Ray Tracing Harmonic Functions

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with

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Harmonic functions

special kind of function



$$\sum_{i} \frac{\partial^2 f}{\partial x_i^2} = 0$$

well-understood mathematically





Harmonic functions

 $\sum_{n=1}^{n} \frac{\partial^2 f}{\partial x^2} = 0$ $\Delta f := \sum_{i=1}^{n}$



Harmonic functions

 $\neg \frac{\partial^2 f}{\partial x^2} = 0$ $\Delta f :=$



S

ar



Intersecting a ray with a level set





Level sets of harmonic functions show up everywhere

Poisson surface reconstruction [Kazhdan *et al.* 2006]

generalized winding numbers [Jacobson *et al.* 2013]



nonplanar polygons [Maxwell 1873]

curve networks de Goes *et al.* 2011]

input



Riemann surfaces [Riemann 1851]

hyperspherical harmonics Fock 1935]





shell structures in architectural geometry [Adiels *et al.* 2022]

space-filling surfaces for digital fabrication 6





... but, they're hard to render with existing techniques





may have singularities



increasing (purported) Lipschitz constant

ray marching (with fixed step size)

sphere tracing (with purported Lipschitz constant)



... but, they're hard to render with existing techniques





may have boundaries





... but, they're hard to render with existing techniques





may have boundaries





Sphere tracing [Hart 1996]

compute intersections for signed distance functions (SDFs)

f(x) = distance to curve



Sphere tracing [Hart 1996]

compute intersections for signed distance functions (SDFs)



Sphere tracing: beyond SDFs [Hart 1996]

• Easy to generalize to *Lipschitz* functions:

(essentially, $|\nabla f| \leq L$)

- Important fact: $|f(x) - f(y)| \le L|x - y|$
- provides a conservative bound on distance

[Inigo Quilez 2015]



Problem: many harmonic functions are not Lipschitz





Problem: many harmonic functions are not Lipschitz







Problem: many harmonic functions are not Lipschitz



$\theta(x, y) = \operatorname{atan2}(y, x)$



+*π* 0

Problem: many harmonic functions are not Lipschitz

No matter how close points get, function values never get closer

$\theta(x, y) = \operatorname{atan2}(y, x)$

no distance bound for sphere tracing

$\begin{array}{c} \theta = \frac{\pi}{4} \\ \bullet \\ \bullet \\ \theta = 0 \end{array}$

 $+\pi$

Main idea: get distance bounds from Harnack's inequality

Let *f* be a positive harmonic function on a ball:



lower bound

upper bound

always safe to take step of size $\frac{R}{2} \left| a+2-\sqrt{a^2+8a} \right|,$ f(x)where a =

We can use the fact that f is harmonic to obtain a distance bound

2D example



Outline

I. HARNACK'S INEQUALITY II. HARNACK TRACING















Harnack's Inequality

$(\operatorname{in} \mathbb{R}^d)$



lower bound









a linear function can change arbitrarily fast

*technically speaking, positive affine functions





f(0)

a linear function can change arbitrarily fast

*technically speaking, positive affine functions





a linear function can change arbitrarily fast

*technically speaking, positive affine functions





a linear function can change arbitrarily fast

but if it changes too fast, it does not stay *positive*

*technically speaking, positive affine functions





a linear function can change arbitrarily fast

but if it changes too fast, it does not stay *positive* f(0) = 1

*technically speaking, positive affine functions



positive linear functions must stay between the upper and lower bounds



The Mean Value Property









mean value property

$$f(x) = \frac{1}{\operatorname{vol}(S)} \int_{S} f(z) \, dz$$





mean value property
$$f(x) = \frac{1}{\operatorname{vol}(S)} \int_{S} f(z) \, dz$$

$$f(y) = \frac{1}{\text{vol}(S)} \int_{S} \frac{R^2 - r^2}{R^{2-d} |y - z|^d} f(z) dz$$

weighted average





mean value property
$$f(x) = \frac{1}{\operatorname{vol}(S)} \int_{S} f(z) \, dz$$

$$f(y) = \frac{1}{\text{vol}(S)} \int_{S} \frac{R^2 - r^2}{R^{2-d} |y-z|^d} f(z) dz$$

$$R - r \le |y - z| \le R + r$$





$$mean value property$$
$$f(x) = \frac{1}{\text{vol}(S)} \int_{S} f(z) dz$$
Poisson kernel
$$f(y) = \frac{1}{\text{vol}(S)} \int_{S} \frac{R^2 - r^2}{R^{2-d} |y-z|^d} f(z) dz$$

