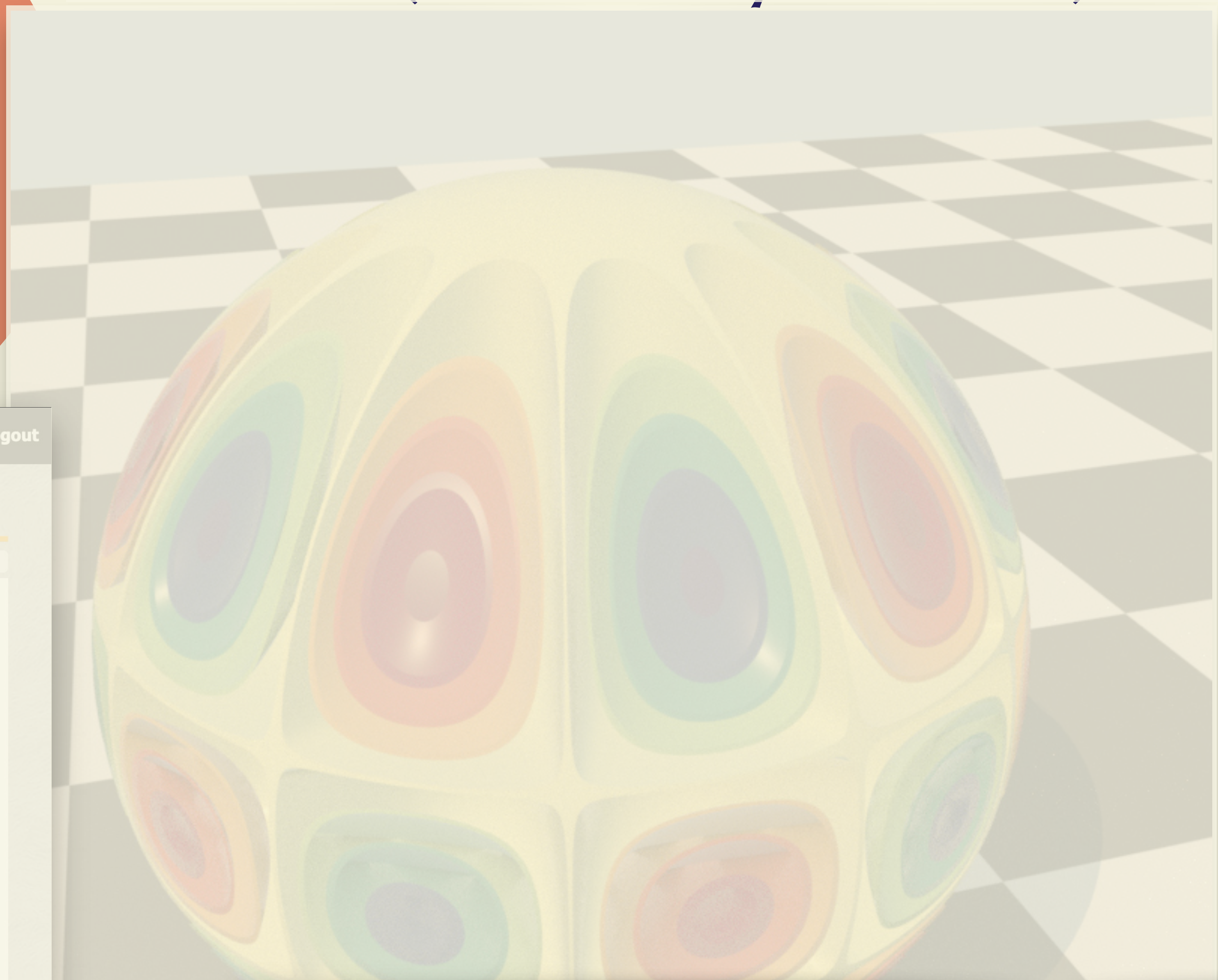




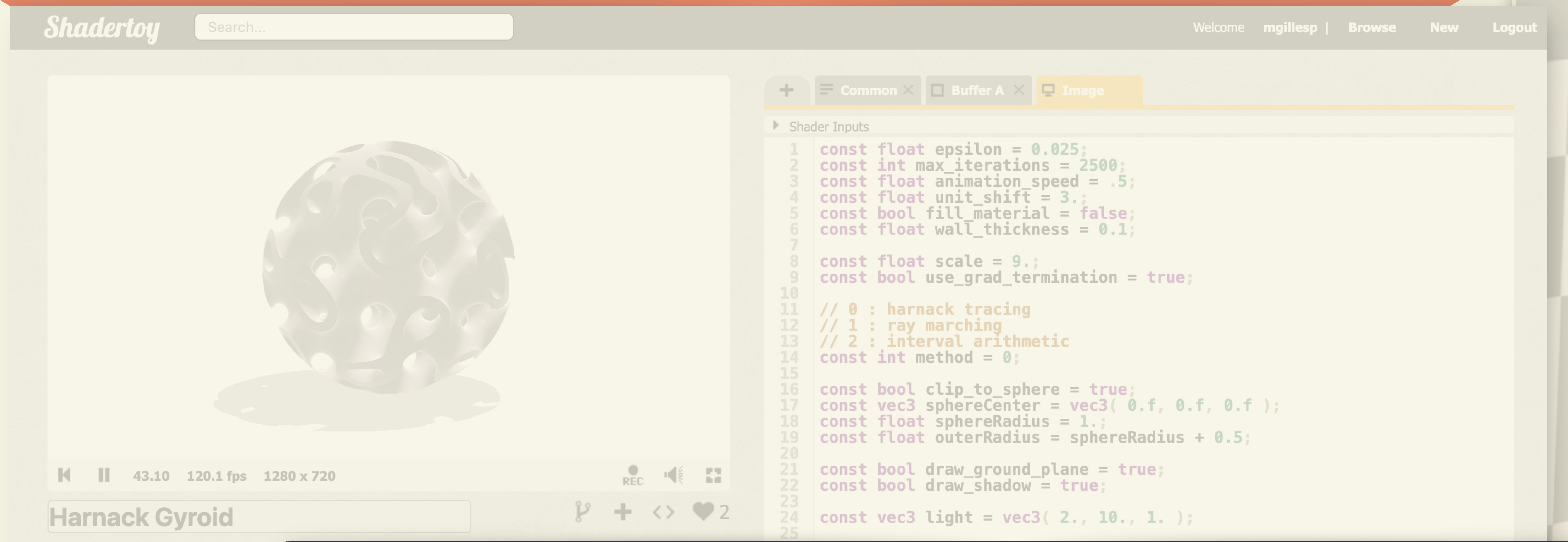
Blender (CPU ray tracer)

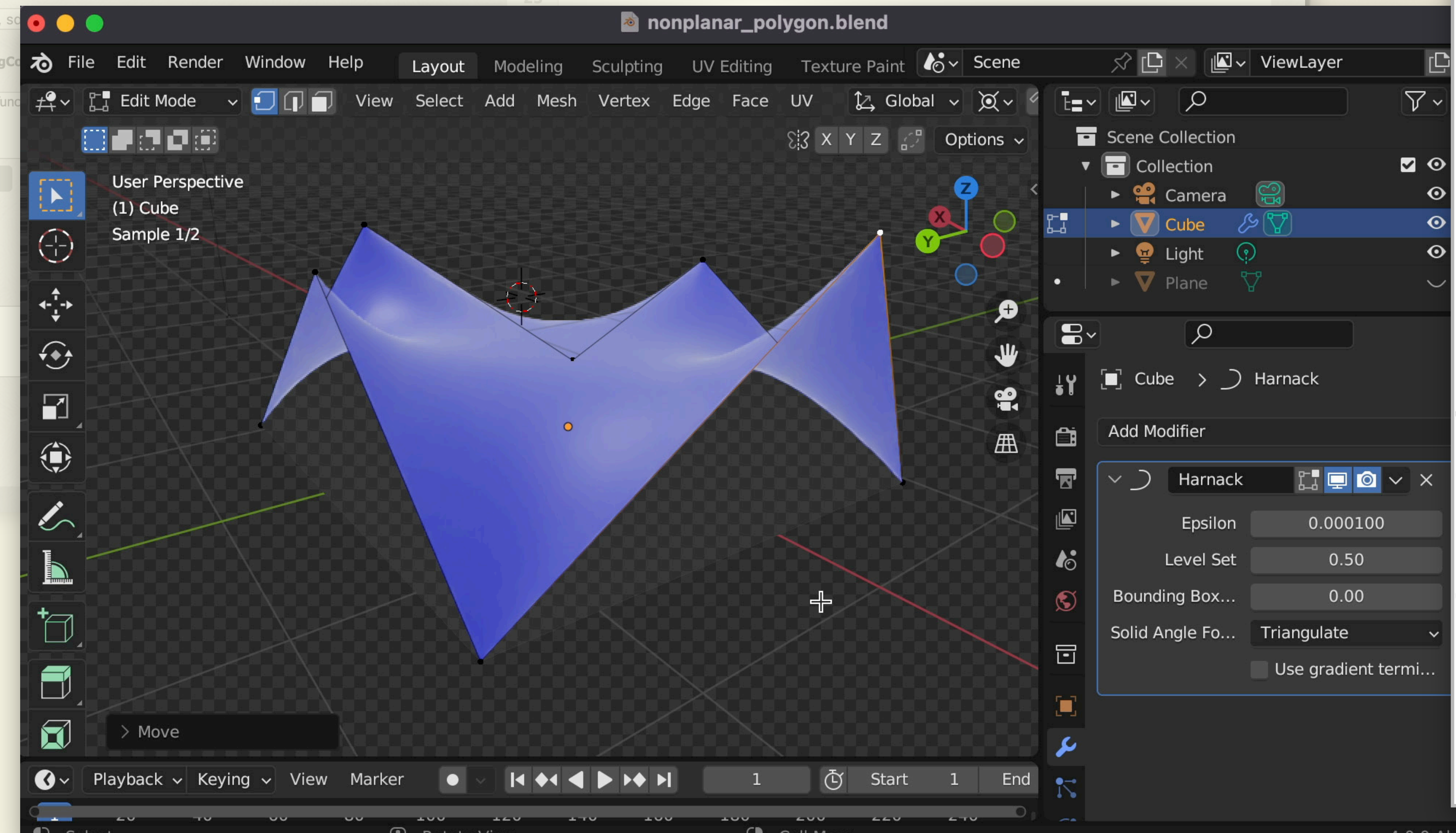
Simple to implement

PBRT (CPU ray tracer)

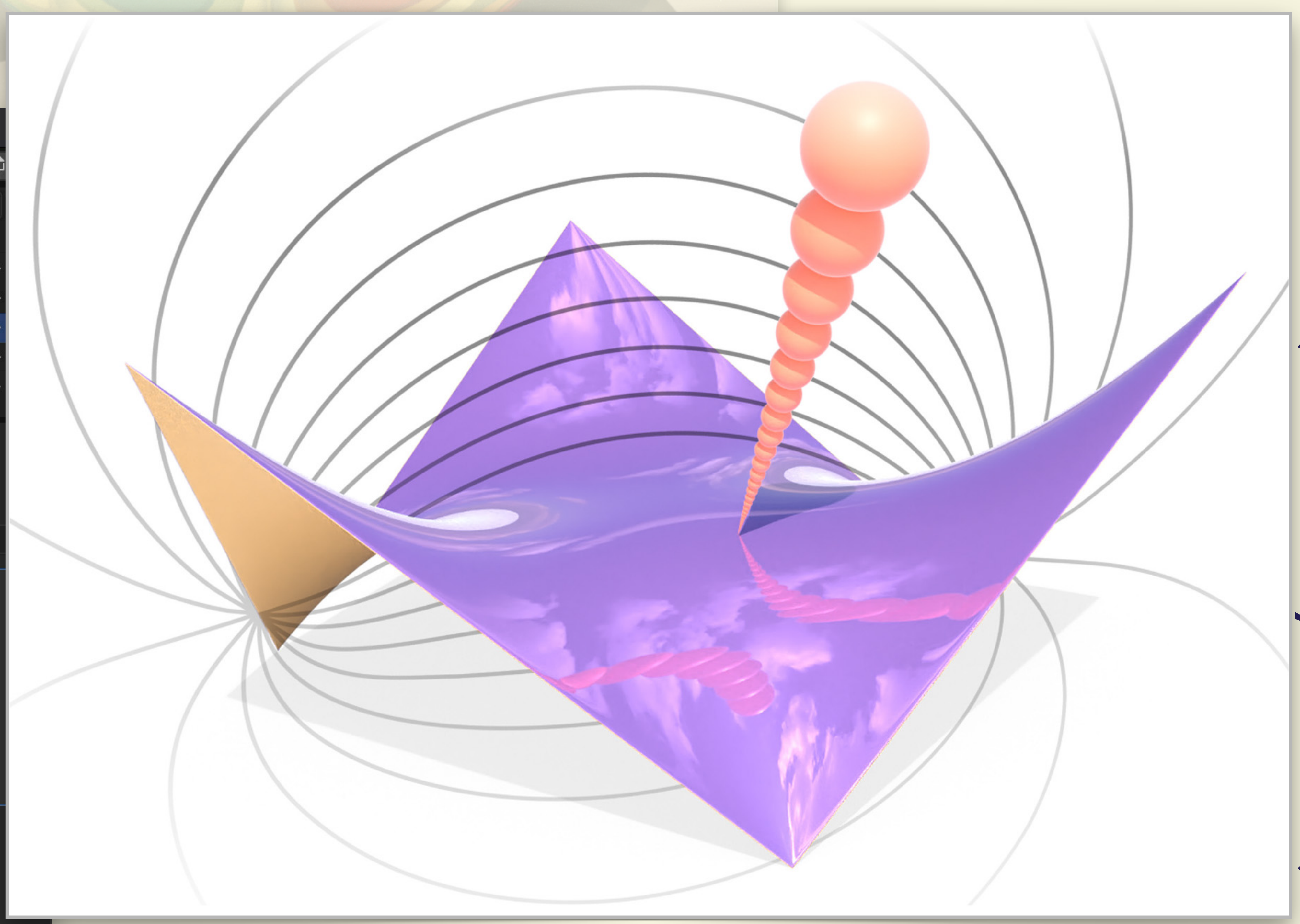


ShaderToy (WebGL shaders)





Blender (CPU ray tracer)



