Hamiltonian Mechanics

Simple Case

We call the total energy of a system $H(x,p) + \frac{1}{2m}p^2 + V(x)$ the Hamiltonian. For example, a harmonic oscillator has Hamiltonian $H(x,p) = \frac{1}{2m}p^2 + \frac{1}{2}kx^2$. The dynamics of this system are described by *Hamilton's equations of motion*

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \frac{\partial H}{\partial p}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}p = -\frac{\partial H}{\partial x}$$

In the case of the harmonic oscillator, this gives us the familiar result

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \frac{p}{m} = v$$
$$F = \frac{\mathrm{d}}{\mathrm{d}t}p = -kx \quad \text{(Hooke's Law)}$$

Preliminary Definitions

Given a configuration space Q, we define *phase space* to be the cotangent bundle $T^\star Q$. The *Hamiltonian* is a function $H : T^\star Q \to \mathbb{R}$. We can define a *canonical one-form* Θ on $T^\star Q$ by

$$\left\langle \Theta(p,q), u_{p_q} \right\rangle = \left\langle (p,q), \mathrm{d}\pi_{T^\star Q} u_{p_q} \right\rangle$$

In coordinates, $\Theta(q,p) = p_i \mathrm{d}q^i$. Using Θ, we can define a *canonical two-form*

$$\Omega = -\mathrm{d}\Theta$$

In coordinates,

$$\Omega = \mathrm{d}q^i \wedge \mathrm{d}p^i$$

Ω gives $T^\star Q$ the structure of a *symplectic manifold* (i.e. Ω is closed and nondegenerate). If $F : T^\star Q \to T^\star Q$ preseves Ω, then F is *symplectic*. If F preserves Θ, then F is *special symplectic*. Since Ω is nondegenerate, it gives us a canonical isomorphism between the space of vector fields and the space of 1-forms. By analogy with the Riemannian case, we will define musical isomorphisms by

$$\Omega(\alpha^\sharp, X) = \alpha(X) \quad X^\flat(Y) = \Omega(X,Y)$$

Then given a Hamiltonian H we can define a unique Hamiltonian vector field X_H by

$$X_H = (\mathrm{d}H)^\sharp \quad \iff \quad \iota_{X_H}\Omega = \mathrm{d}H$$

This vector field describes time evolution according to the Hamiltonian. In coordinates, this equation gives us Hamilton's equations of motion. If $X_H = (X_q, X_p)$, then

$$\iota_{X_H}\Omega = \mathrm{d}H$$
$$-X_{p_i}\mathrm{d}q^i + X_{q^i}\mathrm{d}p_i = \frac{\partial H}{\partial q^i}\mathrm{d}q^i + \frac{\partial H}{\partial p^i}\mathrm{d}p_i$$

Equating the components yields

$$X_{q^i}(q,p) = \frac{\partial H}{\partial p_i}(q,p)$$
$$X_{p_i}(q,p) = -\frac{\partial H}{\partial q^i}(q,p)$$

The Hamiltonian flow is symplectic:

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(F_H^t)^\star\Omega &= \mathcal{L}_{X_H}\Omega \\ &= \mathrm{d}\iota_{X_H}\Omega + \iota_{X_H}\mathrm{d}\Omega \\ &= \mathrm{d}^2 H - \iota_{X_H}\mathrm{d}^2\Theta \\ &= 0\end{aligned}$$

Hamiltonian Momentum Maps

Suppose we have a left action of a Lie group G on Q, $\Phi : G \times Q \to Q$. This induces an action of G on $T^\star Q$ given by $\Phi_g^{T\star Q}(q,p) = \Phi^\star_{g^{-1}}(q,p)$. In coordinates, this is

$$\Phi^{T^\star Q}(g,(q,p)) = \left((\Phi_g^{-1})^i(q), p_j\frac{\partial \Phi_g^j}{\partial q^i}(q)\right)$$

This action gives us an infinitesimal generator

$$\xi_{T^\star Q}(q,p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Phi^{T^\star Q}(\exp(t\xi),(q,p))$$

We say that Φ is a symmetry of the Hamiltonian H if $H \circ \Phi = H$. In this case, it is also an infinitesimal symmetry. i.e. $\left\langle \mathrm{d}H, \xi_{T^\star Q}\right\rangle = 0$.
Actions of Q lifted to $T^\star Q$ are always special symplectic maps, so $(\Phi_g^{T^\star Q})^\star\Theta = \Theta$ for all $g \in G$. This implies the infinitesimal statement that $\mathcal{L}_{\xi_{T^\star Q}}\Theta = 0$. Furthermore, $\Phi^{T^\star Q}$ is always equivatiant.
Suppose our infinitesimal generator is a Hamiltonian vector field, i.e.

$$\xi_{T^\star Q} = \left(\mathrm{d}U_{\xi_{T^\star Q}}\right)^\sharp \quad \left(\iota_{\xi_{T^\star Q}}\Omega = \mathrm{d}U_{\xi_{T^\star Q}}\right)$$

for some $U_{\xi_{T^\star Q}} \in C^\infty(T^\star Q)$. We note that this $U_{\xi_{T^\star Q}}$ is conserved.

$$\begin{aligned}\mathcal{L}_{X_H}U_{\xi_{T^\star Q}} &= \iota_{X_H}\mathrm{d}U_{\xi_{T^\star Q}} \\ &= \iota_{X_H}\iota_{\xi_{T^\star Q}}\Omega \\ &= -\iota_{\xi_{T^\star Q}}\iota_{X_H}\Omega \\ &= -\iota_{\xi_{T^\star Q}}dH \\ &= -\mathcal{L}_{\xi_{T^\star Q}}H \\ &= 0\end{aligned}$$

What's going on here, and how can we generalize it to more group actions? What we have is a conserved quantity for each element $\xi \in \mathfrak{g}$. We could unify this by saying that instead of one conserved scalar for each ξ, we have a conserved map $\mathfrak{g} \to \mathbb{R}$. So our conserved quantity is really an element of $\mathfrak{g}^\star$. Define $\phi : T^\star Q \to \mathfrak{g}^\star$ by

$$\langle \phi(q,p), \xi\rangle = U_{\xi_{T^\star Q}}(q,p)$$

This is our Hamiltonian momentum map. It is itself a conserved category. If Φ is a symmetry transformation, and X_H is the Hamilton flow of hamiltonian H.

$$\begin{aligned}\mathcal{L}_{X_H}\langle\phi,\xi\rangle &= \mathcal{L}_{X_H}U_{\xi_{T^\star Q}}\\ &= 0\end{aligned}$$