$$\frac{r^2}{(r+r)^d} f(z) \, dz \le f(y) \le \frac{1}{\operatorname{vol}(S)} \int_S \frac{R^2 - r^2}{R^{2-d}(R-r)^d} \, f(z)$$







mean value property
$$f(x) = \frac{1}{\operatorname{vol}(S)} \int_{S} f(z) \, dz$$

$$f(y) = \frac{1}{\text{vol}(S)} \int_{S} \frac{R^2 - r^2}{R^{2-d} |y - z|^d} f(z) \, dz$$

$$\frac{1}{(S)} \int_{S}^{f(z)} dz \le f(y) \le \frac{R^2 - r^2}{R^{2-d}(R-r)^d} \frac{1}{\operatorname{vol}(S)} \int_{S}^{f(z)} f(z)$$







mean value property
$$f(x) = \frac{1}{\operatorname{vol}(S)} \int_{S} f(z) \, dz$$

$$f(y) = \frac{1}{\text{vol}(S)} \int_{S} \frac{R^2 - r^2}{R^{2-d} |y-z|^d} f(z) dz$$

$$f(x) \leq f(y) \leq \frac{R^2 - r^2}{R^{2-d}(R-r)^d} \qquad f(x)$$





Harnack's Inequality



mean value property $f(x) = \frac{1}{\operatorname{vol}(S)} \int_{S} f(z) \, dz$

Poisson kernel

$$f(y) = \frac{1}{\text{vol}(S)} \int_{S} \frac{R^2 - r^2}{R^{2-d} |y - z|^d} f(z) \, dz$$

Harnack's inequality

$$\frac{R^2 - r^2}{R^{2-d}(R+r)^d} f(x) \le f(y) \le \frac{R^2 - r^2}{R^{2-d}(R-r)^d} f(x)$$





Harnack's Inequality



mean value property $f(x) = \frac{1}{\operatorname{vol}(S)} \int_{S} f(z) \, dz$

Poisson kernel

$$f(y) = \frac{1}{\text{vol}(S)} \int_{S} \frac{R^2 - r^2}{R^{2-d} |y - z|^d} f(z) dz$$

Harnack's inequality

 $\frac{1 - r/R}{(1 + r/R)^{d-1}} f(x) \le f(y) \le \frac{1 + R/r}{(1 - r/R)^{d-1}} f(x)$












Distance bounds from Harnack's inequality

Let *f* be a positive harmonic function on a ball:



lower bound

upper bound

always safe to take step of size $\frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$ where a =

We can use the fact that f is harmonic to obtain a distance bound



Distance bounds from Harnack's inequality

Let f be a positive harmonic function on a ball:

What if *f* is not positive? Just add a constant to make it positive on the ball

 $\frac{1 + r/R}{r/R} f(x)$

always safe to take step of size $r = \frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$ where $a = \frac{f(x)}{f^*}$

All you need is a valid ball radius and a lower bound on f



Algorithm sketch

Harnack TracingStarting from point x_0 in direction d:Pick ball radiusPick ball radiusShift f to be positive on ballCalculate safe step sizeTake safe step in ray directionRepeat until f is sufficiently close to f^*





Invalid lower bounds





Balancing the radius and shift





Balancing the radius and shift



smaller radius, larger shift



larger radius, smaller shift





Sphere tracing acceleration

[Keinert et al. 2014]: "over-stepping"

conservative steps

valid oversteps







Acceleration: gradient termination

How do you decide when you have "hit" the surface?

$|f(\mathbf{x}) - f^*| < \varepsilon$





Acceleration: gradient termination

How do you decide when you have "hit" the surface?





Acceleration: gradient termination

How do you decide when you have "hit" the surface?



 $|f(\mathbf{x}) - f^*| < \varepsilon$



 $\frac{|f(\mathbf{x}) - f^*|}{|\nabla f(\mathbf{x})|} < \varepsilon$



Angle-valued functions

$\theta(x, y) = \operatorname{atan2}(y, x)$











Angle-valued functions

continuous when viewed modulo 2π



$\theta(x, y) = \operatorname{atan2}(y, x)$







Angle-valued functions → continuous functions



DISCOMINNOOS FUNCTION





In practice: look for level sets above and below

distance to upper level set distance to lower level set distance to *f**





Angle-valued functions allow for boundaries







Angle-valued functions allow for boundaries





shaders)

(WebGL





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PBRT (CPU ray tracer)



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Blender (CPU ray tracer)

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PBRT (CPU ray tracer)







Generalized winding number [Jacobson et al. 2013]

input mesh

a.k.a. signed solid angle

repaired mesh
(directly ray traced)