III. Examples

Generalized winding number

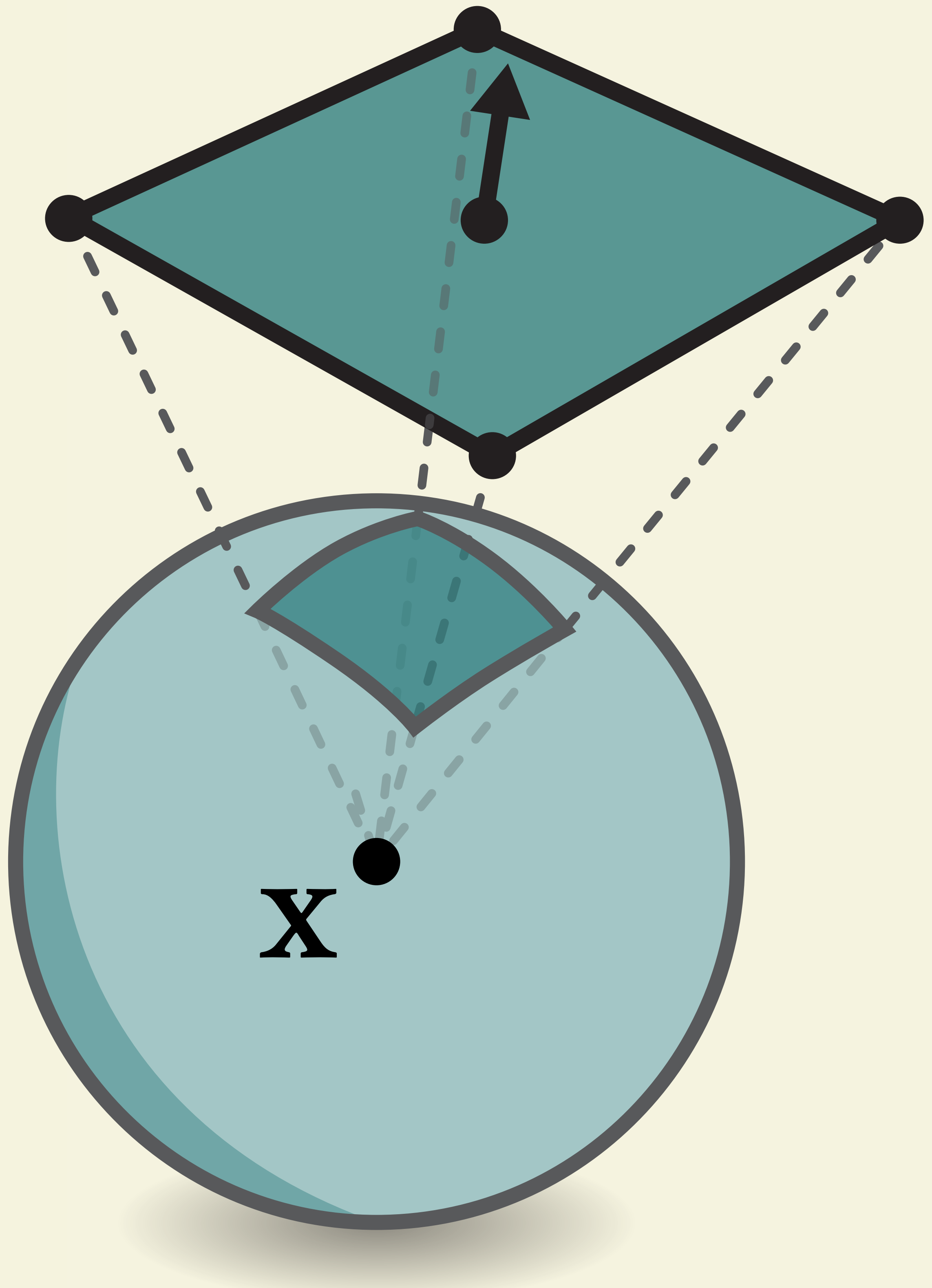
[Jacobson et al. 2013]

a.k.a. ***signed solid angle***

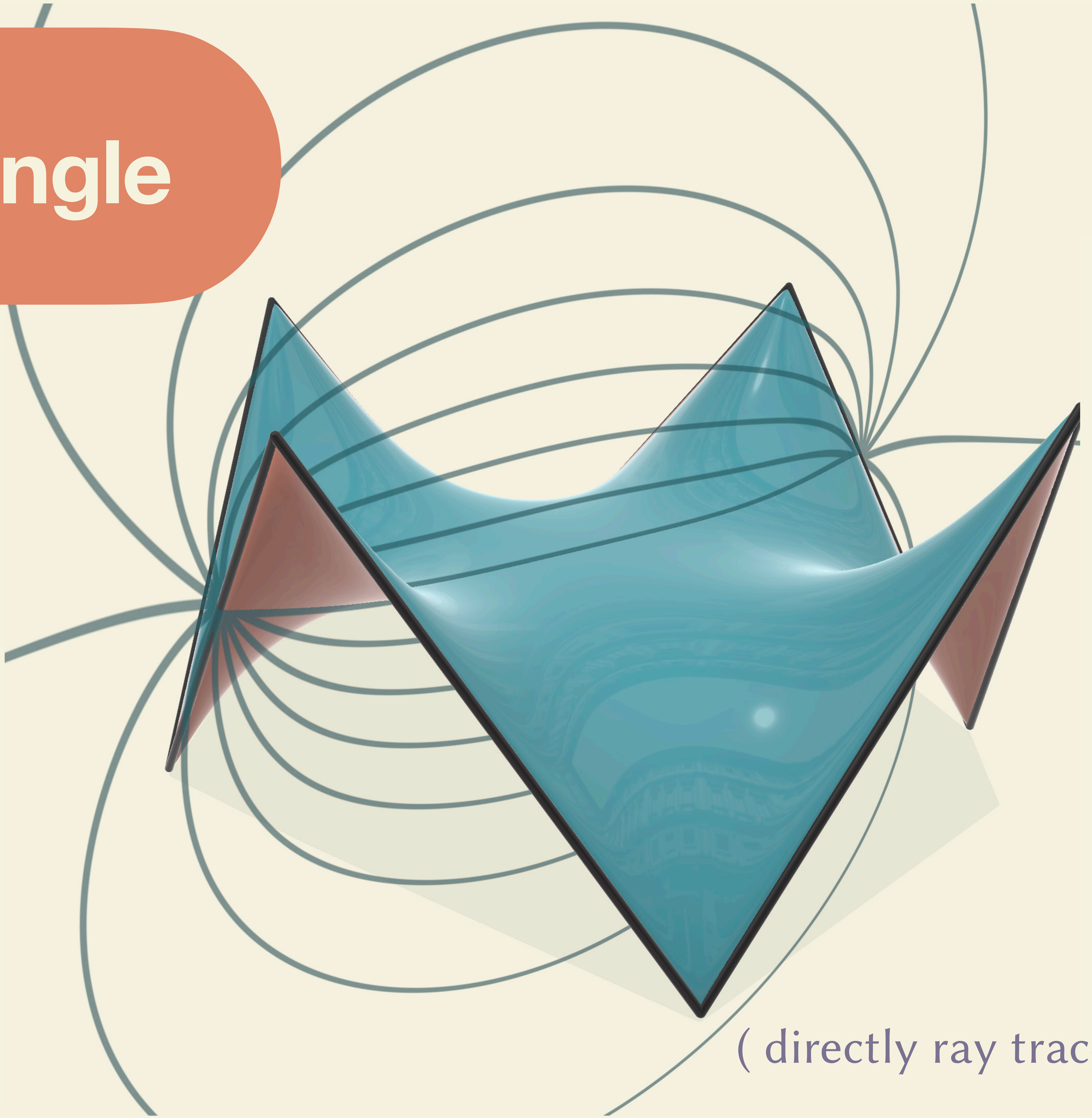
input mesh

repaired mesh
(directly ray traced)

Signed solid angle

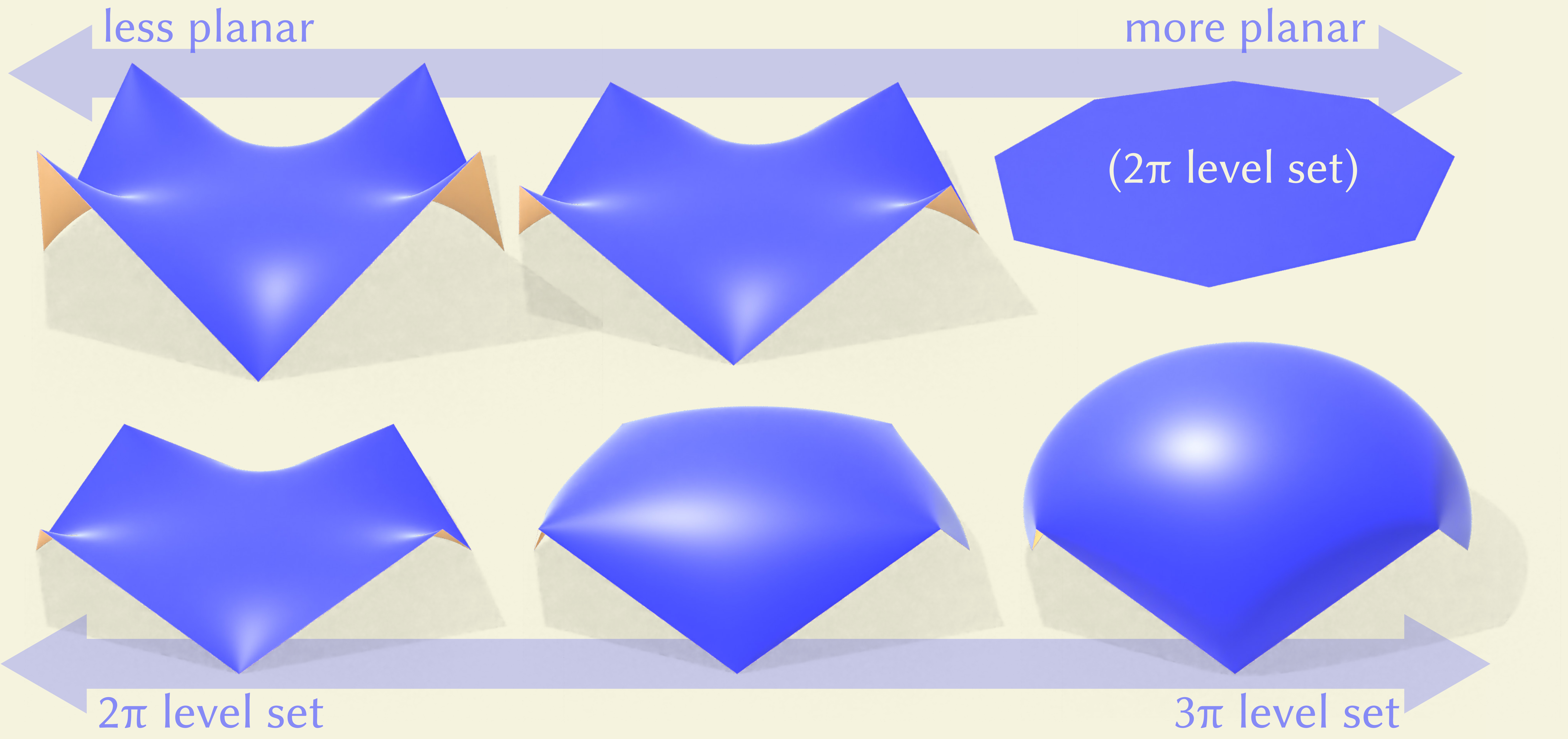


signed solid angle

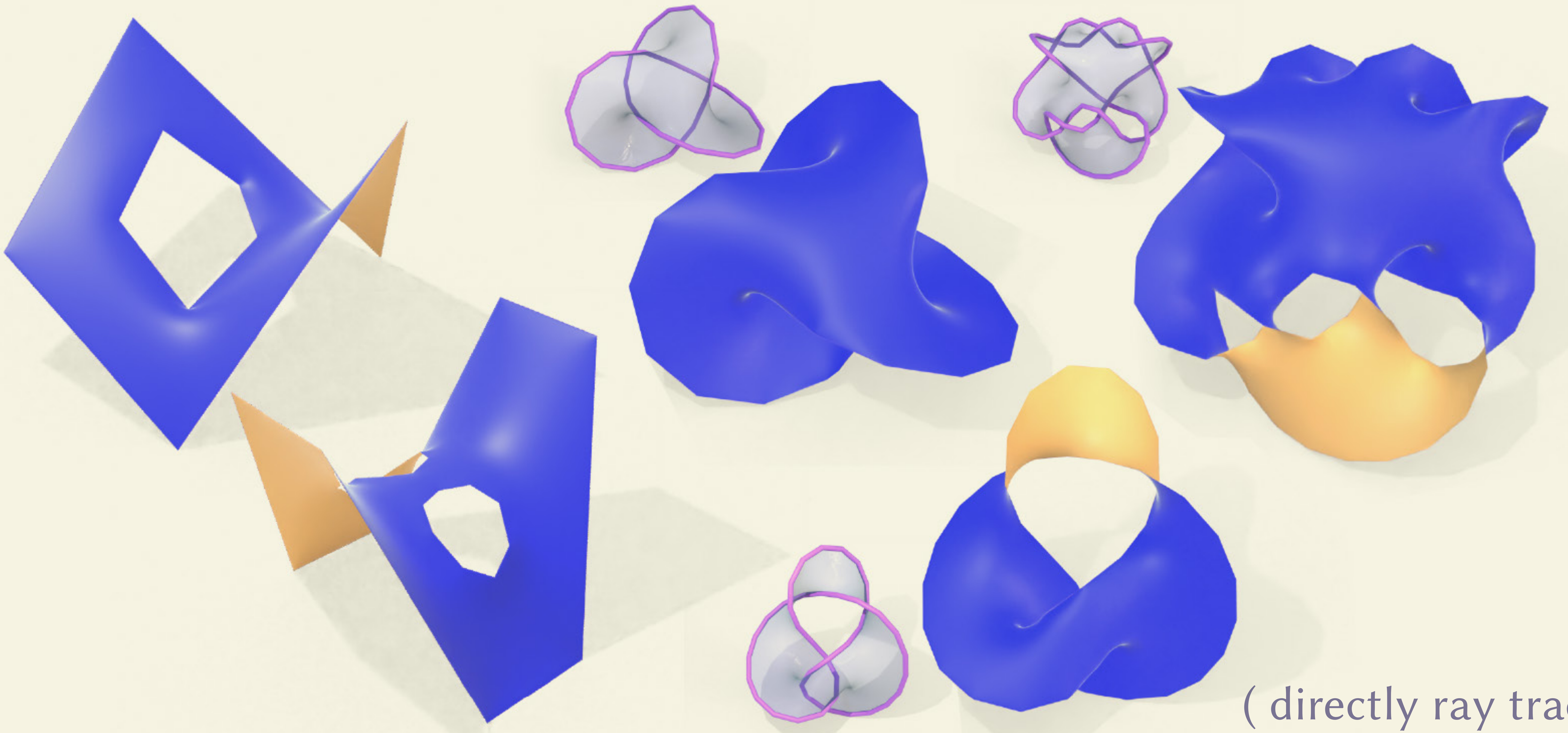


(directly ray traced)

Signed solid angle



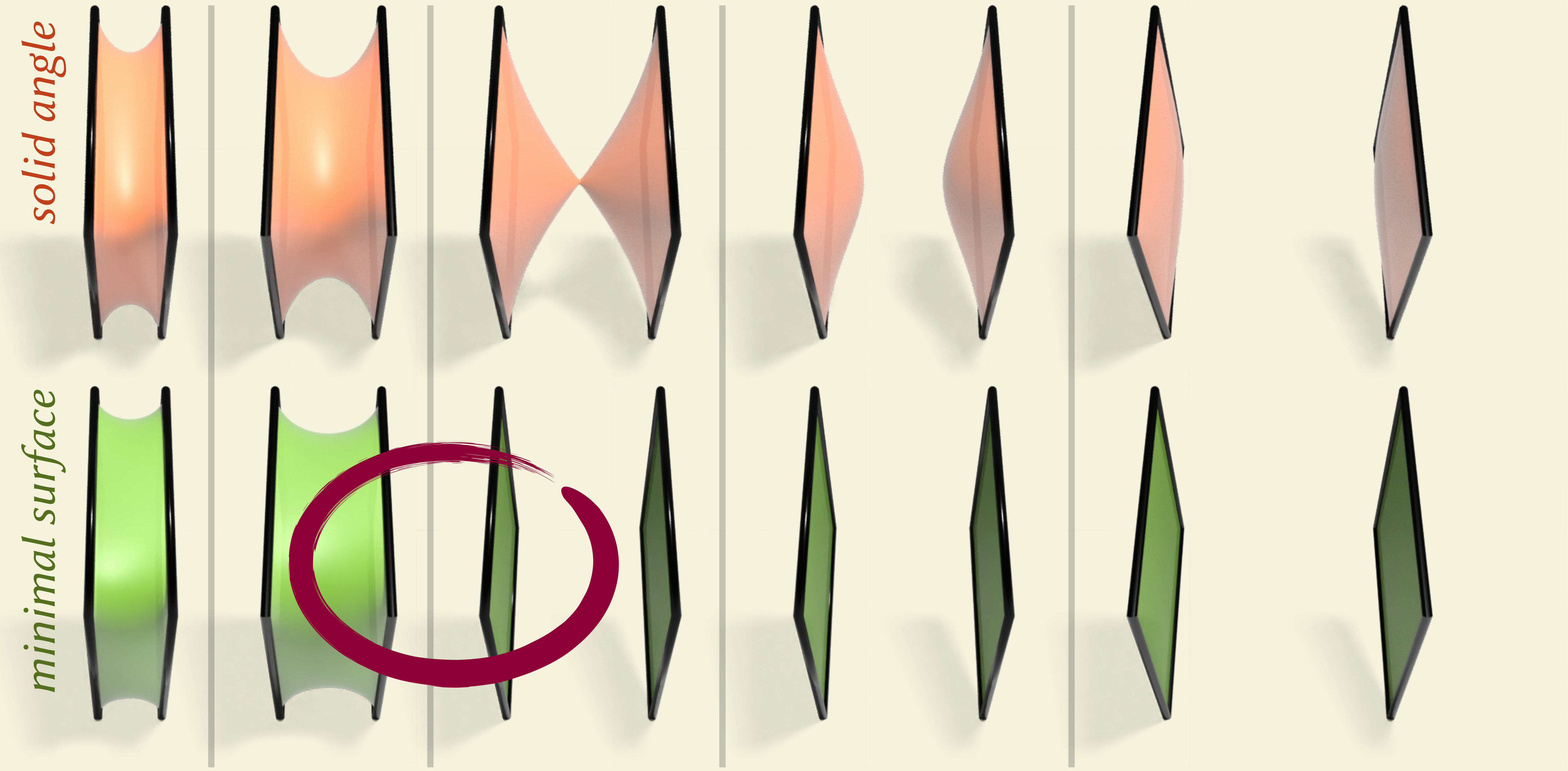
General nonplanar polygons



(directly ray traced)