Now, how do we generalize this? In the proof of conservation, we didn't actually need $\xi_{T^\star Q}$ to be a Hamilton flow, since we took the d of the Hamiltonian anyway. All we needed is that $\mathrm{d}U_{\xi_{T^\star Q}}(\xi) = \iota_{\xi_{T^\star Q}}\Omega$. So we can generalize the idea of a momentum map by saying a momentum map is a map $\phi : T^\star Q \to \mathfrak{g}^\star$ that satisfies

$$\mathrm{d}(\langle\phi,\xi\rangle) = \iota_{\xi_{T^\star Q}}\Omega$$

These more general momentum maps are still conserved by Hamilton flow. We can define a Hamiltonian momentum map $J_H : T^\star Q \to \mathfrak{g}^\star$ by

$$\langle J_H(q,p),\xi\rangle = \big\langle\Theta(q,p),\xi_{T^\star Q}(q,p)\big\rangle = \iota_{\xi_{T^\star Q}}\Theta$$

We can verify that this is indeed a momentum map

$$\begin{aligned}\mathrm{d}\langle J_H(q,p),\xi\rangle &= \mathrm{d}\iota_{\xi_{T^\star Q}}\Theta\\ &= \mathcal{L}_{\xi_{T^\star Q}}\Theta - \iota_{\xi_{T^\star Q}}\mathrm{d}\Theta\\ &= -\iota_{\xi_{T^\star Q}}\mathrm{d}\Theta\\ &= \iota_{\xi_{T^\star Q}}\Omega\end{aligned}$$

Examples

$\mathbb{R}^3$ acting on $\mathbb{R}^3$ by translation

Consider the additive action of $\mathbb{R}^3$ on $\mathbb{R}^3$. We let $G = \mathbb{R}^3$, $Q = \mathbb{R}^3$, $T^\star Q = \mathbb{R}^3 \oplus \mathbb{R}^3$. Our action is

$$\begin{aligned}\Phi &: (g,q) \mapsto q+g\\ \Phi^{TQ} &: (g,(q,v)) \mapsto (q+g,v)\end{aligned}$$

We can dualize to find the cotangent lift.

$$\begin{aligned}\left\langle\Phi_g^{T^\star Q}(q,p),(q+g,v)\right\rangle &= \left\langle(q,p),\Phi_{g^{-1}}^{TQ}(q+g,v)\right\rangle\\ &= \langle(q,p),(q,v)\rangle\\ &= \langle p,v\rangle\\ &= \langle(q+g,p),(q+g),v\rangle\end{aligned}$$

So $\Phi^{T^\star Q}(g,(q,p)) = (q+g,p)$.

$$\begin{aligned}\xi_{T^\star Q} &= (\xi,0)\\ U_{\xi_{T^\star Q}}(q,p) &= \langle p,\xi\rangle\\ \langle\phi(q,p),\xi\rangle &= U_{\xi_{T^\star Q}}(q,p) = \langle p,\xi\rangle\end{aligned}$$

$SO(3)$ acting on $\mathbb{R}^3$ by rotation

First, we explore $\mathfrak{so}(3)$. $SO(3)$ is the space of 3×3 orthogonal matrices with determinant 1. The Lie algebra $\mathfrak{so}(3)$ is the space of matrices ξ such that $\exp(\xi) \in SO(3)$. The orthogonal condition on $SO(3)$ means that ξ must be skew-symmetric. The determinant constraint means that ξ must have trace 0. So

$$\xi = \begin{pmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{pmatrix}$$

This is just the cross product matrix. So for each ξ, we have a vector ω_ξ such that $\xi(v) = \omega_\xi \times v$. Consider the action of $SO(3)$ on $\mathbb{R}^3$.

$$\begin{aligned} \Phi &: (A, q) \mapsto Aq \\ \Phi^{TQ} &: (A, (q, v)) \mapsto (Aq, Av) \end{aligned}$$

Again, we dualize the tangent lift to find the cotangent lift

$$\begin{aligned} \left\langle \Phi_A^{T^\star Q}(q,p), (Aq, v) \right\rangle &= \left\langle (q,p), \Phi_{A^{-1}}^{TQ}(Aq, v) \right\rangle \\ &= \langle (q,p), (q, A^{-1}v) \rangle \\ &= \langle p, A^{-1}v \rangle \\ &= \langle Ap, v \rangle \\ &= \langle (Aq, Ap), (Aq, v) \rangle \end{aligned}$$

Therefore, $\Phi^{T^\star Q}(A, (q,p)) = (Aq, Ap)$. Differentiating tells us that

$$\xi_{T^\star Q}(q,p) = (\xi q, \xi p) = (\omega_\xi \times q, \omega_\xi \times p)$$

To find $\tilde{\phi}$, we need to solve Hamilton's equations

$$\begin{aligned} \omega_\xi \times q &= \frac{\partial U_\xi}{\partial p} \\ \omega_\xi \times p &= -\frac{\partial U_\xi}{\partial q} \end{aligned}$$

This is solved by $U_\xi = (q \times p) \cdot \omega_\xi$. So our momentum map is

$$\phi(q,p)(\omega_\xi) = (q \times p) \cdot \omega_\xi$$

(The dual of standard angular momentum).

Equivariance

One importatant property of montum maps is G*-equivariance*. A momentum map is G-equivariant if $\mathrm{Ad}^\star_{g^{-1}} \circ J_H = J_H \circ \Phi_g^{T^\star Q}$ (i.e. it commutes with the G-action on $T^\star Q$ and $\mathfrak{g}^\star$).
Since $(\Phi_g^{T^\star Q})^{-1} = \Phi_{g^{-1}}^{T^\star Q}$, the Lagrangian momenum map is G-equivariant iff

$$\begin{aligned} J_H &= \mathrm{Ad}^\star_{g^{-1}} \circ J_H \circ (\Phi_{g^{-1}}^{T^\star Q}) \\ \langle J_H(q,v), \xi \rangle &= \left\langle (J_H \circ (\Phi_{g^{-1}}^{T^\star Q}))(q,v), \mathrm{Ad}_{g^{-1}} \xi \right\rangle \end{aligned}$$

Before we show this, we need a lemma: $(Ad_g \xi)_M = \Phi^\star_{g^{-1}} \xi_M$.

$$
\begin{aligned}
(\mathrm{Ad}_g\,\xi)_M(x) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(\exp(t\,\mathrm{Ad}_g\,\xi), x) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(g(\exp t\xi)g^{-1}, x) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi_g \circ \Phi_{\exp t\xi} \circ \Phi_{g^{-1}}(x)) \quad \text{fact abou t exponential map} \\
&= \mathrm{d}\Phi_{g^{-1}}\Phi_g(\xi_M(\Phi_{g^{-1}}(x))) \\
&= (\Phi^\star_{g^{-1}}\xi_M)(x)
\end{aligned}
$$