Signed solid angle

signed solid angle



Signed solid angle





more planar

$(2\pi \text{ level set})$

3π level set



General nonplanar polygons







Interpolating surfaces





Continuous interpolation



Discontinuous Jump



Architectural grid shells [Adiels et al. 2022]

those with curved surfaces like shells and grid shells carry load mainly through membrane action, making t and beams used today. The complex geometry, combine production, spatial and aesthetic aspects, makes this centuries. Early treatises in architectural geometry inc (1512-1570), examining the art of cutting stones in va and applications from the field of differential geometry have experimented with various shapes to balance requi and Félix Candela [4], Eladio Dieste's "Gaussian vaults Other examples include Weingarten surfaces [7], such surfaces. Additional techniques include form finding [8]

Emil Adiels

12-sided polygon





Chalmers University of Technology, Sweden, e-mail: emil.adie Mats Ander

Chalmers University of Technology, Sweden, e-mail: mats.and Chris J. K. Williams

Chalmers University of Technology, Sweden, e-mail: christoph

Surface reconstruction [Kazhdan et al. 2006]

visualize results of Poisson surface reconstruction without requiring volumetric meshing or linear solves

____ �___

[Barill et al. 2018] : evaluate solution as a sum of dipoles





Riemann surfaces



Fig. 24 Relief des 2. Zweiges der Funktion $F(\varphi, k)$ mit k = 0.8. $(\varphi = \varphi_1 + i \varphi_2)$ Fig. 24 Relief of the 2nd branch of the function $F(\varphi, k)$ with k = 0.8. $(\varphi = \varphi_1 + i \varphi_2)$

[Jahnke, Emde & Lösch 1960]



Riemann surfaces as graphs





Riemann surfaces as graphs

f(x, y) - z = 0

level set











Hyperspherical harmonics



 $f(x, y, z, w) = y^3 - 3yz^2$

 $f(x, y, z, w) = x^{3}y + xy^{3} - 3xyw^{2} - 3xyz^{2}$ (directly ray traced) 70



The gyroid

[Diegel 2021]

Metal AM heat exchanger design workflow

| contents | news | ever

not a harmonic function in 3D ... but is a *slice* of a harmonic function in 4D





Fig. 6 Section view of completed heat exchanger, including hot and cold fluid zones (left), and the printed part showing minimal support material requirements (right).



*technically the trigonometric approximation to the gyroid

Laplacian Eigenfunctions

 $\Delta_{\mathbb{R}^3}\phi(x,y,z) = \lambda\,\phi(x,y,z) \implies \Delta_{\mathbb{R}^4}\left(e^{w\sqrt{-\lambda}}\phi(x,y,z)\right) = 0$

Convergence

Same asymptotic rate as sphere tracing











Subharmonic functions

harmonic: $\Delta f = 0$

subharmonic: $\Delta f \leq 0$

less than the harmonic function with the same boundary values obeys upper bounds on harmonic functions

Can we apply Harnack tracing?

Warning: this slide uses the *positive-semidefinite Laplacian* where $\Delta f = -\sum_{i} \frac{\partial^2}{\partial x_i^2} f$

superharmonic: $\Delta f \ge 0$

greater than the harmonic function with the same boundary values obeys *lower* bounds on harmonic functions



Functions with bounded Laplacian

if $|\Delta f| \leq \lambda$, then $f(x) - \frac{1}{2}$

and f(x) +

Warning: this slide uses the *positive-semidefinite Laplacian* where $\Delta f = -\sum_{i} \frac{\partial^2}{\partial x_i^2} f$

$$\frac{\lambda}{2d} \|x\|_{\mathbb{R}^d}^2 \text{ is superharmonic}$$
$$\frac{\lambda}{2d} \|x\|_{\mathbb{R}^d}^2 \text{ is subharmonic}$$



Harnack tracing for other PDEs

Harnack inequalities exist for many PDEs But positivity becomes harder to enforce!



77

Optimization

Signed Distance Functions eikonal condition $||\nabla f|| = 1$ insufficient to ensure f is an SDF Xie et al. 2022, Marschner et al. 2023]





Solid angle bounds

spatial extent of level sets



function value





Solid angle in 4D

Shape interpolation via solid angle







Solid angle in 4D

Shape interpolation via solid angle



3D solid angle formula



Theorem 2.2. Let $\Omega \subseteq \mathbb{R}^n$ be a solid angle spanned by unit vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, let V be the matrix whose ith column is \mathbf{v}_i , and let $\alpha_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$, as above. Let T_{α} be the following infinite multivariable Taylor series:

$$T_{\alpha} = \frac{|\det V|}{(4\pi)^{n/2}} \sum_{\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i < j} a_{ij}}}{\prod_{i < j} a_{ij}!} \prod_{i} \Gamma\left(\frac{1 + \sum_{m \neq i} a_{im}}{2}\right) \right] \alpha^{\mathbf{a}}.$$

The series T_{α} agrees with \tilde{V}_{Ω} , the normalized measure of solid angle Ω , wherever T_{α} converges. [Ribando 2006]

4D solid angle formula?