Interpolating surfaces

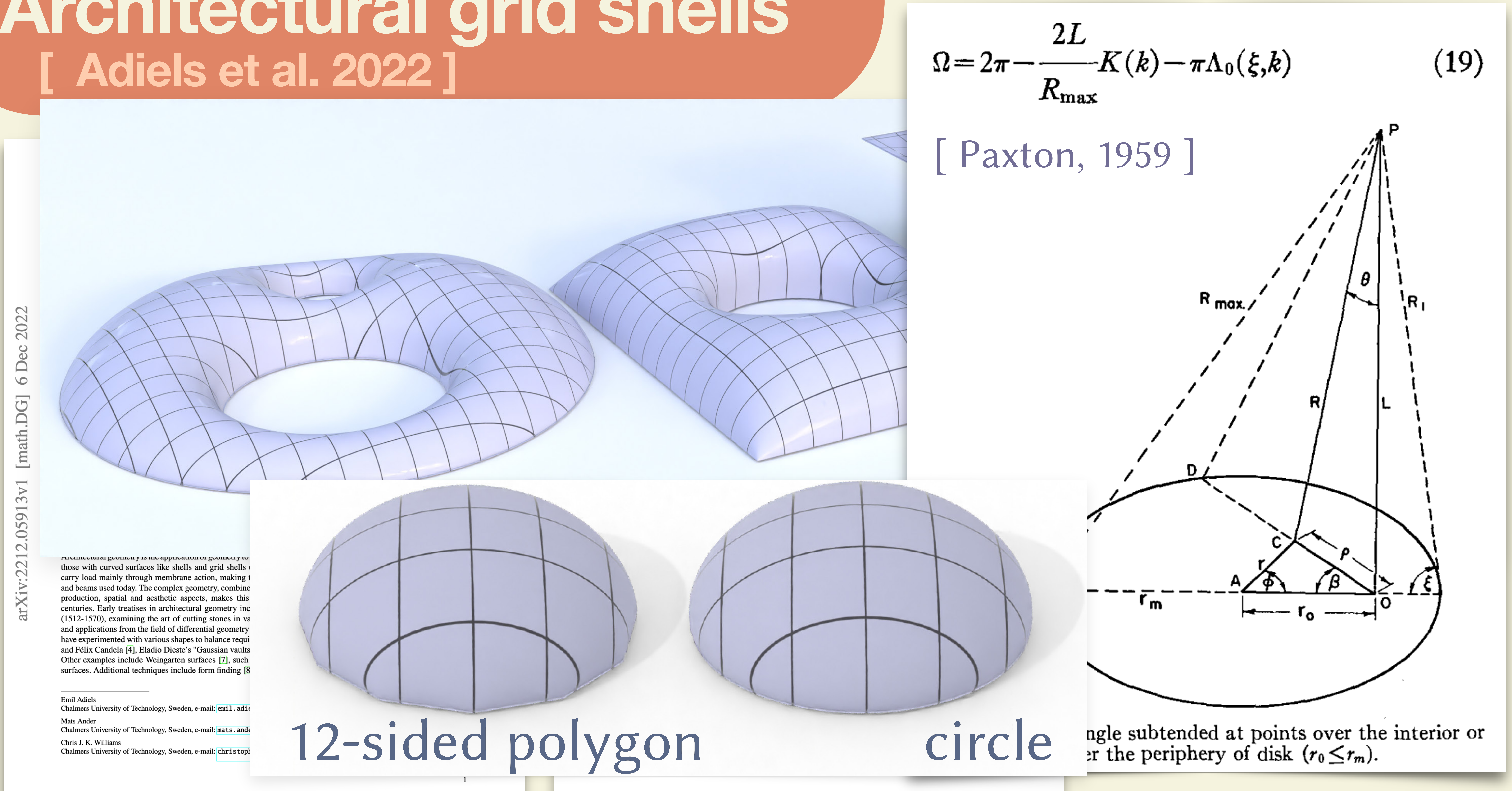
Continuous interpolation



Discontinuous Jump

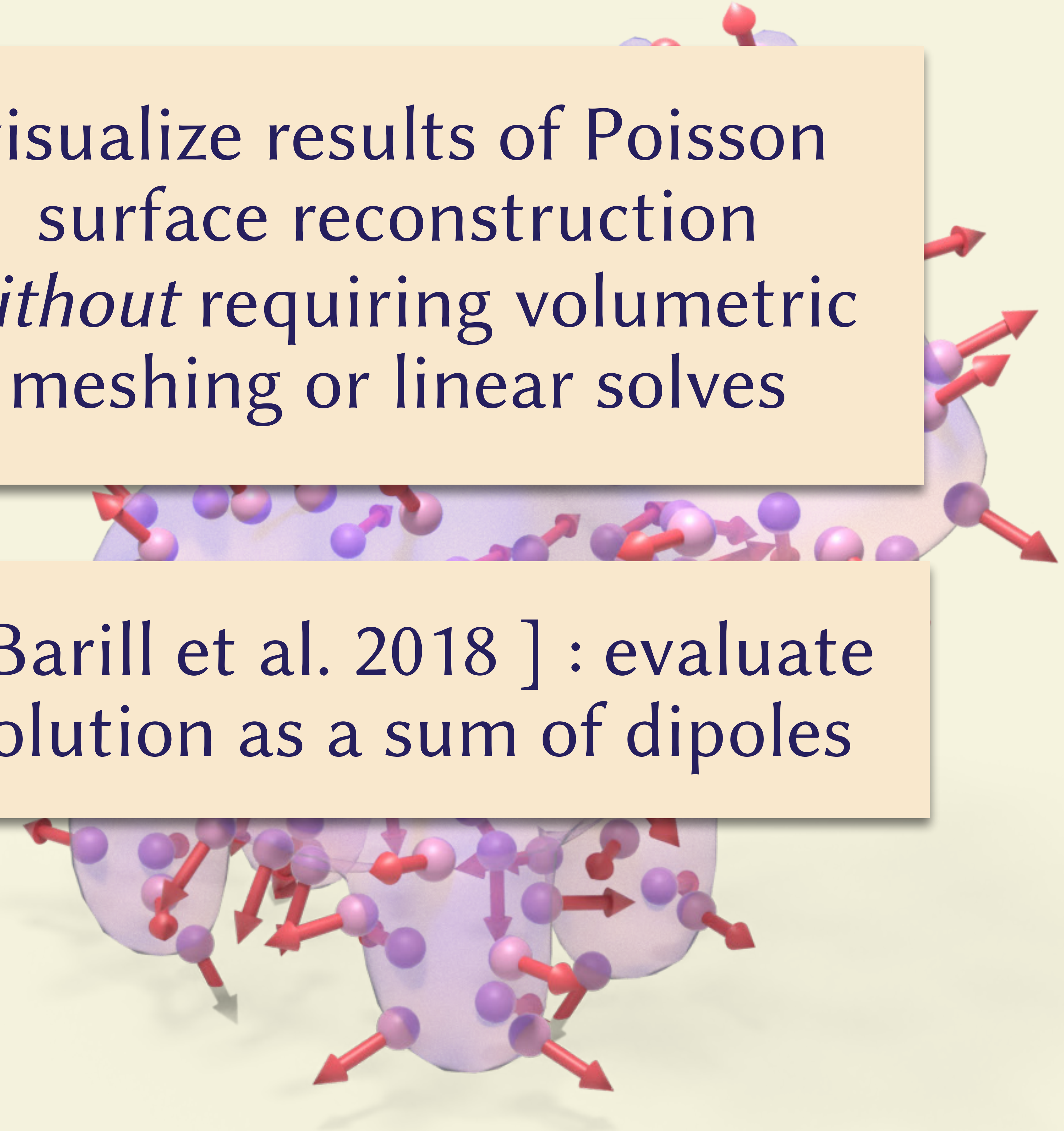
Architectural grid shells

[Adiels et al. 2022]



Surface reconstruction

[Kazhdan et al. 2006]

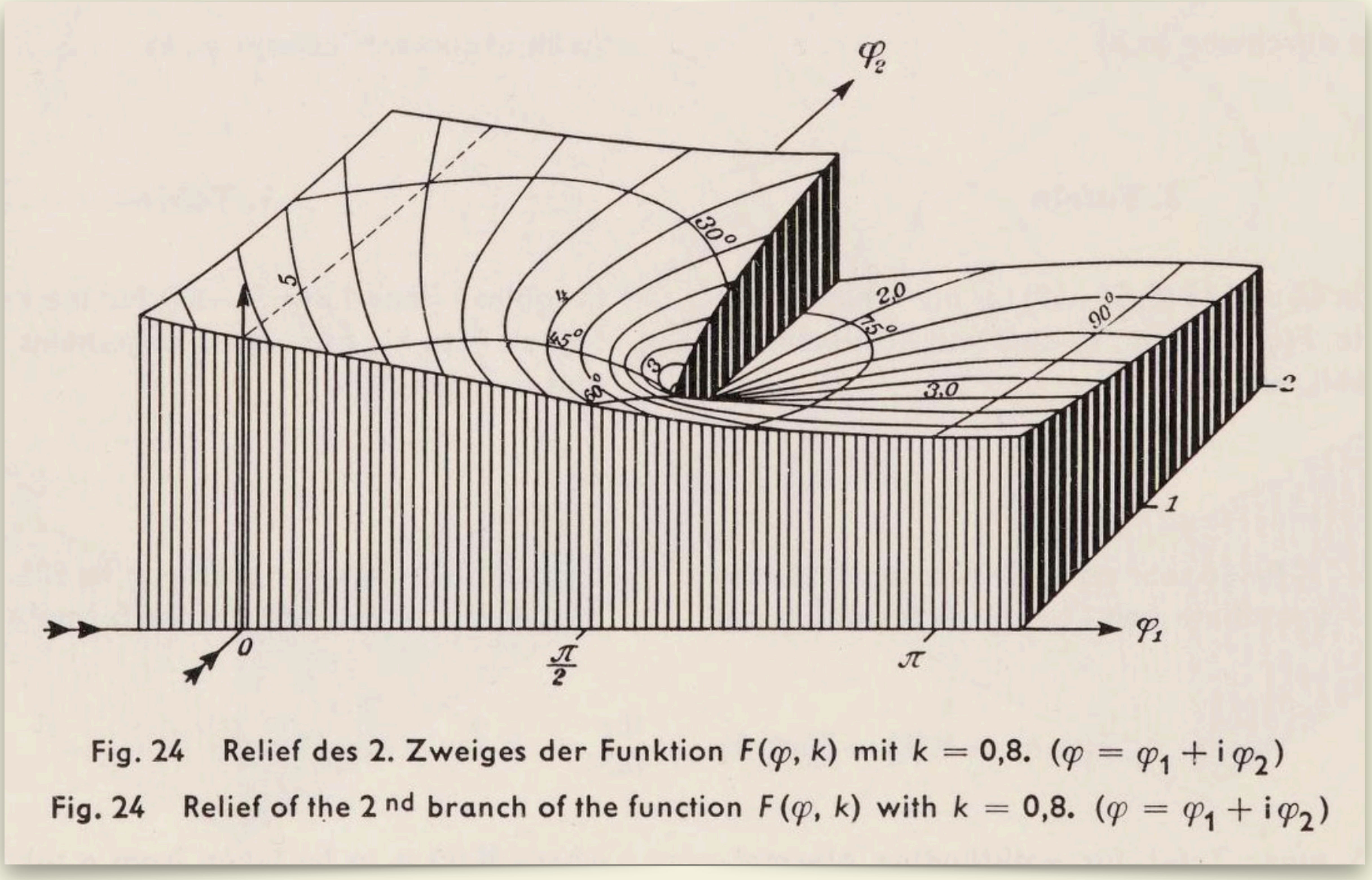


visualize results of Poisson surface reconstruction *without* requiring volumetric meshing or linear solves

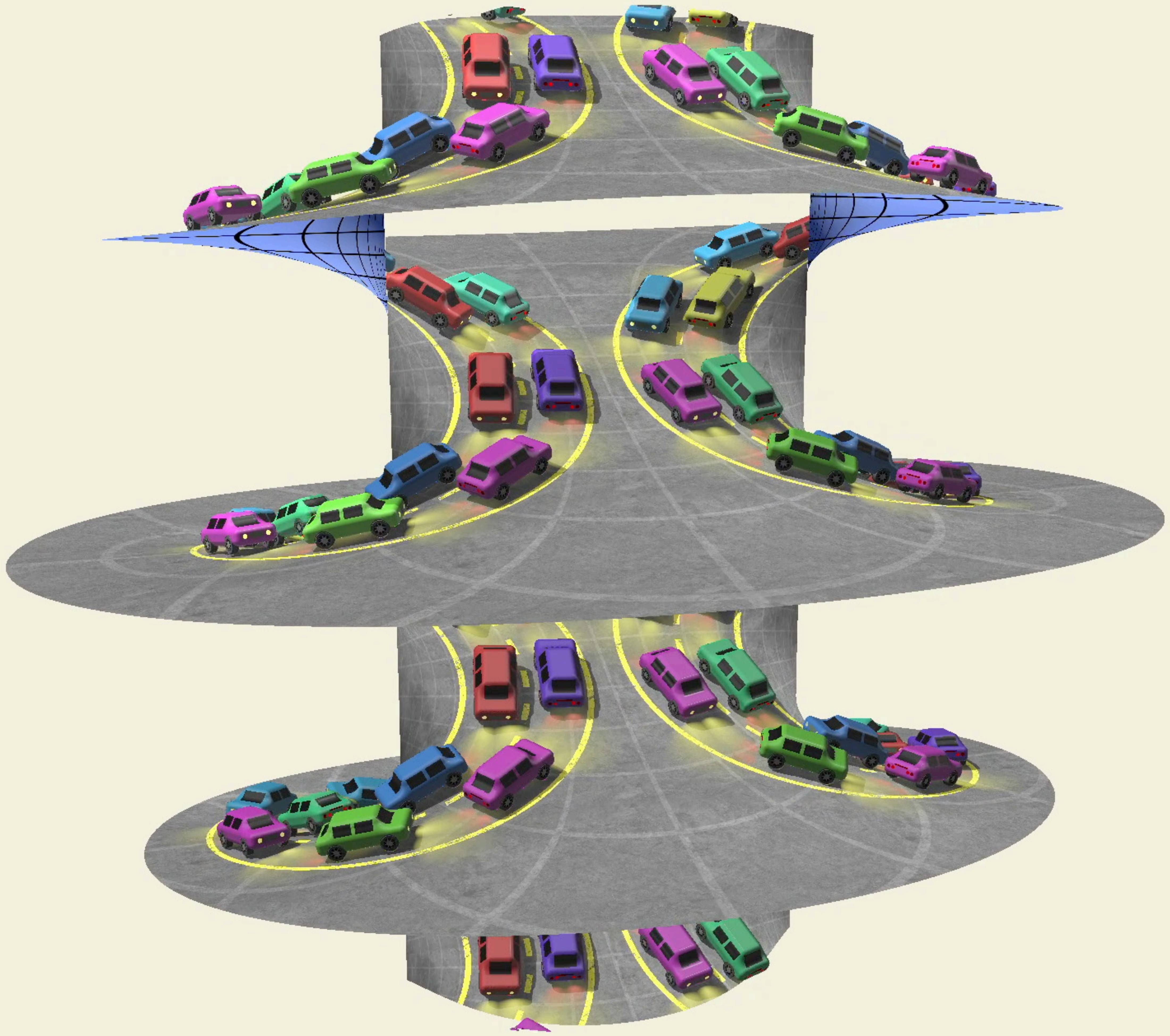
[Barill et al. 2018] : evaluate solution as a sum of dipoles