Computing the right hand side of our desired identity yields

$$
\begin{aligned}
\left\langle (J_H \circ (\Phi^{TQ}_{g^{-1}}))(q,v), \mathrm{Ad}_{g^{-1}}\,\xi \right\rangle &= \left\langle \Theta_H(\Phi^{TQ}_{g^{-1}}(q,v)), (\mathrm{Ad}_{g^{-1}}\,\xi)_{TQ}(\Phi^{TQ}_{g^{-1}}(q,v)) \right\rangle \\
&= \left\langle \Theta_H(\Phi^{TQ}_{g^{-1}}(q,v)), (\Phi^{TQ}_g)^\star \xi_{TQ}(\Phi^{TQ}_{g^{-1}}(q,v)) \right\rangle \\
&= \left\langle \Theta_H(\Phi^{TQ}_{g^{-1}}(q,v)), \xi_{TQ}(q,v)) \right\rangle \\
&= \left\langle ((\Phi^{TQ}_{g^{-1}})^\star \Theta_H)(q,v), \xi_{TQ}(q,v)) \right\rangle \\
&= \langle \Theta_H(q,v), \xi_{TQ}(q,v)) \rangle \quad \Phi^{T^\star Q}_{g^{-1}} \text{ is special sypmlectic} \\
&= \langle J_H(q,v), \xi \rangle
\end{aligned}
$$

Lagrangian Mechanics

Preliminary definitions

We will work with a *configuration manifold* Q with associated *state space* TQ and a *Lagrangian* $L : TQ \to \mathbb{R}$. We let $\pi_Q : TQ \to Q$ be the canonical projection onto Q. We define the *path space* to be

$$
\mathcal{C}(Q) := \{q : [0,T] \to Q \;:\; q \text{ is a } C^2 \text{ curve}\}
$$

and we define the *action map* $\mathcal{S} : \mathcal{C}(Q) \to \mathbb{R}$ by

$$
\mathcal{S}(q) := \int_0^T L(q(t), \dot{q}(t))\,\mathrm{d}t
$$

$\mathcal{C}(Q)$ is a smooth manifold, and the tangent space $T_q\mathcal{C}(Q)$ is the set of C^2 maps $v_q : [0,T] \to TQ$ such that $\pi_Q \circ v_q = q$. We can describe the second derivatives of curves on Q by the *second-oder submanifold* of $T(TQ)$ to be

$$
\ddot{Q} := \{w \in T(TQ) \;:\; \mathrm{d}\pi_Q(w) = \pi_{TQ}(w)\} \subset T(TQ)
$$

To understand this definition, we will compute $\mathrm{d}\pi_Q$. Since $\pi_Q : TQ \to Q$, $\mathrm{d}\pi_Q : T(TQ) \to TQ$. Let $X = ((q,\dot{q}),(r,\dot{r})) \in T(TQ)$, $f \in C^\infty(Q)$. Then

$$
\begin{aligned}
\mathrm{d}\pi_Q(X)(f) &= X(f \circ \pi_Q) \\
&= (q, r)
\end{aligned}
$$

If $\mathrm{d}\pi_Q(q) = \pi_{TQ}(w)$, then $(q,r) = (q,\dot{q})$, so $r = \dot{q}$. Thus, $\ddot{Q}$ is the set of elements of the form $((q,\dot{q}),(\dot{q},\ddot{q})) \in T(TQ)$

The Lagrangian One-Form

Given a Lagrangian L, there exists a unique map $D_{EL}L : \ddot{Q} \to T^{\star}Q$ (the *Euler-Lagrange map*) and a unique one-form Θ_L (the *Lagrangian one-form*) on TQ such that for all variations $\delta q \in T_q\mathcal{C}(Q)$ of $q(t)$, we have

$$\langle \mathrm{d}\mathcal{S}(q), \delta q\rangle = \int_0^T D_{EL}\,\langle L(q,\dot{q},\ddot{q}), \delta q\rangle\ \mathrm{d}t + \left\langle \Theta_L, \hat{\delta} q\right\rangle\Big|_0^T$$

where

$$\hat{\delta} q(t) := \left(\left(q(t), \frac{\partial q}{\partial t}(t)\right), \left(\delta q(t), \frac{\partial \delta q}{\partial t}(t)\right)\right)$$

We can compute these maps by computing the variation of the action map.

$$\begin{aligned}\langle \mathrm{d}\mathcal{S}(q), \delta q\rangle &= \int_0^T \left[\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\frac{\mathrm{d}}{\mathrm{d}t}\delta q^i\right]\ \mathrm{d}t \\ &= \int_0^T \left[\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\right]\delta q^i\ \mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right]_0^T\end{aligned}$$

This gives us expressions for $D_{EL}L$ and Θ_L in coordinates.

$$\begin{aligned}(D_{EL}L)_i &= \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} \\ \Theta_L &= \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i\end{aligned}$$

Lagrangian Vector Fields and Flows

A *Lagrangian Vector Field* is a vector field $X_L : TQ \to T(TQ)$ on TQ such that

$$D_{EL}L \circ X_L = 0$$

and the *Lagrangian flow* $F_L : TQ \times \mathbb{R} \to TQ$ is the flow of X_L. We will denote the flow at time t by F_L^t. A curve $q \in \mathcal{C}(Q)$ is said to be a *solution of the Euler-Lagrange equations* if

$$\int_0^T \langle D_{EL}L(q), \delta q\rangle\ \mathrm{d}t = 0$$

for all variations $\delta q \in T_q\mathcal{C}(Q)$. This is equivalent to $(q, \dot{q})$ being an integral curve of X_L and means that q must satisfy the *Euler-Lagrange equations*

$$\frac{\partial L}{\partial q^i}(q,\dot{q}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}(q,\dot{q})\right) = 0$$

for all $t \in [0, T]$.

The Lagrangian Symplectic Form

We define the *solution space* $\mathcal{C}_L(Q) \subset \mathcal{C}(Q)$ to be the set of solutions to the Euler-Lagrange equations. Since an element $q \in \mathcal{C}_L(Q)$ is an integral curve of a vector field, it is uniquely determined by the initial conditions

$(q(0), \dot{q}(0)) \in TQ$. Therefore, we can identify $\mathcal{C}_L(Q)$ with TQ, the space of initial conditions. We define the *restricted action map* $\hat{\mathcal{S}} : TQ \to \mathbb{R}$ by

$$\hat{\mathcal{S}}(q_0, v_0) = \mathcal{S}(q) \quad \text{where } q \in \mathcal{C}_L(Q) \text{ and } (q(0), \dot{q}(0)) = (q_0, v_0)$$

Since q is a solution of the Euler-Lagrange equations, $\int_0^T \langle D_{EL}L(q), \delta q\rangle \; \mathrm{d}t = 0$ for any variation $\delta q \in T_q\mathcal{C}(Q)$. Given $X = ((q, v), (r, w)) \in T_{(q,v)}(TQ)$, pick δq such that $\hat{\delta} q(t) = (F_L^t)_\star X$. (Recall that $\hat{\delta} q(t) \in T(TQ)$. Picking δq like this ensures that $\delta q(t)$). Then

$$\begin{aligned}
\left\langle \mathrm{d}\hat{\mathcal{S}}(q_0, v_0), w \right\rangle &= \langle \mathrm{d}\mathcal{S}(q), (F_L^t)_\star(X)\rangle \\
&= \int_0^T \langle D_{EL}L(\ddot{q}), \delta q\rangle \; \mathrm{d}t + \left\langle \Theta_L(\dot{q}), (F_L^t)_\star X\right\rangle\Big|_0^T \\
&= \left\langle \Theta_L(\dot{q}), (F_L^t)_\star X\right\rangle\Big|_0^T \\
&= \left\langle \Theta_L(\dot{q}(T)), (F_L^T)_\star X\right\rangle - \langle \Theta_L(\dot{q}(0)), X\rangle \\
&= \left\langle ((F_L^T)^\star \Theta_L)(\dot{q}(0)), X\right\rangle - \langle \Theta_L(\dot{q}(0)), X\rangle \\
&= \left\langle ((F_L^T)^\star \Theta_L)(q, v), X\right\rangle - \langle \Theta_L(q, v), X\rangle
\end{aligned}$$