Thanks for listening

Links to Blender code and ShaderToy examples can be found at: www.markjgillespie.com/Research/harnack-tracing









Supplemental Slides

Level sets of harmonic functions show up everywhere





architectural geometry

mathematics



Poisson gradient termination artifacts

gradient termination

fixed termination

$$f(\vec{x}) \coloneqq \sum_{i=1}^{k} a_{i} \frac{(\vec{p}_{i} - \vec{x}) \cdot \vec{n}_{i}}{\|\vec{p}_{i} - \vec{x}\|^{3}}$$
$$\nabla f(\vec{x}) = \sum_{i=1}^{k} a_{i} \left(3 \frac{(\vec{p}_{i} - \vec{x}) \cdot \vec{n}_{i}}{\|\vec{p}_{i} - \vec{x}\|^{5}} (\vec{p}_{i} - \vec{x}) - \frac{1}{\|\vec{p}_{i} - \vec{x}\|^{5}} \right)$$





Filtering out spurious intersections

Moreover, for angle-valued functions, bisection search one may detect discontinuities in $f(\mathbf{x})$, rather than true geometric intersections. For interval-based methods, one can try to "patch" this issue by, *e.g.*, checking whether the value of $\phi(t)$ at the center of the interval is within ε of zero, but this sort of modification voids any guarantees—causing new interval analysis artifacts (see inset). For level sets of simple functions, like the globally continuous gyroid in Figure 27 (top right), interval analysis can reliably compute the first intersection. But for the broader class of surfaces handled by Harnack tracing, significant artifacts were visible in all root finding methods we tried (Figure 27).





Solid angle numerics



Fig. 12. Not all expressions for the solid angle or its derivative provide accurate results in floating point. *Top*: the solid angle formulas based on triangulation and quaternions work well, but the expression based on angle sums suffers from numerical instability. *Bottom*: The Biot-Savart law and its rearrangement by Adiels et al. [2022] both yield accurate normals, but finite differences give incorrect results due to jumps in the angle-valued function.



Off-centered envelopes



Fig. 5. If $f(\mathbf{x})$ is a signed distance function, then terminating intersection queries when $|f(\mathbf{x}) - f^*| < \varepsilon$ ensures that \mathbf{x} is within ε of the chosen level set. But, when $f(\mathbf{x})$ is a general function, this condition loses its geometric meaning and produces an uneven profile along the target surface (left). We can obtain a more meaningful stopping condition using the gradient $\nabla f(\mathbf{x})$, to relate changes in function value to changes in position (center, right).



In practice: look for level sets above and below



Algorithm 2 TRACEANGLEVALUED $(\mathbf{r}_0, \mathbf{v}, f^*, f(\mathbf{x}), R(\mathbf{x}), c(\mathbf{x}), \varepsilon, t_{\max})$

2: while $t < t_{max}$ do

3: $\mathbf{r}_t \leftarrow \mathbf{r}_0 + t\mathbf{v}$

4: Find the two level set values bracketing the current value of f $f_0 \leftarrow (f(\mathbf{r}_t) - f^*)/(2\pi)$ $f_- \leftarrow 2\pi |f_0| + f^*$

$$f_{+} \leftarrow 2\pi [f_{0}] + f^{*}$$

8: Stop if close to either surface (S.

if $\min(f(\mathbf{r}_t) - f_{-}, f_{+} - f(\mathbf{r}_t)) \le \varepsilon \|\nabla f(\mathbf{r}_t)\|$ then return t

11: Compute step size bound for each of the two closest level sets

$$a_{-} \leftarrow (f(\mathbf{r}_{t}) - c(\mathbf{r}_{t}))/(f_{-} - c(\mathbf{r}_{t}))$$

$$a_{+} \leftarrow (f(\mathbf{r}_{t}) - c(\mathbf{r}_{t}))/(f_{+} - c(\mathbf{r}_{t}))$$

$$\rho_{-} \leftarrow \frac{1}{2}R(\mathbf{r}_{t}) |a_{-} + 2 - \sqrt{a_{-}^{2} + 8a_{-}}|$$

$$\rho_{+} \leftarrow \frac{1}{2}R(\mathbf{r}_{t}) |a_{+} + 2 - \sqrt{a_{+}^{2} + 8a_{+}}|$$

$$\rho \leftarrow \min(\rho_{-}, \rho_{+}) \qquad \triangleright Take the smaller of the two steps$$

$$t \leftarrow t + \rho$$





Ray tracing harmonic functions in 3D

Harnack's inequality in 3D:







lower bound

upper bound

given isovalue f^* , safe to take step of size $r = \frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$ where $a = \frac{b}{f^*}$

complete algorithm:

Algorithm 1 HARNACKTRACE($\mathbf{r}_0, \mathbf{v}, f^*, f(\mathbf{x}), R(\mathbf{x}), c(\mathbf{x}), \varepsilon, t_{\max}$)

1: $t \leftarrow 0$ 2: while $t < t_{max}$ do current point along ray 3: $\mathbf{r}_t \leftarrow \mathbf{r}_0 + t\mathbf{v}$ 4: if $|f(\mathbf{r}_t) - f^*| \le \varepsilon \|\nabla f(\mathbf{r}_t)\|$ then ▶ stopping condition return t 5: if $f^* \leq c(\mathbf{r}_t)$ then $\triangleright if f^*$ lies below the lower bound... 6: $\rho \leftarrow R(\mathbf{r}_t) \triangleright \dots$ we can safely take the maximum step of R 7: else 8: $a \leftarrow (f(\mathbf{r}_t) - c(\mathbf{r}_t))/(f^* - c(\mathbf{r}_t)) \triangleright otherwise, shift and...$ 9: $\rho \leftarrow \frac{1}{2}R(\mathbf{r}_t) | a + 2 - \sqrt{a^2 + 8a} | > ... compute safe step size$ 10: *▶take step* $t \leftarrow t + \rho$ 11: ▷ray does not hit surface 12: **return** −1



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Ray tracing harmonic functions in 3D

Harnack's inequality in 3D:







lower bound

upper bound

given isovalue f^* , safe to take step of size $r = \frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$ where $a = \frac{f}{f^*}$

Algorithm 2 TRACEANGLEVALUED $(\mathbf{r}_0, \mathbf{v}, f^*, f(\mathbf{x}), R(\mathbf{x}), c(\mathbf{x}), \varepsilon, t_{\max})$

1: $t \leftarrow 0$ 2: while $t < t_{\max}$ do *current point along ray* $\mathbf{r}_t \leftarrow \mathbf{r}_0 + t\mathbf{v}$ 3: 4: Find the two level set values bracketing the current value of f 5: $f_0 \leftarrow (f(\mathbf{r}_t) - f^*)/(2\pi)$ 6: $f_{-} \leftarrow 2\pi \lfloor f_0 \rfloor + f^*$ 7: $f_+ \leftarrow 2\pi [f_0] + f^*$ 8: ⊳*Stop if close to either surface (§3.1.2)* if $\min(f(\mathbf{r}_t) - f_{-}, f_{+} - f(\mathbf{r}_t)) \le \varepsilon \|\nabla f(\mathbf{r}_t)\|$ then 9: return t 10: 11: Compute step size bound for each of the two closest level sets $a_{-} \leftarrow (f(\mathbf{r}_{t}) - c(\mathbf{r}_{t}))/(f_{-} - c(\mathbf{r}_{t}))$ 12: 13: $a_+ \leftarrow (f(\mathbf{r}_t) - c(\mathbf{r}_t))/(f_+ - c(\mathbf{r}_t))$ 14: $\rho_{-} \leftarrow \frac{1}{2}R(\mathbf{r}_{t}) | a_{-} + 2 - \sqrt{a_{-}^{2} + 8a_{-}} |$ 15: $\rho_+ \leftarrow \frac{1}{2}R(\mathbf{r}_t) \left| a_+ + 2 - \sqrt{a_+^2 + 8a_+} \right|$ $\rho \leftarrow \min(\rho_{-}, \rho_{+})$ ► Take the smaller of the two steps 16: $t \leftarrow t + \rho$ 17: *▶ray does not hit surface* 18: **return** −1





2D Harnack Tracing

Let *f* be a positive harmonic function on a 2D ball



lower bound

upper bound

We can use the fact that f is harmonic to obtain a distance bound





3D Harnack Tracing

Harnack's inequality in 3D:





lower bound

upper bound

given isovalue f^* , safe to take step of size $r = \frac{R}{2} \left| a + 2 - \sqrt{a^2 + 8a} \right|,$ where $a = \frac{f(x)}{f^*}$