(directly ray traced)

Riemann surfaces



[Jahnke, Emde & Lösch 1960]



(directly ray traced)

Riemann surfaces as graphs

graph

$$z = f(x, y)$$

(directly ray traced)

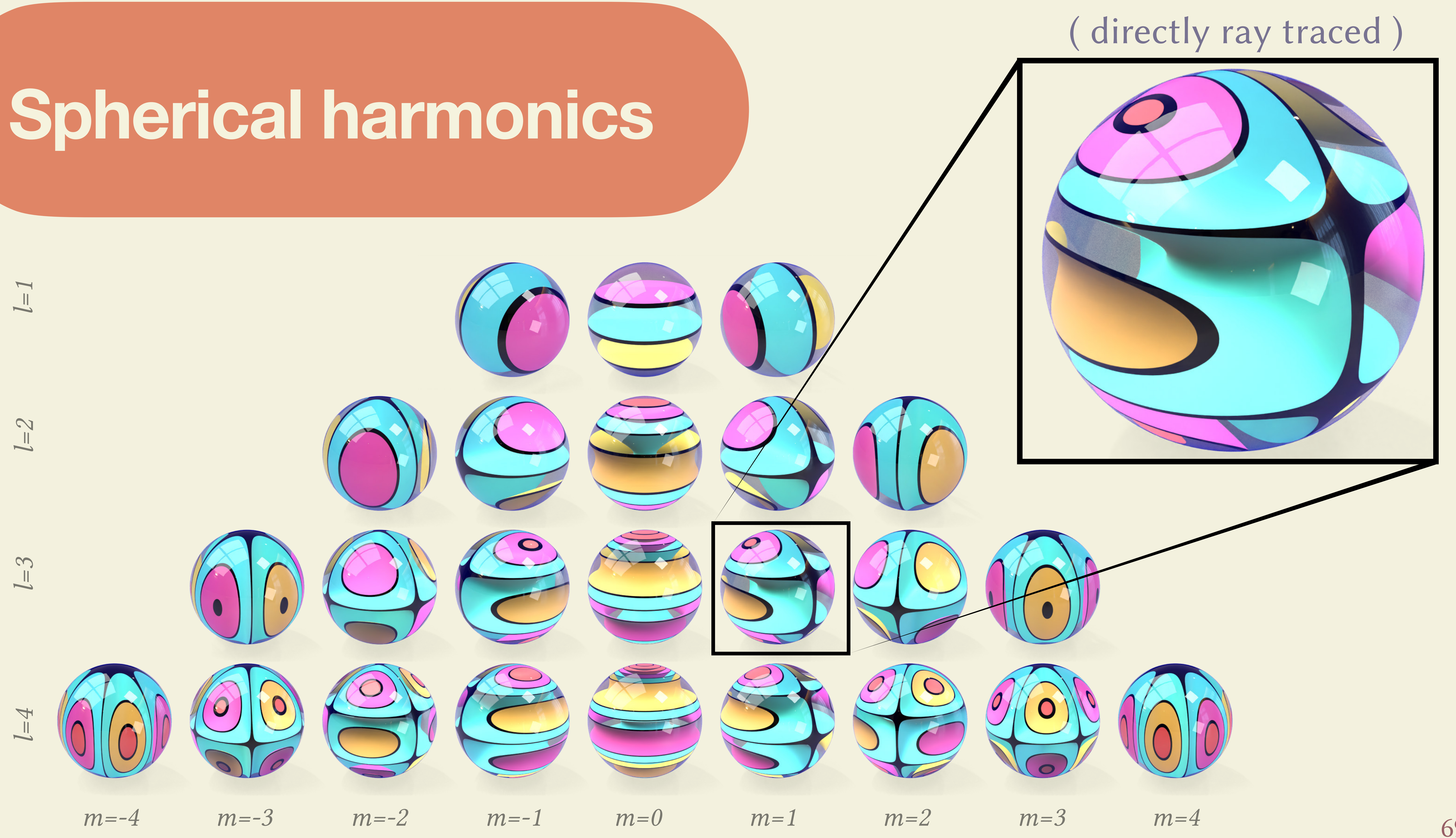
Riemann surfaces as graphs

$$f(x, y) - z = 0$$

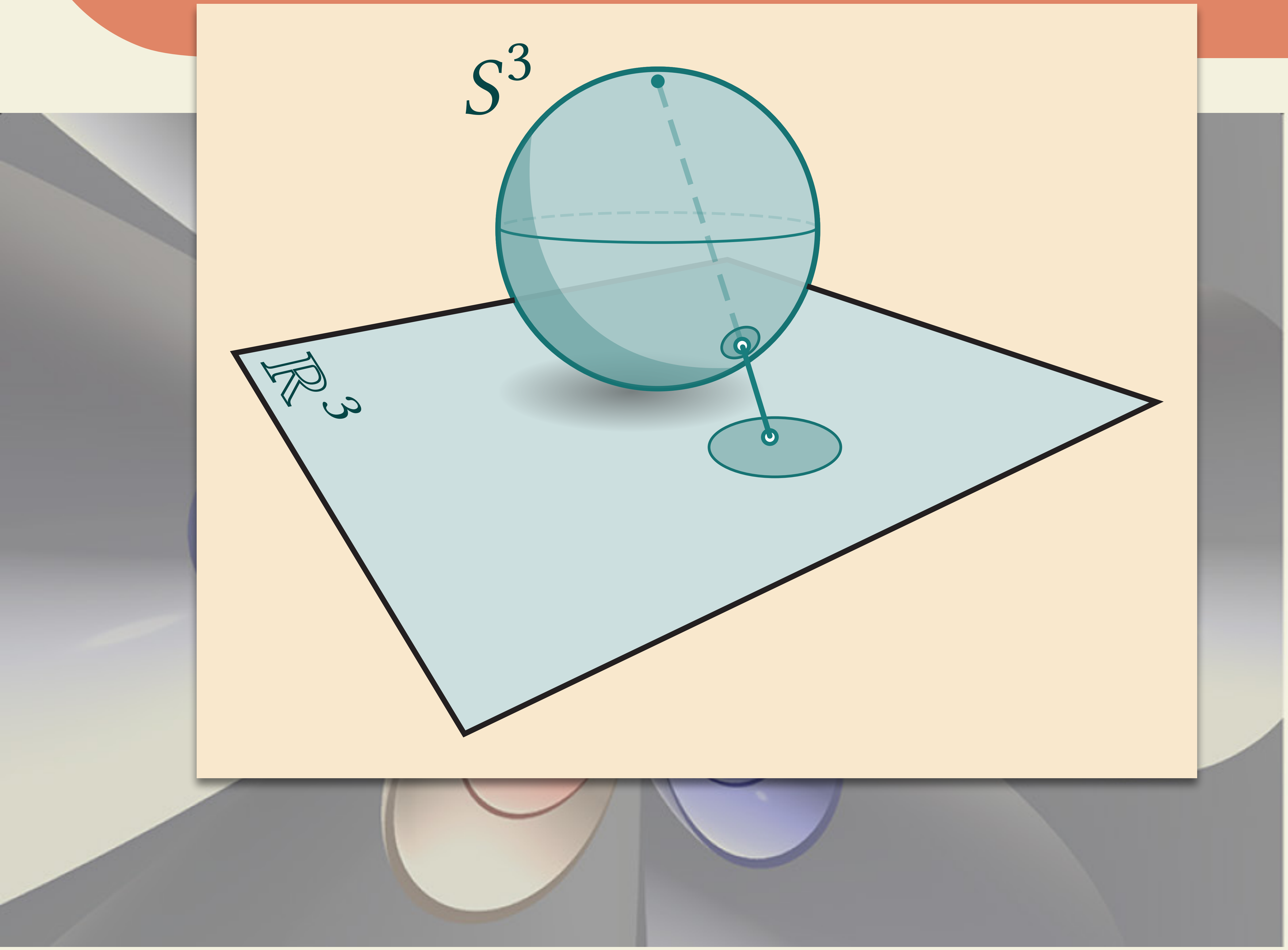
level set

(directly ray traced)

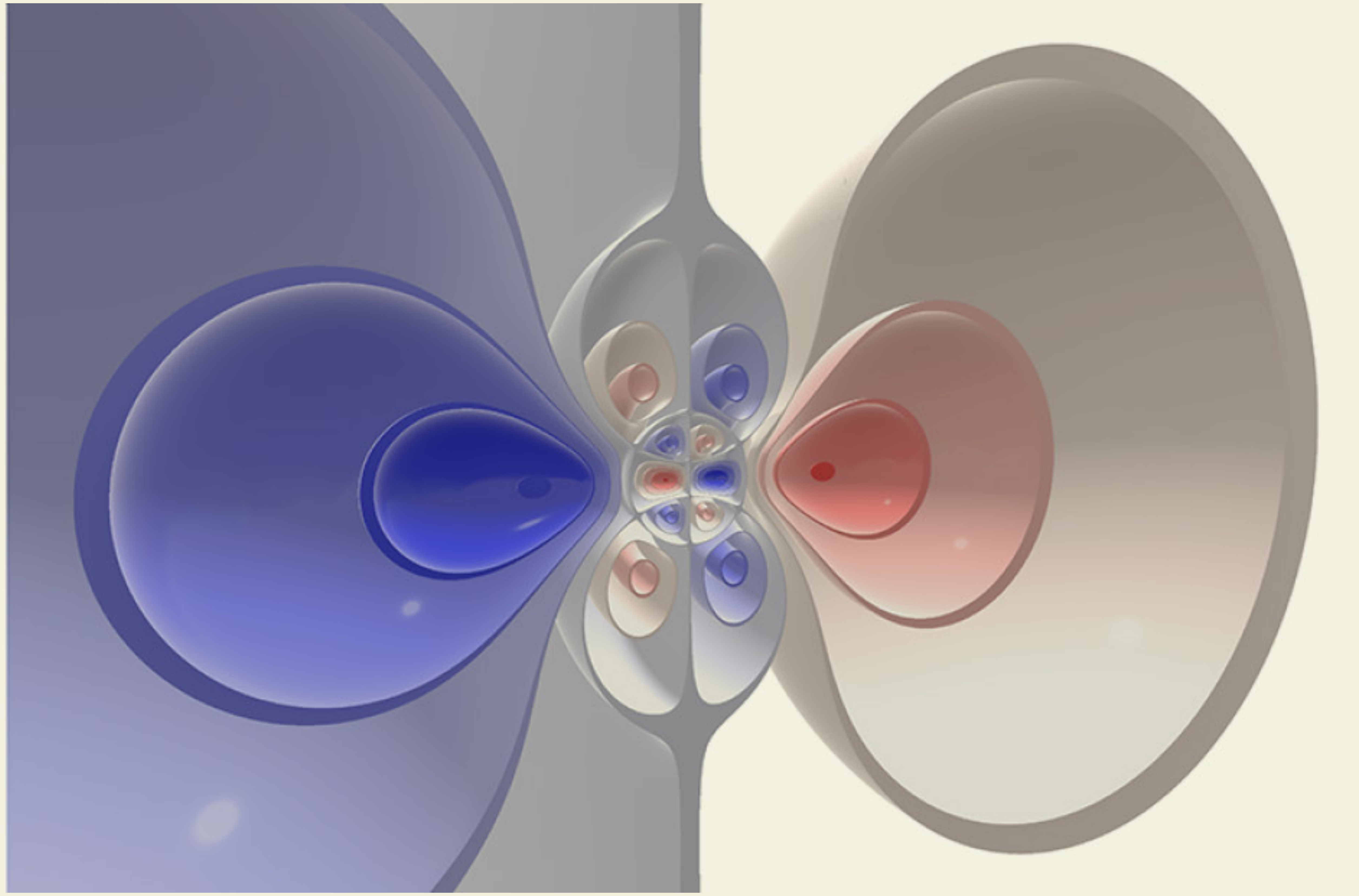
Spherical harmonics



Hyperspherical harmonics



$f(x, y, z, w) = y^3 - 3yz^2$



$f(x, y, z, w) = x^3y + xy^3 - 3xyw^2 - 3xyz^2$

(directly ray traced)

The gyroid

[Diegel 2021]