Since $\mathrm{d}^2 = 0$, differentiating both sides reveals that

$$(F_L^T)^\star \mathrm{d}\Theta_L = \mathrm{d}\Theta_L$$

Thus, Lagrangian flow preserves the 2-form $\mathrm{d}\Theta_L$. We define the *Lagrangian symplectic form* $\Omega_L = d\Theta_L$. It is given in coordinates by

$$\Omega_L(q, \dot{q}) = \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \mathrm{d}\dot{q}^i \wedge \mathrm{d}q^j$$

The Lagrangian Momentum Map

Suppose we have a Lie group G with a left action on Q, $\Phi : G \times Q \to Q$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^\star$ be its dual. We can lift Φ to an action $\Phi^{TQ} : G \times TQ \to TQ$ by

$$\Phi^{TQ}(g, X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(g, \exp(tX))$$

In coordinates,

$$\Phi^{TQ}(g, (q, v)) = \left(\Phi^i(g, q), \frac{\partial \Phi^i}{\partial q^j}(g, q)\dot{q}^j\right)$$

Any tangent vector ξ in $\mathfrak{g}$ induces a vector field ξ_Q on Q by

$$\xi_Q(q) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(\exp(t\xi), q)$$

Similarly, ξ induces a vector field ξ_{TQ} on TQ by

$$\xi_{TQ}(q, v) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^{TQ}(\exp(t\xi), (q, v))$$

These induced vector fields are called *infinitesimal generators*.

We define the *Lagrangian momentum map* $J_L : TQ \to \mathfrak{g}^\star$ by

$$\langle J_L(q,v), \xi \rangle = \big\langle \Theta_L(q,v), \xi_{TQ}(q,v) \big\rangle = (\iota_{\xi_{TQ}} \Theta_L)(q,v)$$

In coordinates,

$$\begin{aligned} \big\langle \Theta_L, \xi_{TQ}(q,v) \big\rangle &= \frac{\partial L}{\partial \dot{q}^i} \mathrm{d}q^i \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \Phi^{TQ}(\exp(t\xi), (q,v)) \right) \\ &= \frac{\partial L}{\partial \dot{q}^i} \mathrm{d}q^i \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \Phi(\exp(t\xi), q), \text{something} \right) \\ &= \left\langle \frac{\partial L}{\partial \dot{q}}, \xi_Q(q) \right\rangle \end{aligned}$$

Symmetries of the Lagrangian

If $L \circ T_g^{TQ} = L$ for all $g \in G$, then L is *invariant* under Φ^{TQ} and the group action is a *symmetry* of the Lagrangian. Invariance of the Lagrangian implies *infinitesimal invariance*

$$\big\langle \mathrm{d}L, \xi_{TQ} \big\rangle = 0 \; \forall \xi \in \mathfrak{g}$$

If L is invariant under a G action, then

$$L(\Phi_g(q), \partial_q \Phi_g(q) \cdot \dot{q}\,) = L(q, \dot{q}\,)$$

Differentiating both sides with respect to $\dot{q}$ in the δq direction yields

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}}(\Phi_g(q), \partial_q \Phi_g(q) \cdot \dot{q}\,) \cdot \partial_q \Phi_g(q) \cdot \delta q &= \frac{\partial L}{\partial \dot{q}}(q, \dot{q}\,) \cdot \delta q \\ (\Phi_g^{TQ})^\star \Theta_L &= \Theta_L \end{aligned}$$

Noether's Theorem for Lagrangian Mechanics

If the action of G on TQ is a symmetry of the Lagrangian, the Lagrangian flow preserves the momentum map. We can see this in the following computation:
The action of G on Q induces a pointwise action of G on $\mathcal{C}(Q)$ (i.e. $\Phi_g(q)(t) = \Phi_g(q(t))$). This gives us an infinitesimal generator on $\mathcal{C}(Q)$.

$$\xi_{\mathcal{C}(Q)}(q)(s) = \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \Phi(\exp(t\xi), q)(s) = \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \Phi(\exp(t\xi), q(s)) = \xi_{TQ}(q(s))$$

Since L is invariant under the G-action, so is S. Since $\mathcal{S}$ is invariant under the G-action, we also know that Φ_g maps solution curves to solution curves. So $\xi_{\mathcal{C}(Q)}(q) \in T_q(\mathcal{C}_L)$. So we can look at the restricted action map and find that

$$\begin{aligned} 0 &= \left\langle \hat{\mathcal{S}}(q,v), \xi_{TQ}(q,v) \right\rangle \\ &= \big\langle \Theta_L(\dot{q}(T)), \xi_{TQ}(\dot{q}(T)) \big\rangle - \big\langle \Theta_L(q,v), \xi_{TQ}(q,v) \big\rangle \\ &= \big\langle J_L(F_L^T(v,q)) - J_L(q,v), \xi \big\rangle \end{aligned}$$

Legendre Transforms

We tie Hamiltonian and Lagrangian mechanics together using the *Legendre transform* (or *fibre derivative*) $\mathbb{F}L : TQ \to T^\star Q$.

$$\langle \mathbb{F}L(q,v),(q,w)\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} L((q,v)+\epsilon(q,w))$$

In coordinates, this is

$$\mathbb{F}L : (q,\dot{q}) \mapsto (q,p) = \left(q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right)$$

We call L *regular* if $\mathbb{F}L$ is a local isomorphism and *hyperregular* if $\mathbb{F}L$ is a global isomorphism.
The *fibre derivative* of the Hamiltonian is the map $\mathbb{F}H : T^\star Q \to TQ$.

$$\langle (q,\alpha), \mathbb{F}H(q,\beta)\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} H((q,\beta)+\epsilon(q,\alpha))$$

In coordinates, this is given by

$$\mathbb{F}H : (q,p) \to (q,\dot{q}) = \left(q, \frac{\partial H}{\partial p}(q,p)\right)$$

Like with the Lagrangian, we say that H is *regular* if $\mathbb{F}H$ is a local isomorphism and *hyperregular* if it is a global isomorphism.
The canonical one- and two-forms and Hamiltonian momentum maps are related to the Lagrangian one- and two-forms and the Lagrangian momentum maps by the fibre derivative.

$$\begin{aligned}
\mathbb{F}L^\star\Theta &= \mathbb{F}L^\star(p_i \mathrm{d}q^i) \\
&= \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i \\
&= \Theta_L \\
\mathbb{F}L^\star\Omega &= -\mathbb{F}L^\star \mathrm{d}\Theta \\
&= -\mathrm{d}\mathbb{F}L^\star\Theta \\
&= -\mathrm{d}\Theta_L \\
&= -\Omega_L \\
\langle \mathbb{F}L^\star J_H(q,p),\xi\rangle &= \left\langle J_H\left(q,\frac{\partial H}{\partial p}\right),\xi\right\rangle \\
&= \left\langle \Theta\left(q,\frac{\partial H}{\partial p}\right), \xi_{T^\star Q}\left(q,\frac{\partial H}{\partial p}\right)\right\rangle \\
&= \langle \mathbb{F}L^\star\Theta(q,p), \mathbb{F}L^\star\xi_{T^\star Q}(q,p)\rangle \\
&= \langle \Theta_L(q,\dot{q}), \xi_{TQ}\rangle \\
&= \langle J_L(q,\dot{q}),\xi\rangle
\end{aligned}$$

Fact: If L is hyperregular, then H will also be hyperregular and $\mathbb{F}H = (\mathbb{F}L)^{-1}$.

Discrete Mechanics

Discrete Variational Mechanics: Lagrangian Viewpoint

Starting from a configuration space Q, we can define the *discrete state space* to be $Q \times Q$. A *discrete Lagrangian* is a function $L_d : Q \times Q \to \mathbb{R}$. If we fix a series of times $\{t_k = kh \ : \ k = 0, \ldots, N\}$, we can define the *discrete path space* as

$$\mathcal{C}_d(Q) = \{q_d : \{t_k\}_{k=0}^N \to Q\}$$

The *discrete action map* $\mathcal{S}_d : \mathcal{C}_d(Q) \to \mathbb{R}$ is defined to be

$$\mathcal{S}_d(q_d) = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$$

The discrete path space is a product manifold and its tangent space $T_{q_d}\mathcal{C}_d(Q)$ is

$$T_{q_d}\mathcal{C}_d(Q) = \{v_{q_d} : \{t_k\}_{k=0}^N \to TQ \mid \pi_Q \circ v_{q_d} = q_d\}$$

The discrete analogue of $T(TQ)$ is $(Q \times Q) \times (Q \times Q)$. We define π as the projection onto the first copy of $Q \times Q$ and σ as the projection onto the second copy of $Q \times Q$. The *discrete second-order submanifold* is the subset of points of the form $((q_0, q_1), (q_1, q_2))$.
Given this discrete Lagrangian structure, we have discrete versions of the Euler-Lagrange map and the Lagrangian one-form. We can compute them by using discrete integration by parts (rearranging terms) on the discrete action map.

$$\begin{aligned}\langle \mathrm{d}\mathcal{S}_d(q_d), \delta q_d \rangle &= \sum_{k=0}^{N-1} [D_1 L_d(q_k, q_{k+1}) \cdot \delta q_k + D_2 L_2(q_k, q_{k+1}) \cdot \delta q_{k+1}] \\ &= \sum_{k=1}^{N-1} [D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k)]\delta q_k + D_1 L_d(q_0, q_1)\delta q_0 + D_2 L_d(q_{N_1}, q_N)\delta q_N\end{aligned}$$

So our *discrete Euler-Lagrange map* is given by

$$D_{DEL} L_d((q_{k-1}, q_k), (q_k, q_{k+1})) = D_2 L_d(q_{k-1}, q_k) + D_1 L_2(q_k, q_{k+1})$$

And we have two *discrete Lagrangian one-forms*

$$\begin{aligned}\Theta^+_{L_d}(q_0, q_1) = D_2 L_d(q_0, q_1)\mathrm{d}q_1 &= \frac{L_d}{q_1^i}\mathrm{d}q_1^i \\ \Theta^-_{L_d}(q_0, q_1) = -D_1 L_2(q_0, q_1)\mathrm{d}q_0 &= -\frac{L_d}{q_0^i}\mathrm{d}q_0^i\end{aligned}$$

And

$$\langle \mathrm{d}\mathcal{S}(q_d), \delta q_d \rangle = \sum_{k=1}^{N-1} D_{DEL} L_d((q_{k-1}, q_k), (q_k, q_{k+1}))\delta q_k + \Theta^+_{L_d}(q_{N-1}, q_N) \cdot (\delta q_{N-1}, \delta q_N) - \Theta^-_{L_d}(q_0, q_1) \cdot (\delta q_0, \delta q_1)$$

Note that $\mathrm{d}L_d = \Theta^+_{L_d} - \Theta^-_{L_d}$. Since $\mathrm{d}^2 = 0$, $\mathrm{d}\Theta^+_{L_d} = \mathrm{d}\Theta^-_{L_d}$, so we have a well-defined *discrete Lagrangian two-form.*

Discrete Lagrangian Time Evolution

A *discrete evolution operator* X is a map $X : Q \times Q \to (Q \times Q) \times (Q \times Q)$ such that $\pi \circ X = id$. The *discrete map* is $F = \sigma \circ X$. We will require that $X(Q \times Q) \subset \ddot{Q}_d$ (i.e. X has the form $X(q_0, q_1) = (q_0, q_1, q_1, q_2)$). A *discrete Lagrangian operator* X_{L_d} is a second-order discrete evolution operator such that

$$D_{DEL}L_d \circ X_{L_d} = 0$$

The associated *discrete Lagrangian map* is

$$F_{L_d} = \sigma \circ X_{L_d}$$

We define the *discrete solution space* $C_{L_d}(Q) \subset C_d(Q)$ as the set of solutions to the discrete Euler-Lagrange equations. Again, solutions are uniquely determined by initial conditions (since they can be computed by applying F_{L_d} repeatedly). So we can identify $C_{L_d}(Q)$ with $Q \times Q$, the space of initial conditions. This gives us a *restriced discrete action map* $\hat{\mathcal{S}}_d : Q \times Q \to \mathbb{R}$. Let $v_d = (q_0, q_1) \in Q \times Q$ and $w_{v_d} \in T_{v_d}(Q \times Q)$. Since elements of $\mathcal{C}_{L_d}$ are solutions to the discrete Euler-Lagrange equations,

$$\left\langle \mathrm{d}\hat{\mathcal{S}}(v_d), w_{v_d} \right\rangle = \Theta^+_{L_d}(F^{N-1}_{L_d}(v_d)((F^{N-1}_{L_d})_\star(w_{v_d})) - \Theta^-_{L_d}(v_d)(w_d)$$

Differentiating again and recalling that $\mathrm{d}^2\hat{\mathcal{S}} = 0$ yields that

$$(F^{N-1}_{L_d})^\star(\Omega_{L_d}) = \Omega_{L_d}$$

So the discrete Lagrangian map is *discretely symplectic*

Discrete Lagrangian Noether's Theorem

Let G be a Lie group with a left action $\Phi : G \times Q \to Q$ on Q. The infinitesimal generator ξ_Q is defined in the same way as before. The action induces an action on $Q \times Q$ by acting component-wise $\Phi^{Q\times Q}_g(q_0, q_1) = (\Phi_g(q_0), \Phi_g(q_1))$. This action has infinitesimal generator

$$\xi_{Q\times Q}(q_0, q_1) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^{Q\times Q}_{\exp(t\xi)}(q_0, q_1) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi^Q_{\exp(t\xi)}(q_0), \Phi^Q_{\exp(t\xi)}(q_1)) = (\xi_Q(q_0), \xi_Q(q_1))$$