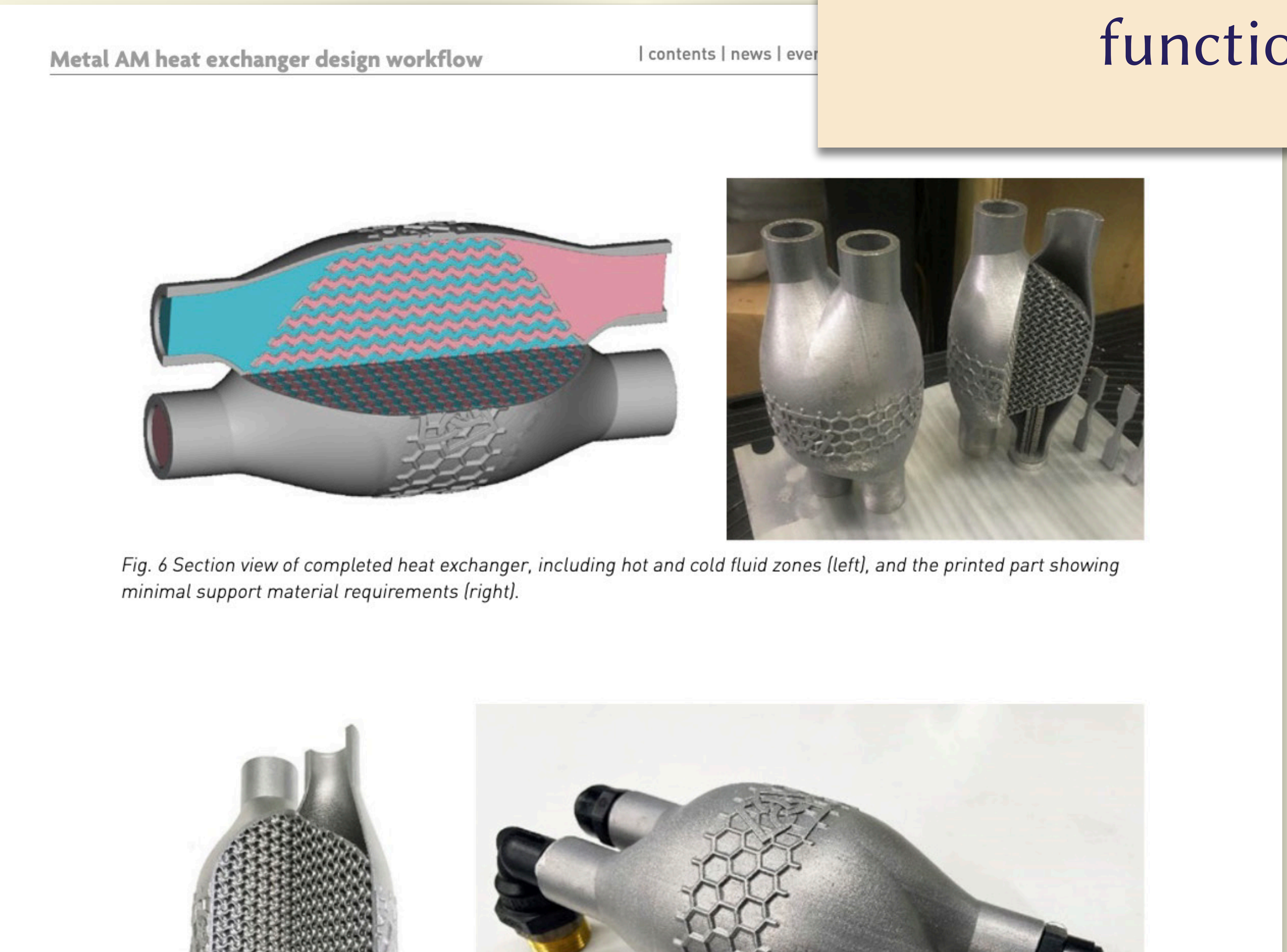
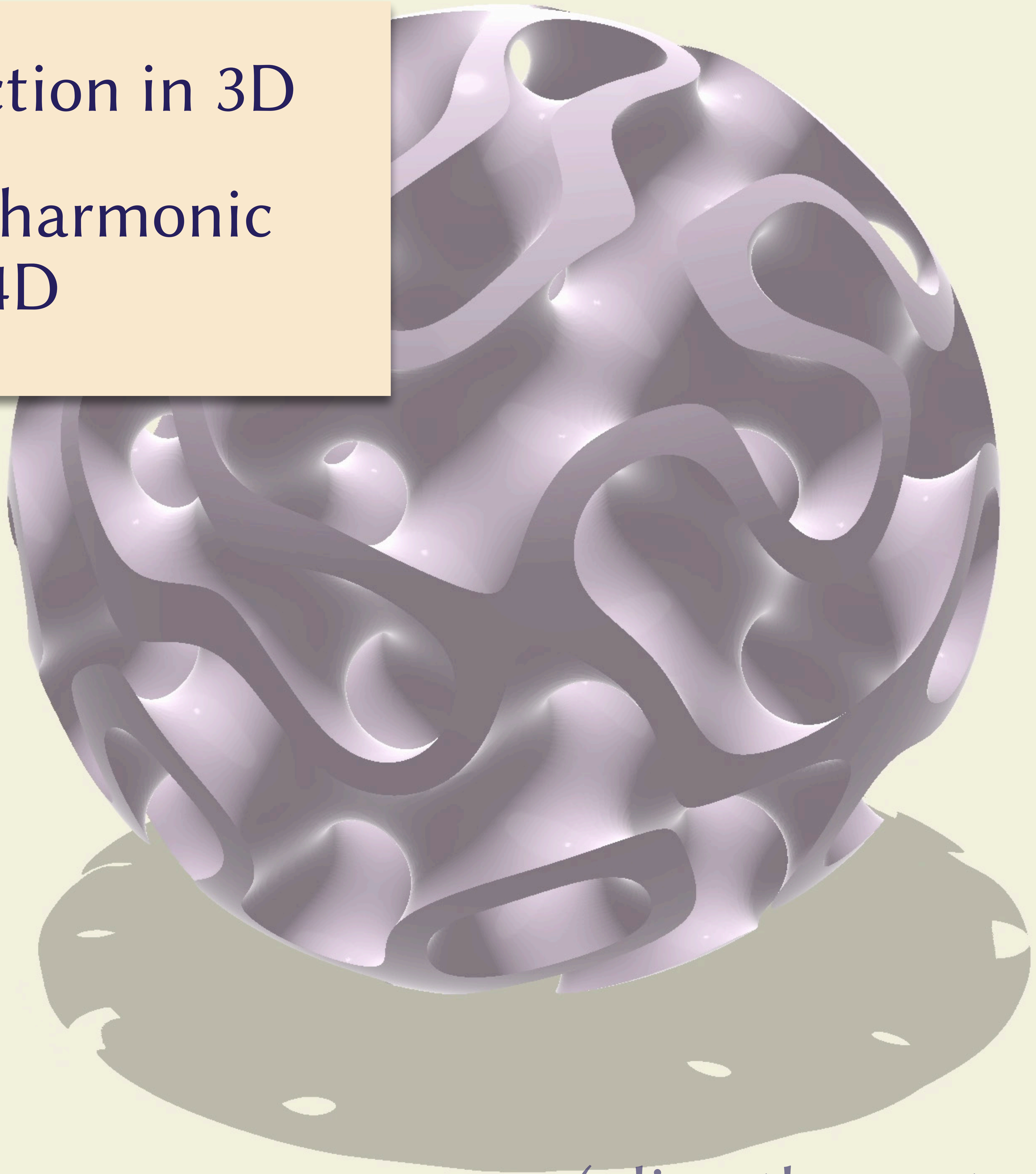

Metal AM heat exchanger design workflow | contents | news | ever

Fig. 6 Section view of completed heat exchanger, including hot and cold fluid zones (left), and the printed part showing minimal support material requirements (right).

not a harmonic function in 3D

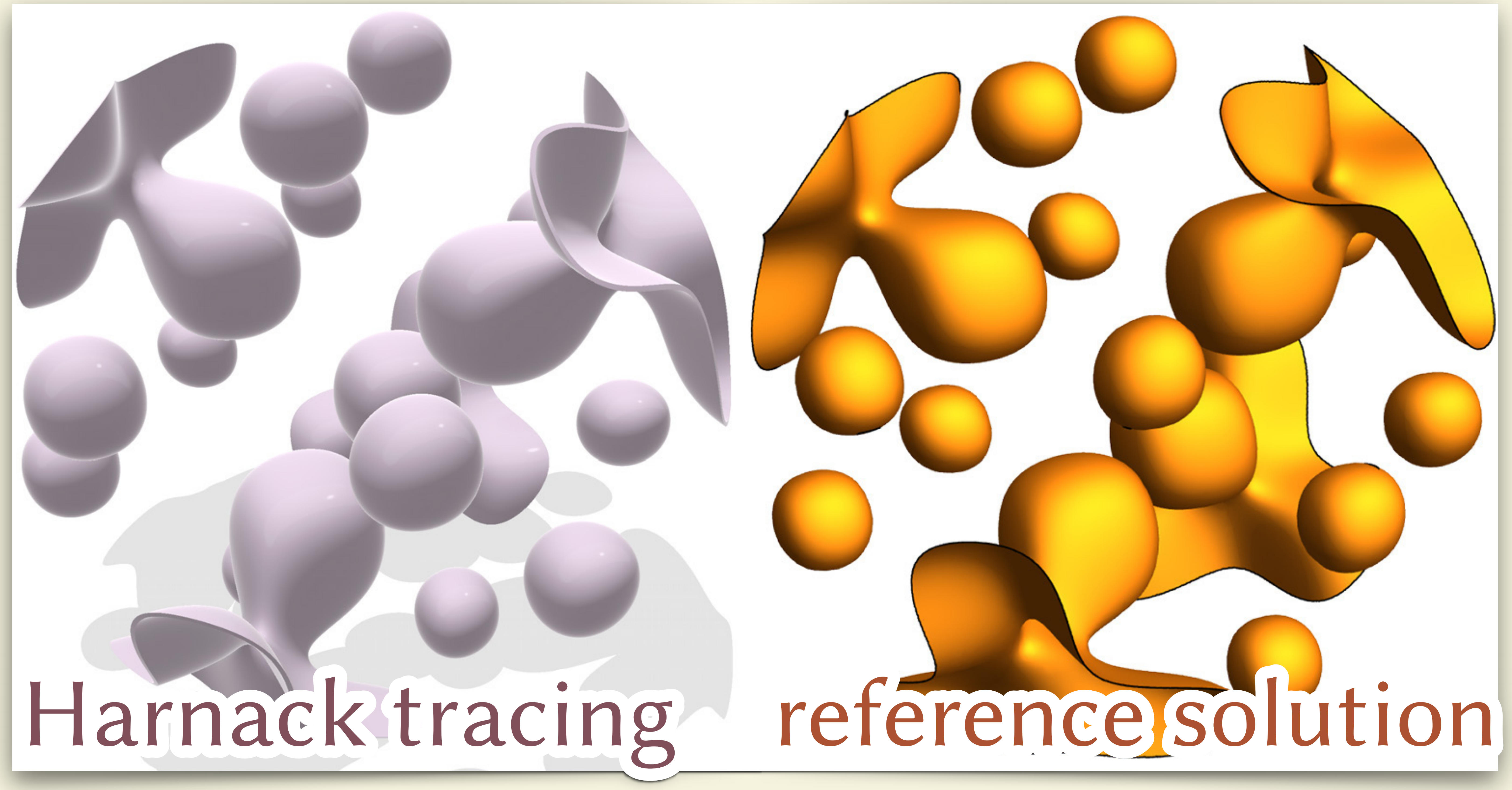
… but is a *slice* of a harmonic function in 4D



(directly ray traced)

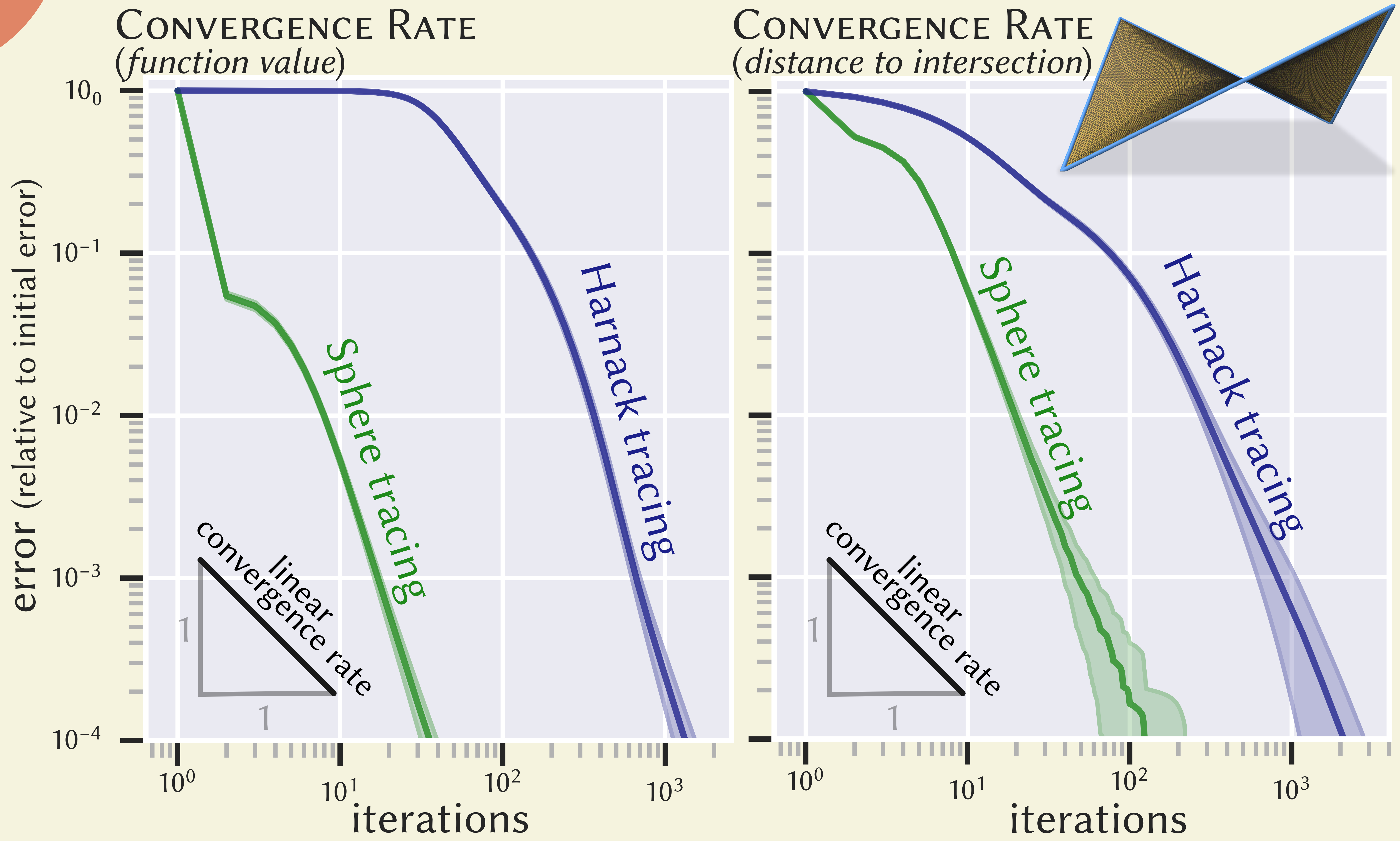
Laplacian Eigenfunctions

$$\Delta_{\mathbb{R}^3}\,\phi(x,y,z) = \lambda\,\phi(x,y,z) \quad \Longrightarrow \quad \Delta_{\mathbb{R}^4}\left(e^{w\sqrt{-\lambda}}\phi(x,y,z)\right) = 0$$



Convergence

Same asymptotic rate as sphere tracing



IV. Future Work

Subharmonic functions

harmonic: $\Delta f = 0$

subharmonic: $\Delta f \leq 0$

less than the harmonic function
with the same boundary values

obeys *upper* bounds on
harmonic functions

superharmonic: $\Delta f \geq 0$

greater than the harmonic function
with the same boundary values

obeys *lower* bounds on
harmonic functions

Can we apply Harnack tracing?

Warning: this slide uses the *positive-semidefinite Laplacian* where $\Delta f = -\sum_i \frac{\partial^2}{\partial x_i^2} f$

Functions with bounded Laplacian

$$\text{if } |\Delta f| \leq \lambda, \qquad \text{then } f(x) - \frac{\lambda}{2d}\|x\|^2_{\mathbb{R}^d} \text{ is superharmonic}$$

$$\text{and } f(x) + \frac{\lambda}{2d}\|x\|^2_{\mathbb{R}^d} \text{ is subharmonic}$$

Warning: this slide uses the *positive-semidefinite Laplacian* where $\Delta f = -\sum_i \frac{\partial^2}{\partial x_i^2} f$

Harnack tracing for other PDEs

Harnack inequalities exist for many PDEs

But positivity becomes harder to enforce!

Optimization

Signed Distance Functions

eikonal condition $||\nabla f|| = 1$

✗ nonconvex, nondifferentiable

✗ insufficient to ensure f is an SDF

[Xie et al. 2022, Marschner et al. 2023

Harmonic Functions

$\Delta f = 0$

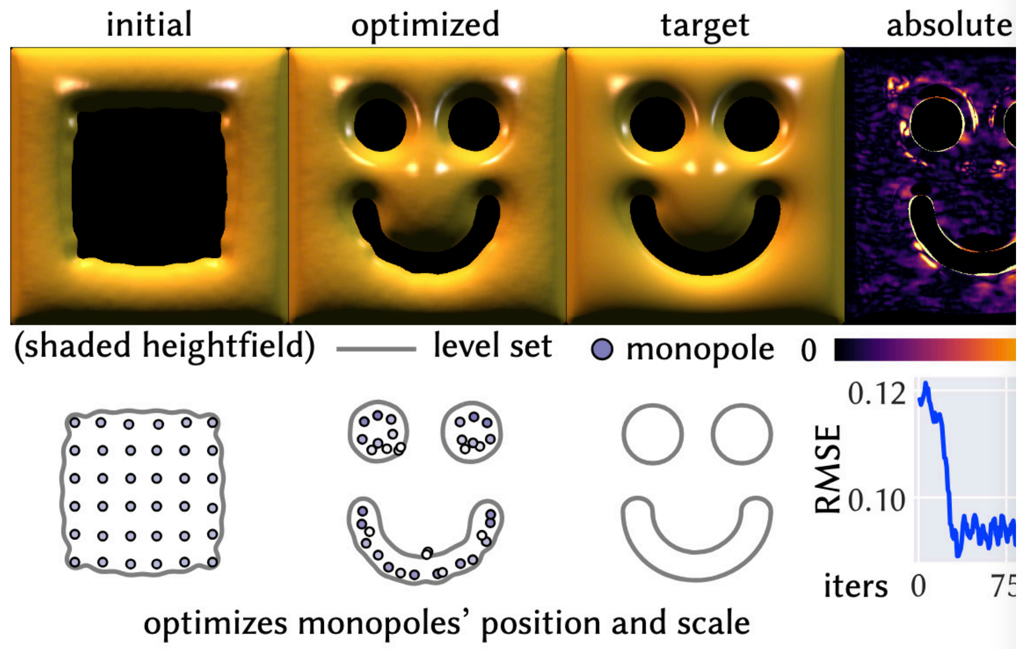


Figure 12. Baran and Lehtinen [2009] define smooth heightfields domain as the solution to a Poisson equation with zero Dirichlet bo conditions. We optimize an implicit boundary, specifically the positi scale of harmonic radial basis functions, to achieve a target heig visualized as a shaded surface (*top row*).

[Miller et al. 2024]

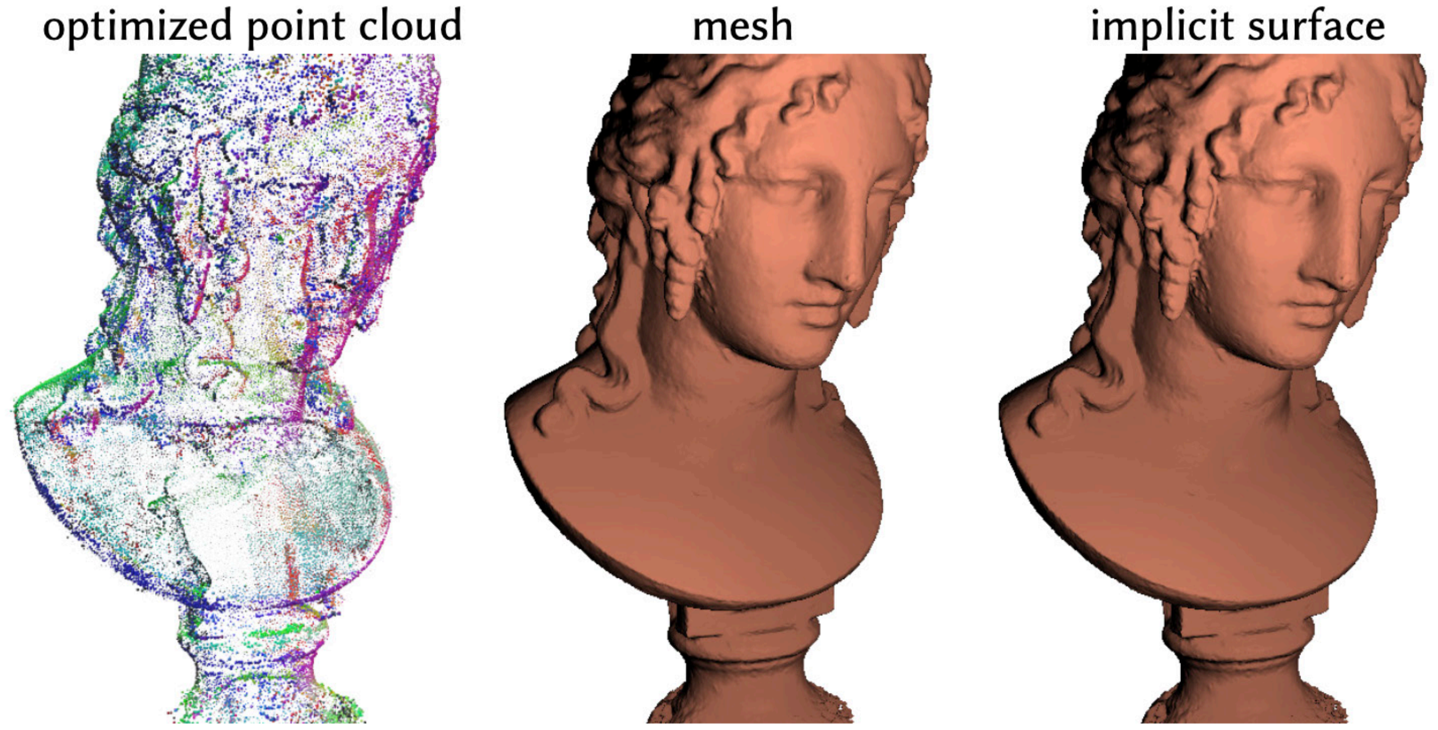
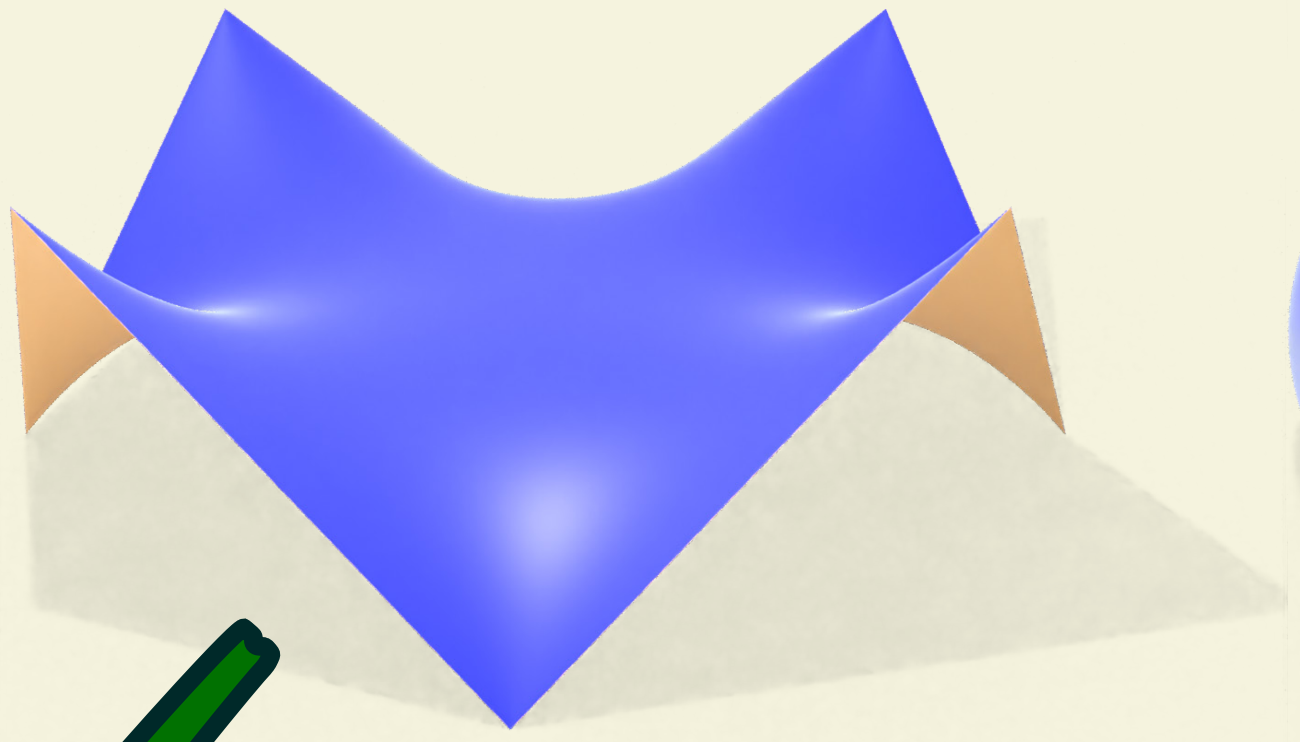


Figure 5. Our regularized dipole sum representation allows us to directly ray trace the optimized point cloud (where we use color to visualize normals, and size to visualize geometry attributes), achieving the same results as ray tracing a mesh without the need to extract one.

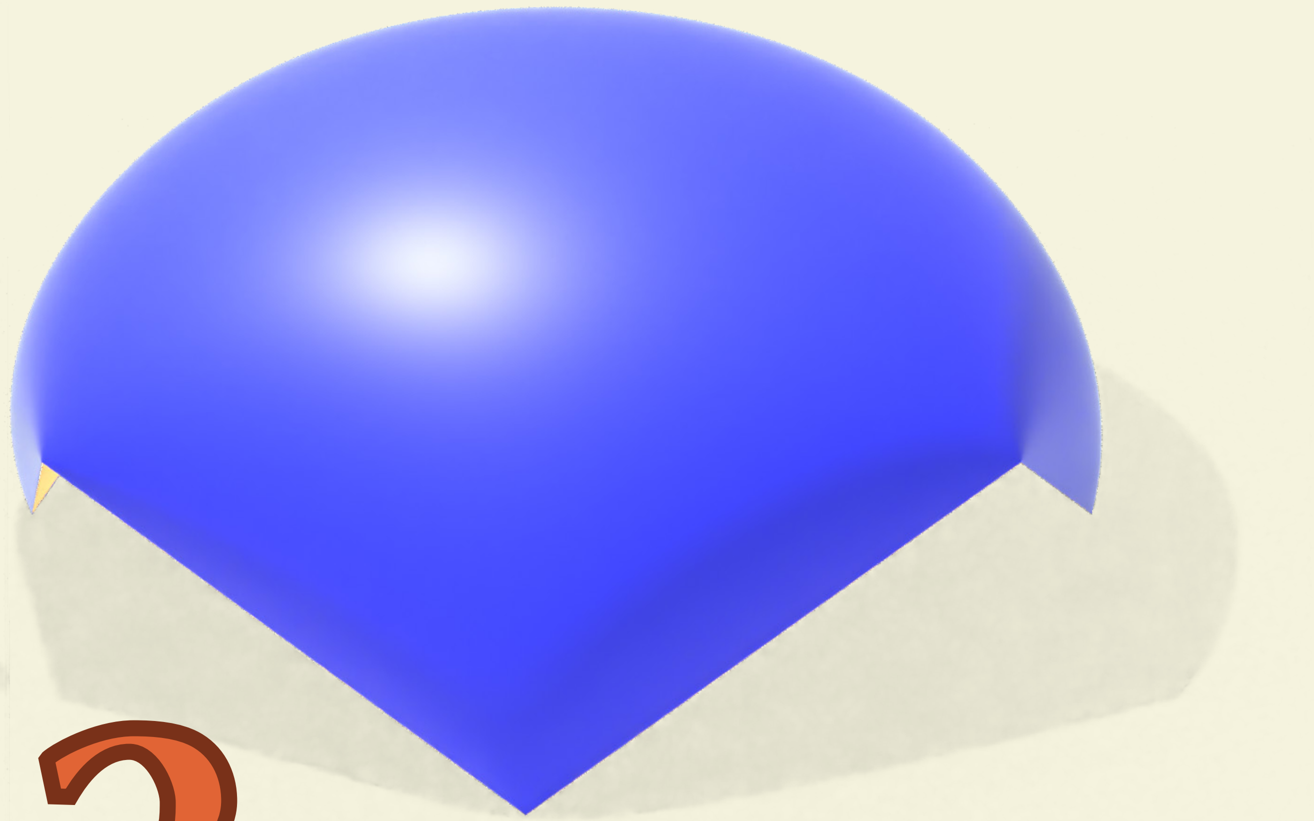
[Chen et al. 2024]