Because we have two discrete Lagrangian one-forms, we get two discrete Lagrangian momentum maps

$$\begin{aligned} \left\langle J^+_{L_d}, \xi \right\rangle &= \left\langle \Theta^+_{L_d}, \xi_{Q\times Q}(q_0, q_1) \right\rangle \\ \left\langle J^-_{L_d}, \xi \right\rangle &= \left\langle \Theta^-_{L_d}, \xi_{Q\times Q}(q_0, q_1) \right\rangle \end{aligned}$$

In coordinates, these are given by

$$\begin{aligned} \left\langle J^+_{L_d}, \xi \right\rangle &= \langle D_2 L_d(q_0, q_1), \xi_Q(q_1) \rangle \\ \left\langle J^-_{L_d}, \xi \right\rangle &= \langle -D_1 L_d(q_0, q_1), \xi_Q(q_0) \rangle \end{aligned}$$

Again, the discrete Lagrangian momentum maps are equivariant if G acts on $Q \times Q$ by a special discrete symplectic map. The same proof from before works.
If $L_d \circ \Phi^{Q\times Q}_g = L_d$, then L_d is *invariant* under Φ and Φ is a symmetry of the discrete Lagrangian. If L_d is invariant, then it is also infinitesimaly invariant (i.e. $\left\langle \mathrm{d}L_d, \xi_{Q\times Q} \right\rangle = 0$). Since $\mathrm{d}L_d = \Theta^+_{L_d} - \Theta^-_{L_d}$, the discrete momentum maps of a symmetry are equal. For symmetries, we will write $J_{L_d} : Q \times Q \to \mathfrak{g}^\star$ for both disrete momentum maps.
The proof is Noether's theorem in the discrete case is similar to the proof in the continuous case.

$$\langle \mathrm{d}\mathcal{S}_d(q_d), \xi_{\mathcal{C}_d(Q)}(q_d) \rangle = \sum_{k=0}^{N-1} \left\langle \mathrm{d}L_d, \xi_{Q\times Q} \right\rangle$$

By infinitesimal invariance, this is 0, so Φ_g maps solution curves to solutinon curves. Using our reduced discrete action map and the fact that solution curves solve the Euler-Lagrange equations,

$$\begin{aligned} 0 &= \left\langle \mathrm{d}\mathcal{S}_d(q_d), \xi_{\mathcal{C}_d(Q)}(q_d) \right\rangle \\ &= \left\langle \mathrm{d}\hat{\mathcal{S}}_d(q_0, q_1), \xi_{Q\times Q}(q_0, q_1) \right\rangle \\ &= \left\langle ((F_{L_2}^N)^\star(\Theta_{L_d}^+) - \Theta_{L_d}^-)(q_0, q_1), \xi_{Q\times Q}(q_0, q_1) \right\rangle \end{aligned}$$

So $F_{L_d}^N$ preserves the discrete momentum map. In particular, this means that F_{L_d} preserves the discrete momentum map.

Discrete Variational Mechanics: Hamiltonian Viewpoint

We can define *discrete Legendre transforms* $\mathbb{F}^+ L_d, \mathbb{F}^- L_d : Q \times Q \to T^\star Q$. These are

$$\begin{aligned} \left\langle \mathbb{F}^+ L_d(q_0, q_1), \delta q_1 \right\rangle &= \langle D_2 L_d(q_0, q_1), \delta q_1 \rangle \\ \left\langle \mathbb{F}^- L_d(q_0, q_1), \delta q_0 \right\rangle &= \langle -D_1 L_d(q_0, q_1), \delta q_0 \rangle \end{aligned}$$

In coordinates, these are written

$$\begin{aligned} \mathbb{F}^+ L_d(q_0, q_1) &= (q_1, D_2 L_d(q_0, q_1)) \\ \mathbb{F}^- L_d(q_0, q_1) &= (q_0, -D_1 L_d(q_0, q_1)) \end{aligned}$$

The discrete fibre derivatives relate the canonical one- and two-forms and Hamiltonian momentum maps to the discrete Lagrangian one- and two-forms and the discrete Lagrangian momentum maps. We will often consider discrete Lagrangians that do not correspond exactly to a Hamiltonian. This means that we will not always have this nice pull-back relationship.

Momentum Matching

The discrete fibre derivatives provide a different interpretation of the discrete Euler Lagrange equation. Define

$$\begin{aligned} p_{k,k+1}^+ &= p^+(q_k, q_{k+1}) = \mathbb{F}^+ L_d(q_k, q_{k+1}) \\ p_{k,k+1}^- &= p^-(q_k, q_{k+1}) = \mathbb{F}^- L_d(q_k, q_{k+1}) \end{aligned}$$

The discrete Euler-Lagrange equations are

$$D_2 L_d(q_{k-1}, q_k) = -D_1 L_d(q_k, q_{k+1})$$

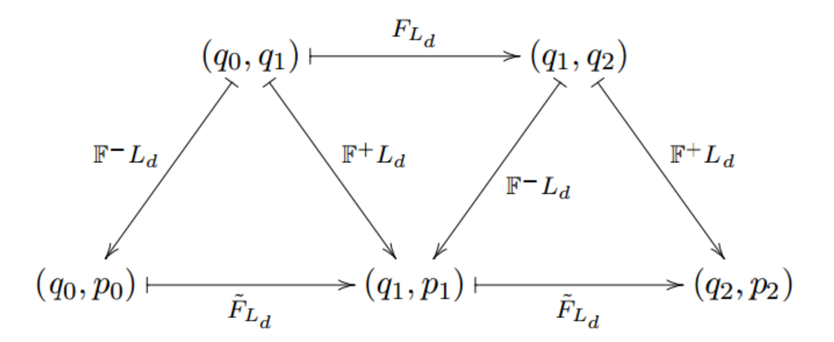
which can be written as

$$\mathbb{F}^+ L_d(q_{k-q}, q_k) = \mathbb{F}^- L_d(q_k, q_{k+1}) \quad \text{or} \quad \mathbb{F}^+ L_d = \mathbb{F}^- L_d \circ F_{L_d}$$

So the discrete Euler-Lagrange equations just say that the momentum at the end of the interval $[k-1, k]$ equals the momentum at the beginning of the interval $[k, k+1]$. This means that each point on the solution curve has a well-defined momentum p_k.