Solid angle bounds

spatial extent of level sets

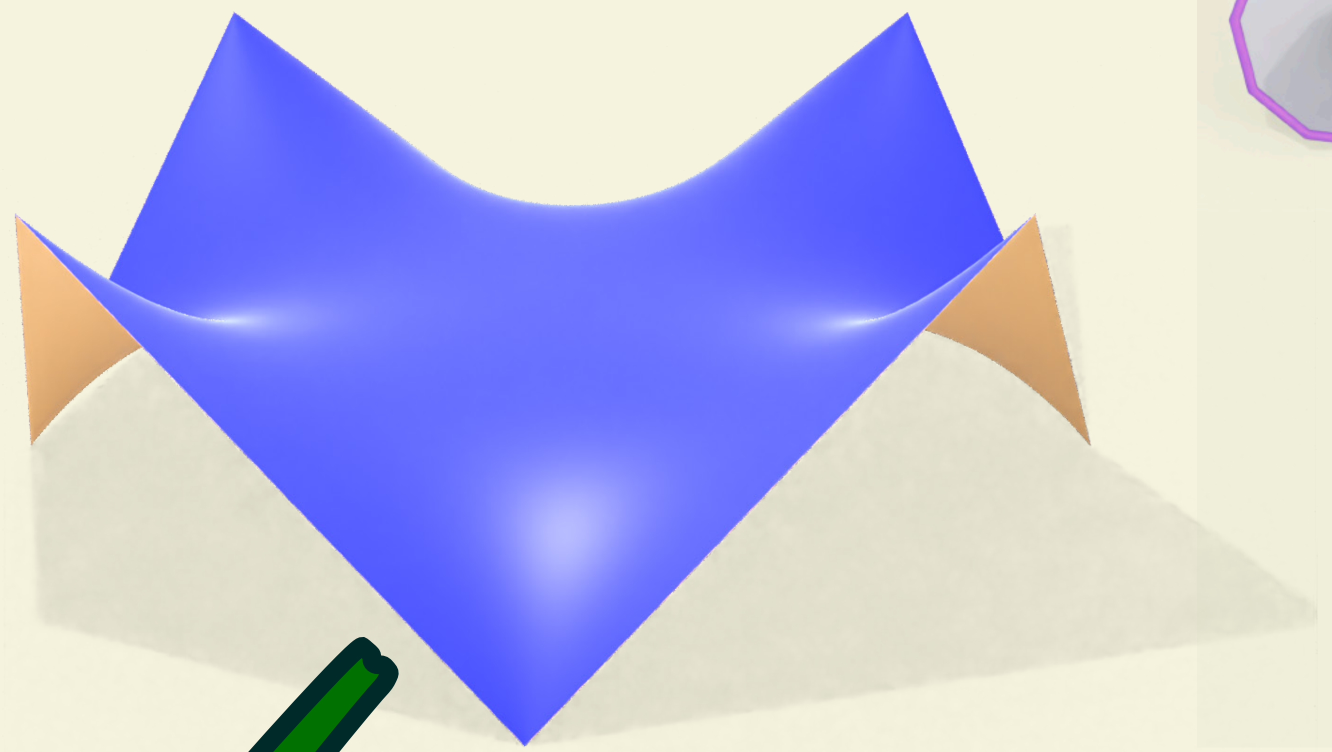


2π level set

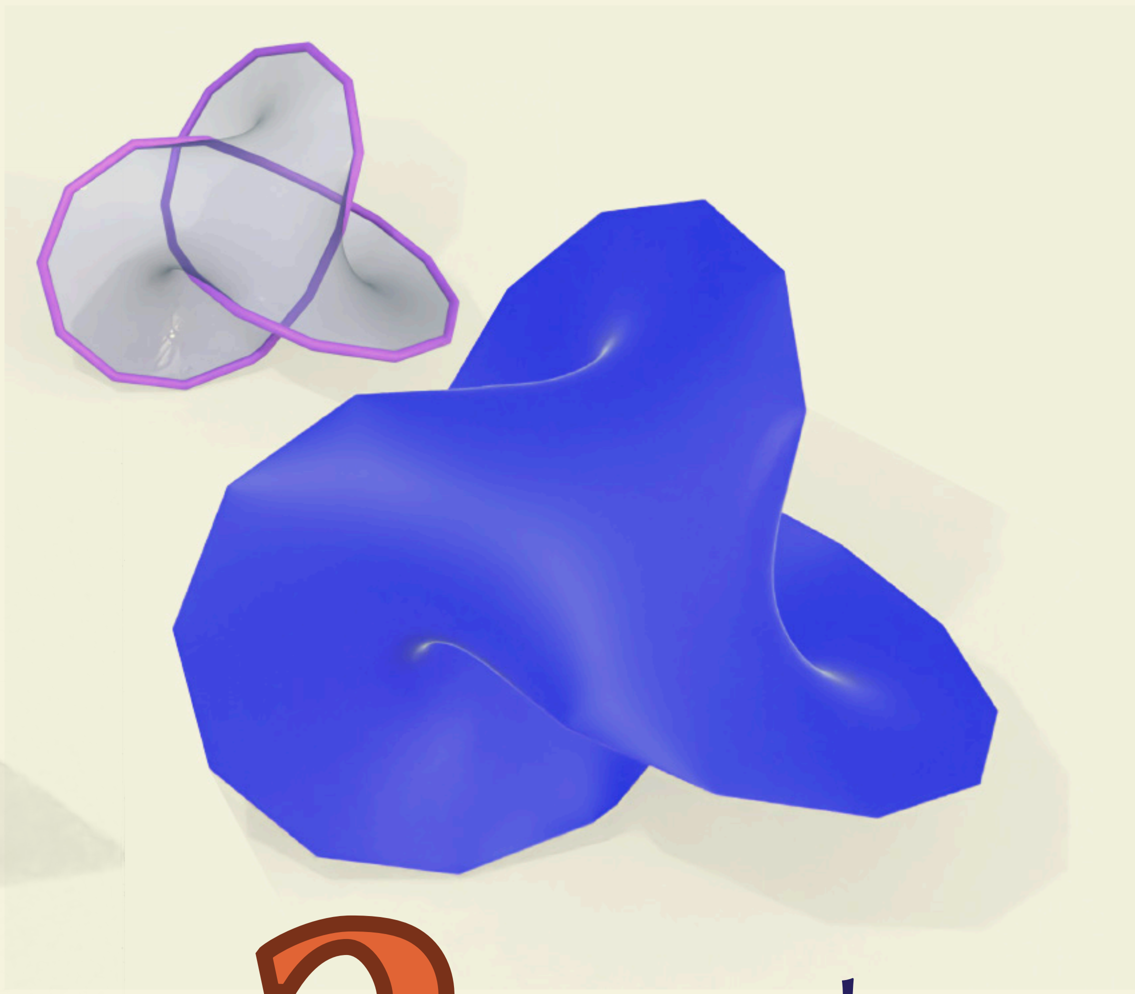


3π level set

function value



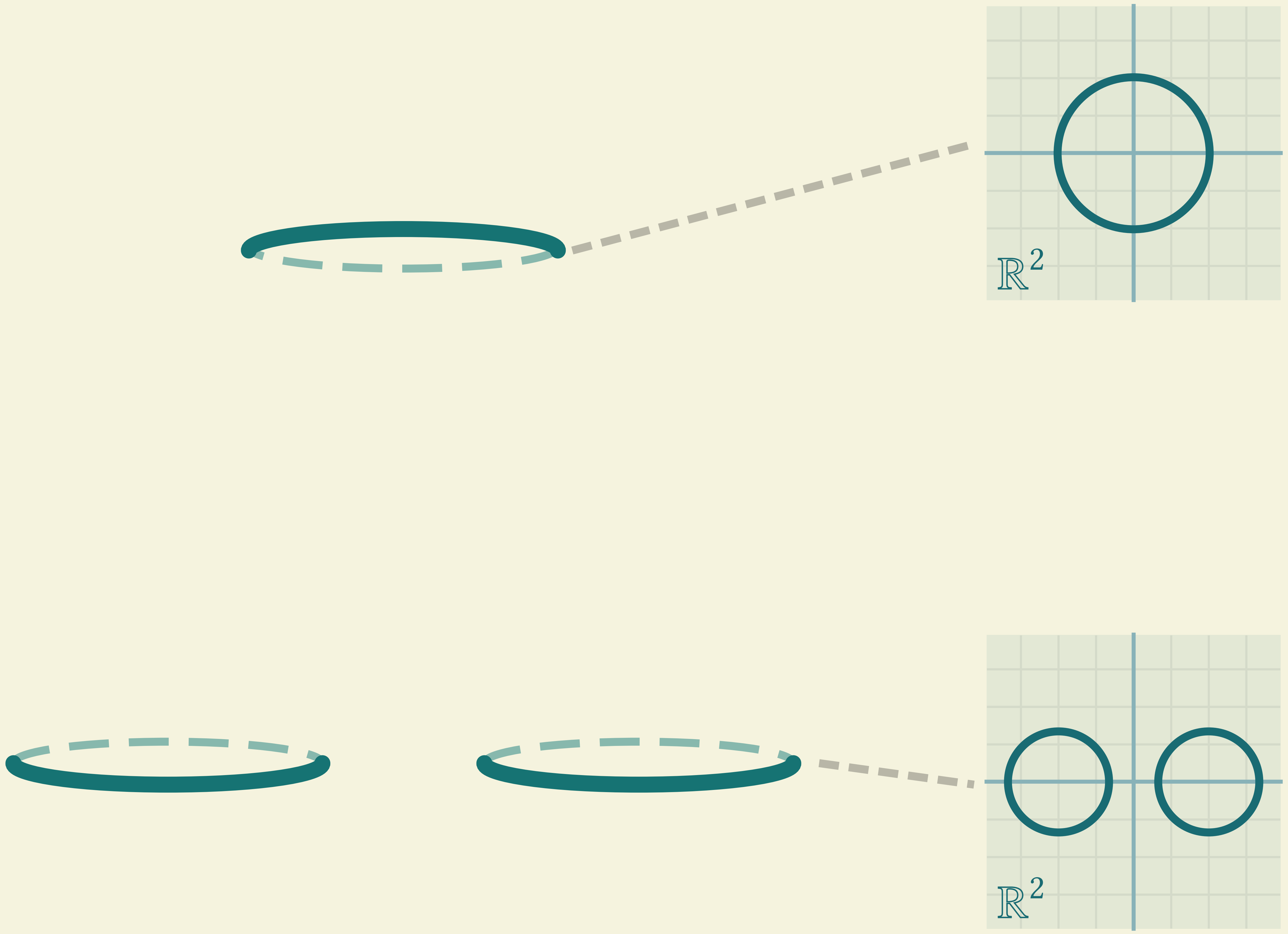
“convex”



general knots

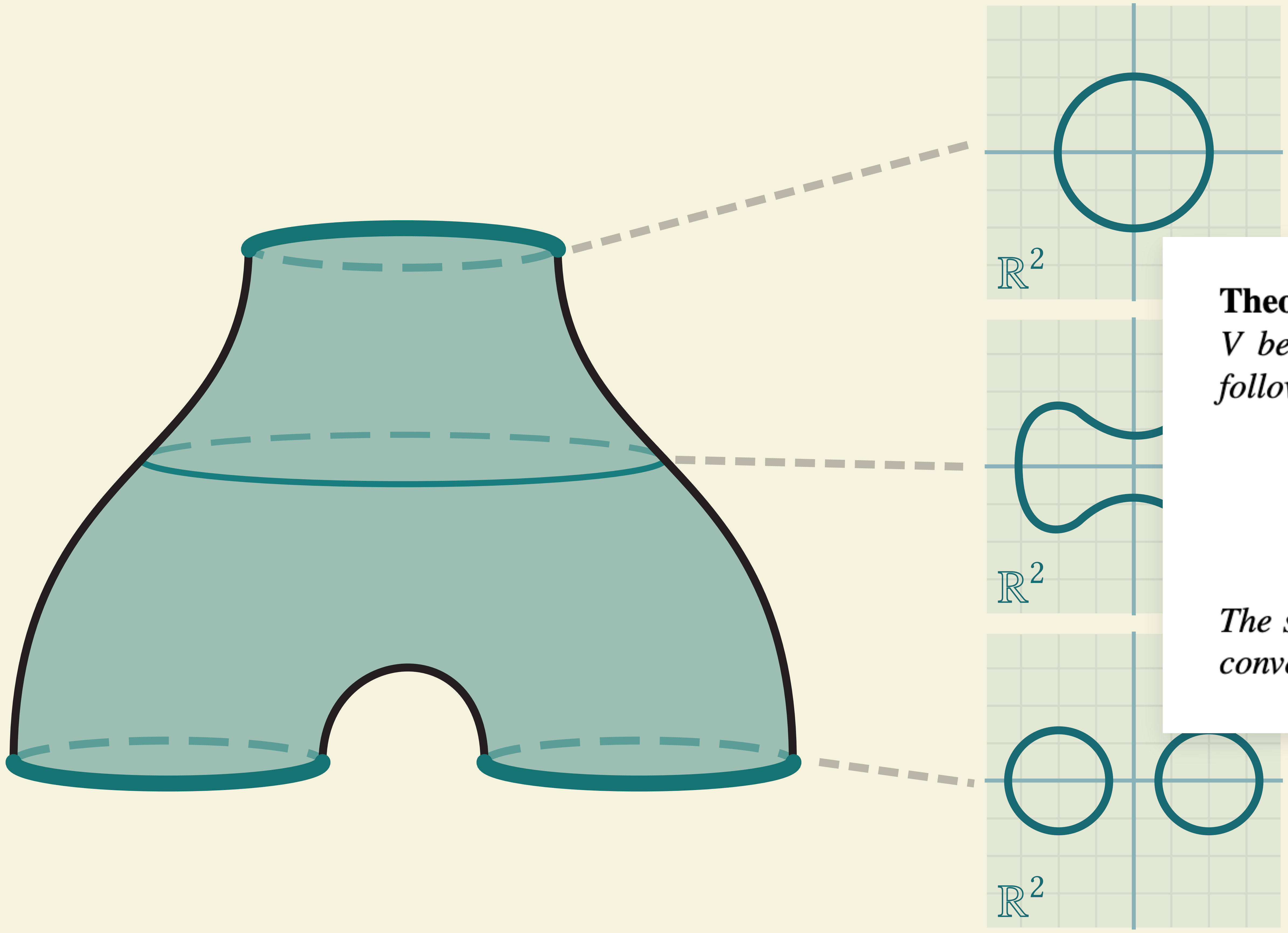
Solid angle in 4D

Shape interpolation via solid angle



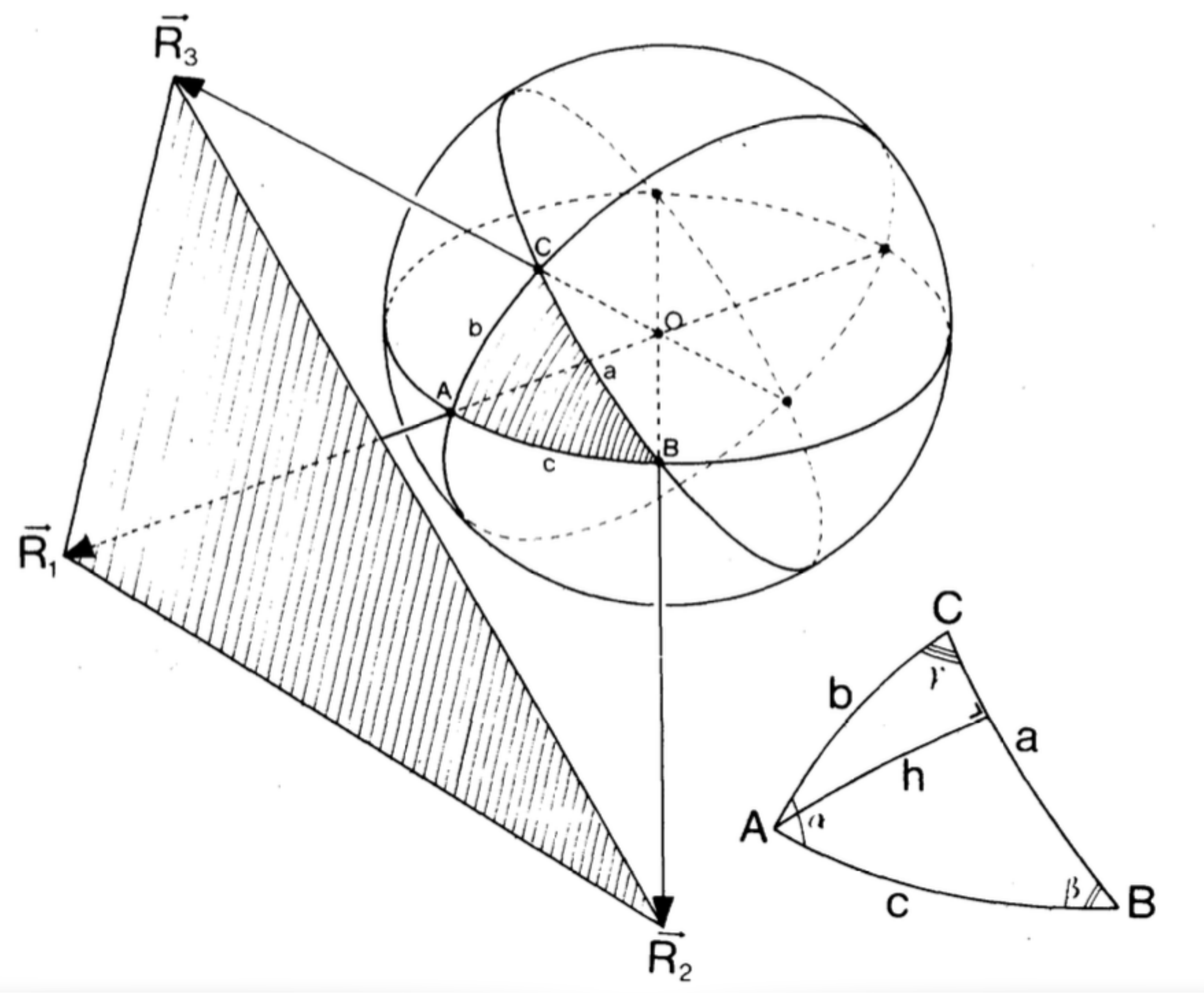
Solid angle in 4D

Shape interpolation via solid angle



3D solid angle formula

[van Oosterom and Strackee 1983]



Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a solid angle spanned by unit vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, let V be the matrix whose ith column is $\mathbf{v}_i$, and let $\alpha_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$, as above. Let T_α be the following infinite multivariable Taylor series:*

$$T_\alpha = \frac{|\det V|}{(4\pi)^{n/2}} \sum_{\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_i \Gamma\left(\frac{1 + \sum_{m \neq i} a_{im}}{2} \right) \right] \alpha^{\mathbf{a}}.$$

The series T_α agrees with $\tilde{V}_\Omega$, the normalized measure of solid angle Ω, wherever T_α converges.

[Ribando 2006]

4D solid angle formula?

Thanks for listening

Links to Blender code and ShaderToy examples can be found at:

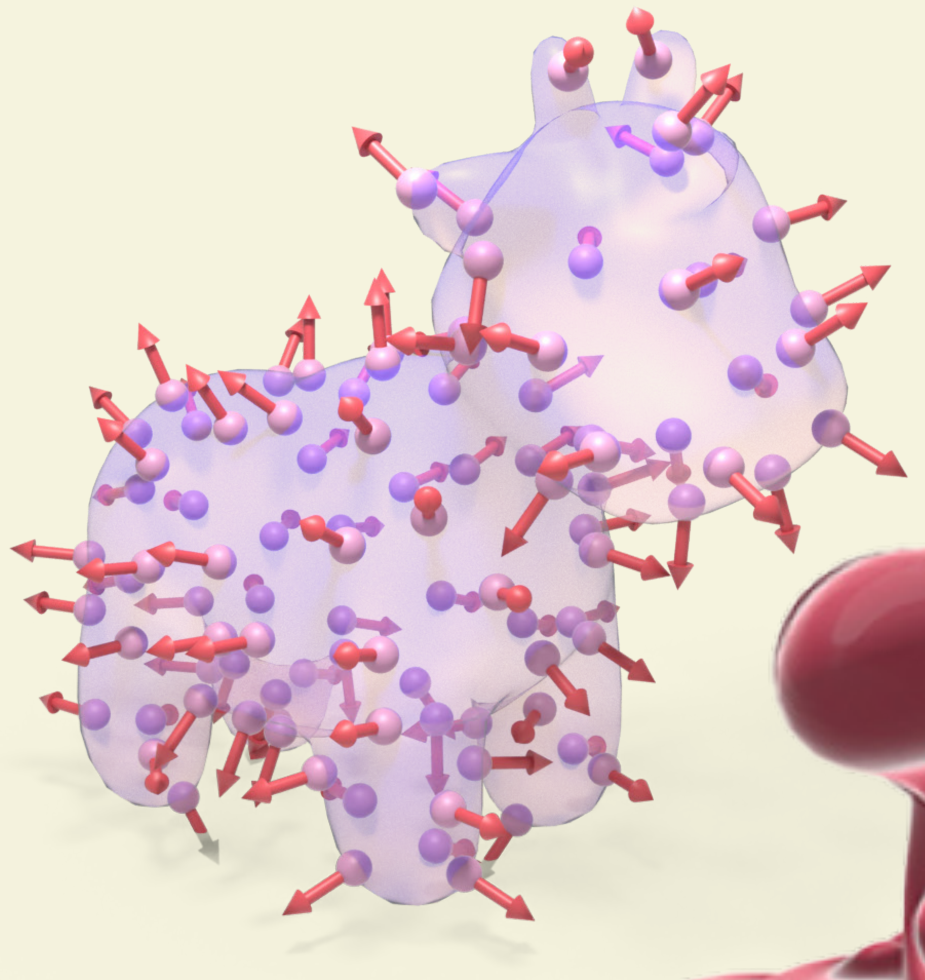
www.markjgillespie.com/Research/harnack-tracing