Discrete Hamiltonian Maps

The discrete fibre derivatives let us translate the discrete Lagrangian map $F_{L_d} : Q \times Q \to Q \times Q$ into the Hamiltonian setting. We define the *discrete Hamiltonian map* $\tilde{F}_{L_d} : T^\star Q \to T^\star Q$ by $\tilde{F}_{L_d} = \mathbb{F}^+ L_d \circ F_{L_d} \circ (\mathbb{F}^+ L_d)^{-1}$. With this definition, the following diagram commutes:



The fact that the middle triangle commutes is our previous observation that $\mathbb{F}^+L_d = \mathbb{F}^-L_d \circ \mathbb{F}_{L_d}$. The fact that the right-hand parallelogram commutes is the definition of $\tilde{F}_{L_d}$. Thus, the right-hand triangle commutes. Since the right- and left-hand triangles are identical (up to re-indexing), the left-hand triangle must also commute.
In coordinates, the discrete Hamiltonian maps are given by

$$\begin{aligned}\tilde{F}_{L_d} : (q_0, p_0) \mapsto (q_1, p_1)\\ p_0 = -D_1 L_d(q_0, q_1)\\ p_1 = D_2 L_d(q_0, q_1)\end{aligned}$$

We can see this from the diagram. According to the diagram, $\tilde{\mathbb{F}}_{L_d} = \mathbb{F}^+L_d \circ (\mathbb{F}^-L_d)^{-1}$. Suppose the state starts out at $(q_0, p_0) = \mathbb{F}^-L_d(q_0, q_1)$. Then $\tilde{F}_{L_d}(q_0, p_0) = \mathbb{F}^+L_d(q_0, q_1) = (q_1, p_1)$. By the definition of $\mathbb{F}^-L_d$, $p_0 = -D_1L_d(q_0, q_1)$. By the definition of $\mathbb{F}^+L_d$, $p_1 = D_2L_d(q_0, q_1)$.

Correspondence Between Discrete and Continuous Mechanics

Suppose we have a configuration space Q, a regular Lagrangian L, points $q_0, q_1 \in Q$ and a time-step $h \in \mathbb{R}$. If q_0 and q_1 are sufficiently close and h is sufficiently small, there exists a unique solution to the Euler-Lagrange equations such that $q(0) = q_0$ and $q(h) = q_1$.
We can define the *exact discrete Lagrangian* as

$$L_d^E(q_0, q_1, h) := \int_0^h L(q_{0,1}(t), \dot{q}_{0,1}(t))\mathrm{d}t$$

where $q_{0,1}$ is the unique solution to the Euler-Lagrange equations.
The Legendre transforms of a regular Lagrangian L and its exact discrete Lagrangian L_d^E are related by

$$\begin{aligned}\mathbb{F}^+L_d^E(q_0, q_1, h) = \mathbb{F}L(q_{0,1}(h), \dot{q}_{0,1}(h))\\ \mathbb{F}^-L_d^E(q_0, q_1, h) = \mathbb{F}L(q_{0,1}(0), \dot{q}_{0,1}(0))\end{aligned}$$

We will show that this is true for $\mathbb{F}^-L_d^E$.

$$\begin{aligned}\mathbb{F}^-L_d^E(q_0, q_1, h) &= -\int_0^h \left[\frac{\partial L}{\partial q} \cdot \frac{\partial q_{0,1}}{\partial q_0} + \frac{\partial L}{\partial \dot{q}} \cdot \frac{\partial \dot{q}_{0,1}}{\partial q_0}\right] \mathrm{d}t\\ &= -\int_0^h \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right] \frac{\partial q_{0,1}}{\partial q_0}\mathrm{d}t - \left[\frac{\partial L}{\partial \dot{q}}\frac{\partial q_{0,1}}{\partial q_0}\right]_0^h\\ &= -\left[\frac{\partial L}{\partial \dot{q}}\frac{\partial q_{0,1}}{\partial q_0}\right]_0^h\end{aligned}$$

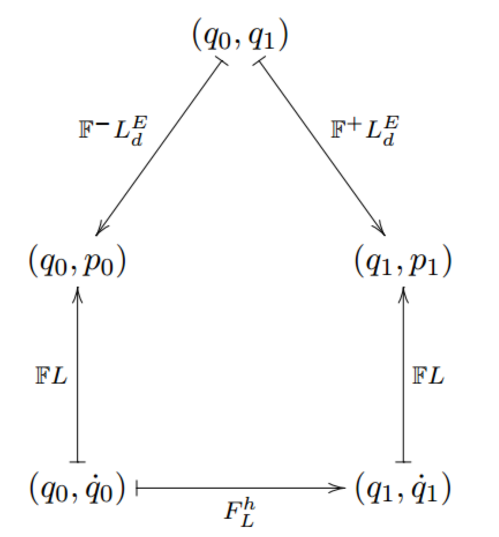
Since $q_{0,1}(0) = q_0, q_{0,1}(h) = q_1$, we know that

$$\frac{\partial q_{0,1}}{\partial q_0}(0) = 1, \quad \frac{\partial q_{0,1}}{\partial q_0}(h) = 0$$

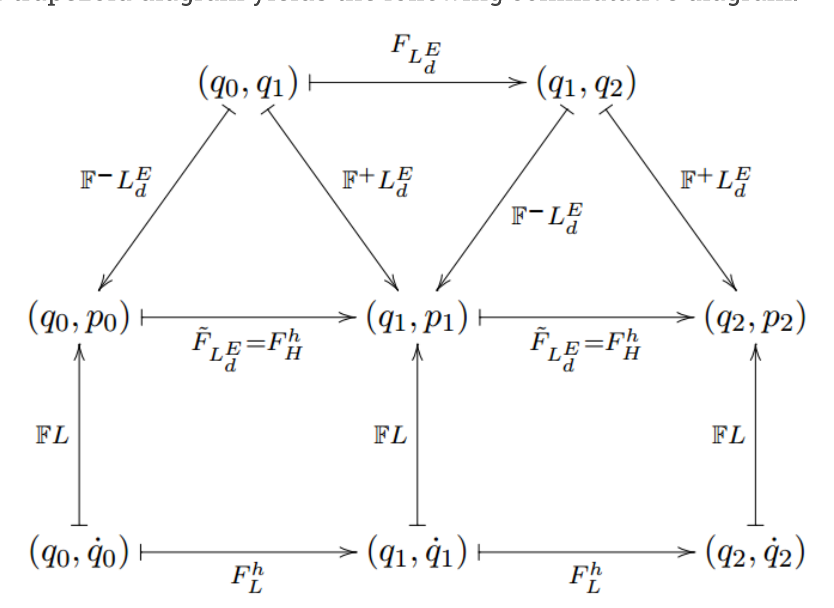
Therefore,

$$\mathbb{F}^- L_d^E(q_0, q_1, h) = \frac{\partial L}{\partial \dot{q}}(q_{0,1}(0), \dot{q}_{0,1}(0)) = \mathbb{F}L(q_{0,1}(0), \dot{q}_{0,1}(0))$$

Since $(q_{0,1}(h), \dot{q}_{0,1}(h)) = \mathbb{F}_L^h(q_{0,1}(0), \dot{q}_{0,1}(0))$, we can draw this equality as a commutative diagram



Combining this with our trapezoid diagram yields the following commutative diagram.