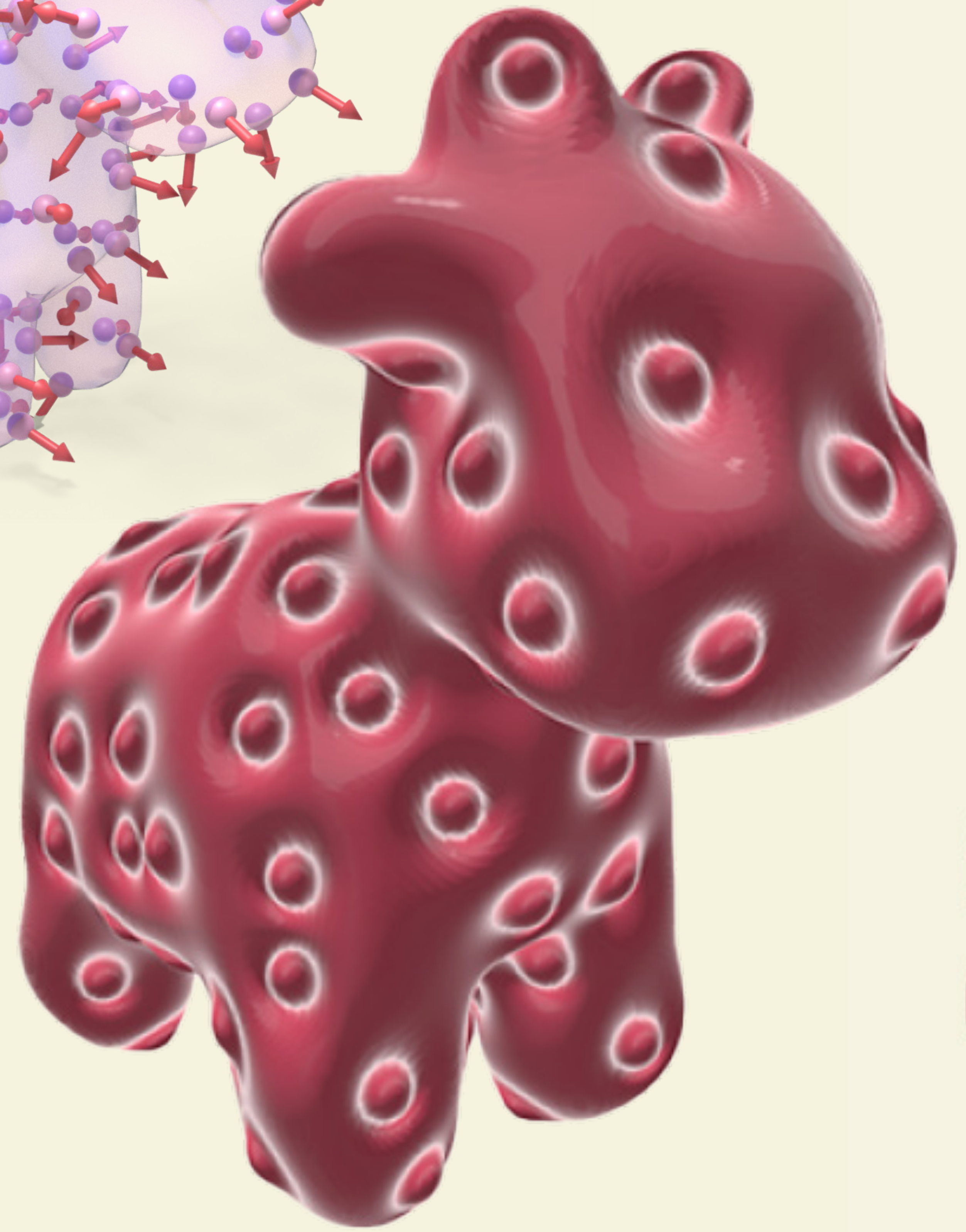
Supplemental Slides

Poisson gradient termination artifacts

$$f(\vec{x}) := \sum_{i=1}^{k} a_i \frac{(\vec{p}_i - \vec{x}) \cdot \vec{n}_i}{\|\vec{p}_i - \vec{x}\|^3}$$

$$\nabla f(\vec{x}) = \sum_{i=1}^{k} a_i \left(3 \frac{(\vec{p}_i - \vec{x}) \cdot \vec{n}_i}{\|\vec{p}_i - \vec{x}\|^5} (\vec{p}_i - \vec{x}) - \frac{1}{\|\vec{p}_i - \vec{x}\|^3} \vec{n}_i \right)$$



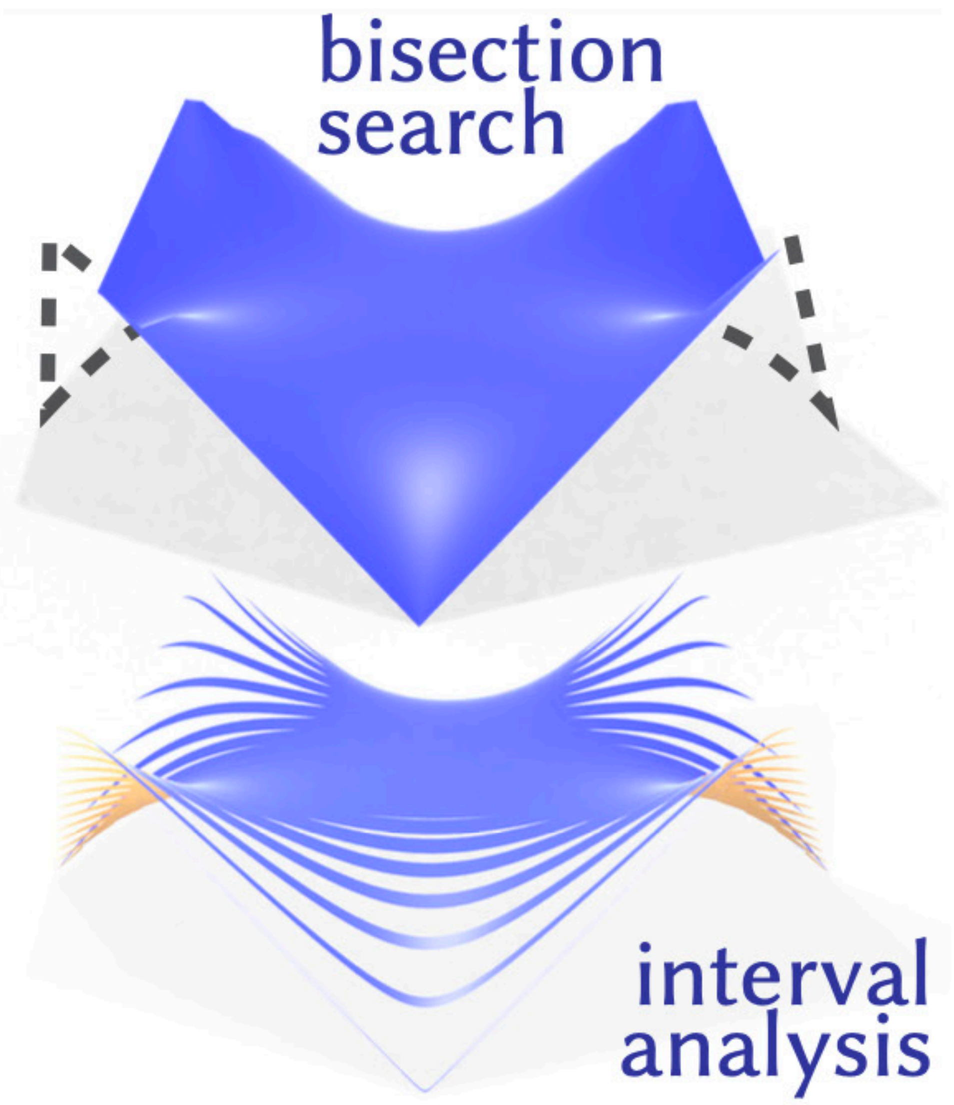


gradient termination

fixed termination

Filtering out spurious intersections

Moreover, for angle-valued functions, one may detect discontinuities in $f(\mathbf{x})$, rather than true geometric intersections. For interval-based methods, one can try to "patch" this issue by, *e.g.*, checking whether the value of $\phi(t)$ at the center of the interval is within ε of zero, but this sort of modification voids any guarantees—causing new artifacts (see inset). For level sets of simple



functions, like the globally continuous gyroid in Figure 27 *(top right)*, interval analysis can reliably compute the first intersection. But for the broader class of surfaces handled by Harnack tracing, significant artifacts were visible in all root finding methods we tried (Figure 27).

Solid angle numerics

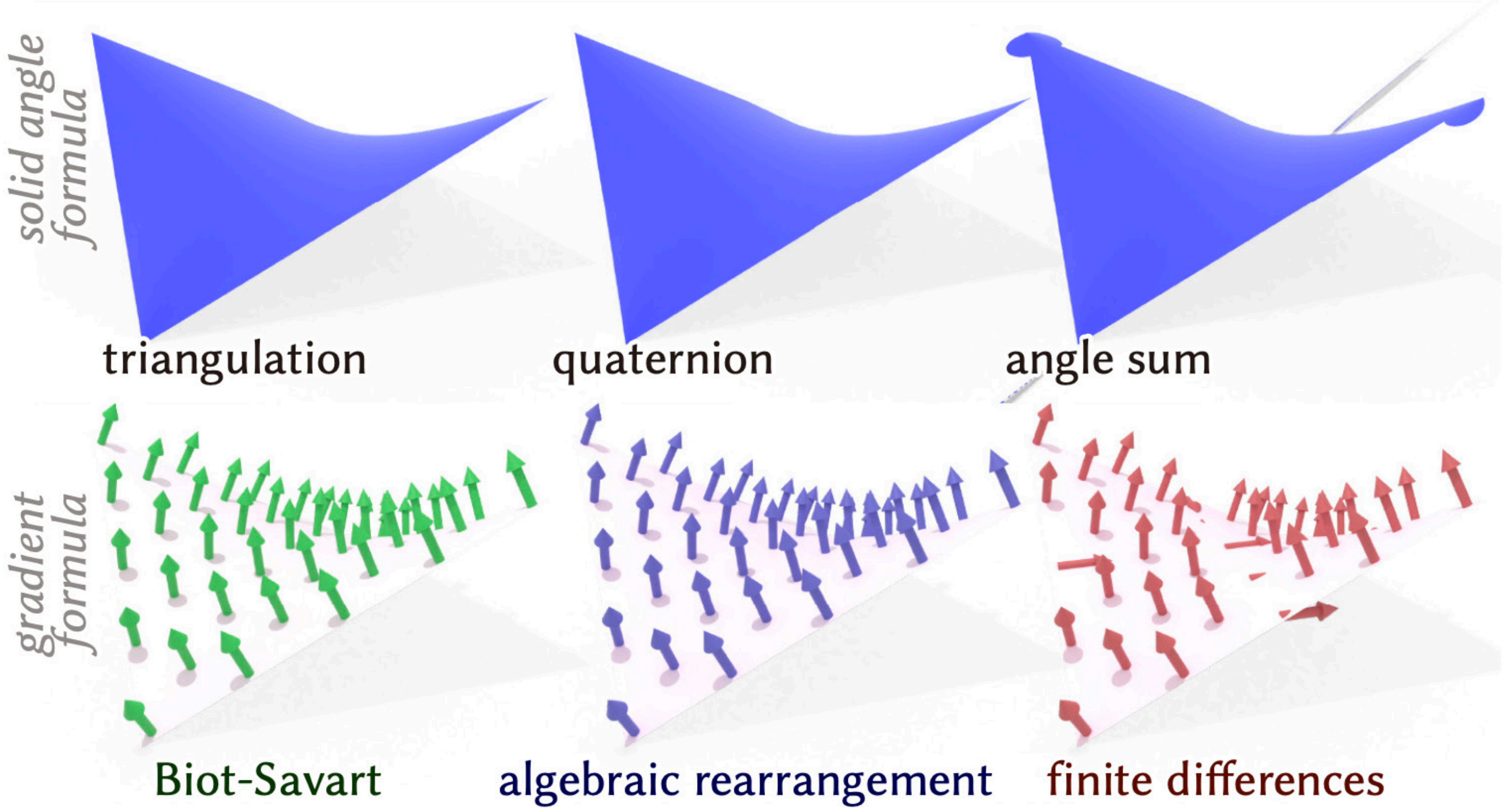


Fig. 12. Not all expressions for the solid angle or its derivative provide accurate results in floating point. *Top*: the solid angle formulas based on triangulation and quaternions work well, but the expression based on angle sums suffers from numerical instability. *Bottom*: The Biot-Savart law and its rearrangement by Adiels et al. [2022] both yield accurate normals, but finite differences give incorrect results due to jumps in the angle-valued function.

Off-centered envelopes

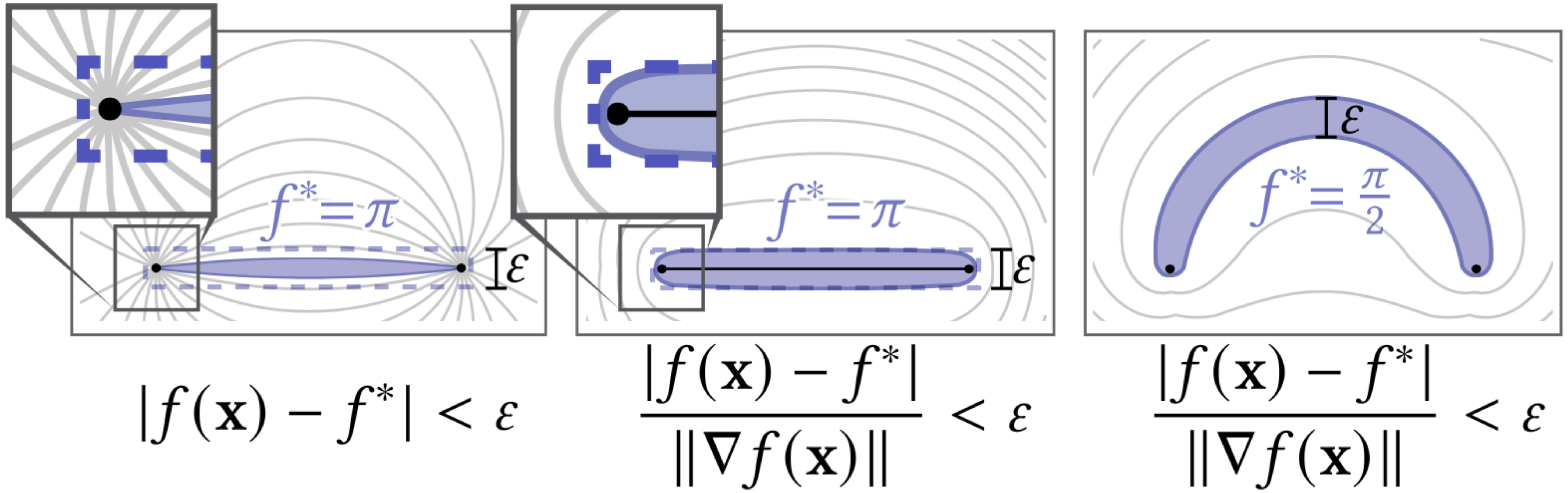


Fig. 5. If $f(\mathbf{x})$ is a signed distance function, then terminating intersection queries when $|f(\mathbf{x}) - f^*| < \varepsilon$ ensures that $\mathbf{x}$ is within ε of the chosen level set. But, when $f(\mathbf{x})$ is a general function, this condition loses its geometric meaning and produces an uneven profile along the target surface *(left)*. We can obtain a more meaningful stopping condition using the gradient $\nabla f(\mathbf{x})$, to relate changes in function value to changes in position *(center, right)*.