You can also look at the correspondence between discrete and continuous systems in terms of trajectories. With Lagrangians, this perspective says that the solutions $\{q_k\}$ of the discrete Lagrangian and $q(t)$ of the continuous Lagrangian are related by

$$q_k = q(t_k) \text{ for } k = 0, \ldots, N$$
$$q(t) = q_{k,k+1}(t) \text{ for } t \in [t_k, t_{k+1}]$$

(Proof in paper)

Variational Integrators

Idea: use approximations to the exact discrete Lagrangian.
Note: when working on implementation, the functions $F_{L_d} : Q \times Q \times \mathbb{R} \to Q \times Q$ and $\tilde{F}_{L_d} : T^\star Q \times \mathbb{R} \to T^\star Q$ are pretty much the same. Given a trajectory $q_0, q_1, \dots, q_{k-1}, q_k$, F_{L_d} computes q_{k+1} by solving

$$D_2 L_2(q_{k-1}, q_k, h) = -D_1 L_d(q_k, q_{k+1}, h)$$

Defining momenta $p_k = D_2 L_d(q_{k-1}, q_k, h)$ lets us write the equation as

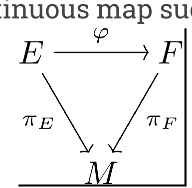
$$p_k = -D_1 L_d(q_k, q_{k+1}, h)$$

The defintion $p_{k+1} = D_2 L_d(q_k, q_{k+1})$ and this equation are the defintion of our map $\tilde{F}_{L_d} : T^\star Q \times \mathbb{R} \to T^\star Q$. It is convenient to implement variational integrators using the map $\tilde{F}_{L_d}$.

Forces

Forced Lagrangian Systems

A morphism of fibre bundles $\phi : E \to F$ is a continuous map such that the following diagram commutes:



(i.e. it preserves base points). A *Lagrangian force* is a morphism of fibre bundles $f_L : TQ \to T^\star Q$. In coordinates, we write

$$f_L : (q, \dot{q}) \mapsto (q, f_L(q, \dot{q}))$$

Given a force, we can modify Hamilton's principle of least action and obtain the *Lagrange-d'Alembert principle*, which states that for any variations δq that are 0 at the endpoints,

$$\delta \int_0^T L(q(t), \dot{q}(t)) \mathrm{d}t + \int_0^T \langle f_L(q(t), \dot{q}(t)), \delta q(t) \rangle \, \mathrm{d}t$$

The usual integration by parts trick yields the *forced Euler-Lagrange equations*

$$\frac{\partial L}{\partial q}(q, \dot{q}) - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) + f_L(q, \dot{q}) = 0$$

Forced Hamiltonian Systems

A *Hamiltonian force* is a morphism $f_H : T^\star Q \to T^\star Q$. We can define an associated horizontal one-form on $T^\star Q$ by

$$\langle f'_H(q, p), u_{p_q} \rangle = \langle f_H(q, p), \mathrm{d}\pi_Q(u_{p_q}) \rangle$$

In coordinates,

$$\langle f'_H(q, p), (\delta q, \delta p) \rangle = \langle f_H(q, p), \delta q \rangle$$

Thus, f'_H maps vectors tangent to the fibers of $T^\star Q$ to 0, which means it is horizontal.
We define the *forced Hamiltonian vector field* by

$$\iota_{X_H} \Omega = \mathrm{d}H - f'_H$$

In coordinates, we get the *forced Hamilton's equations*

$$X_q(q,p) = \frac{\partial H}{\partial p}(q,p)$$
$$X_p(q,p) = -\frac{\partial H}{\partial q}(q,p) + f_H(q,p)$$

The fact that only the second equation is changed is a consequence of f'_H being horizontal.

Legendre Transform with Forces

The forced Lagrangian and forced Hamiltonian perspectives are still connected by the standard Legendre transform

$$f_L = f_H \circ \mathbb{F}L$$

We'll check a simple case. If $L = \frac{1}{2}\dot{q}^T M\dot{q} - V(q)$ and $H = \frac{1}{2}\dot{q}\,M\dot{q} + V(q)$, plugging this into the Euler-Lagrange equation gives

$$\frac{\partial L}{\partial q}(q,\dot{q}) - \frac{\mathrm{d}}{\mathrm{d}t}p + f_H(q,p) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}p = -\frac{\partial V}{\partial q}(q) + f_H(q,p)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}p = -\frac{\partial H}{\partial q}(q,p) + f_H(q,p)$$

Which is just Hamilton's second equation.

(Lagrangian) Noether's Theorem with Forces

Suppose $\Phi : G \times Q \to Q$ is a symmetry of L. We still define the momentum map $J_L : TQ \to \mathfrak{g}^\star$ by

$$\langle J_L(q,v), \xi\rangle = \big\langle \Theta_L(q,v), \xi_{TQ}(q,v)\big\rangle$$

We consider a variation of the form $\delta q(t) = \xi_Q(q(t))$. This variation does not necessarily vanish at the endpoints, so we cannot directly apply the Lagrange d'Alembert principle. We can evaluate the integral on the left hand side of the Lagrange-d'Alembert equation in two ways. Since the group action is a symmetry of L

$$\delta\int_0^T L\mathrm{d}t + \int_0^T \langle f_L, \delta q\rangle\,\mathrm{d}t = \int_0^T \big\langle \mathrm{d}L, \xi_{TQ}\big\rangle\,\mathrm{d}t + \int_0^T \big\langle f_L, \xi_Q\big\rangle\,\mathrm{d}t = \int_0^T \big\langle f_L, \xi_Q\big\rangle\,\mathrm{d}t$$

We can also do the standard integration by parts on the interval to find that

$$\delta\int_0^T L\mathrm{d}t + \int_0^T \langle f_L, \delta q\rangle\,\mathrm{d}t = \int_0^T \left\langle \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + f_L, \xi_Q\right\rangle + \big\langle \Theta_L, \xi_{TQ}\big\rangle\Big|_0^T$$

The first term is 0 since q is a solution to the forced Euler-Lagrange equations. Thus,

$$\begin{aligned}\delta\int_0^T L\mathrm{d}t + \int_0^T \langle f_L, \delta q\rangle\,\mathrm{d}t &= \big\langle \Theta_L, \xi_{TQ}\big\rangle\Big|_0^T \\ &= \big\langle (J_L \circ F_L^T)(q(0),\dot{q}(0)) - J_L(q(0),\dot{q}(0)), \xi\big\rangle\end{aligned}$$

Therefore,

$$\left\langle (J_L \circ F_L^T)(q(0), \dot{q}(0)) - J_L(q(0), \dot{q}(0)), \xi \right\rangle = \int_0^T \left\langle f_L(q(t), \dot{q}(t)), \xi_Q(q(t)) \right\rangle$$

So in general, the momentum map is not preserved. However, if the force is orthogonal to the group action, then the momenum map is preserved.
Note: when external forces are present, Lagrangian and Hamiltonian flows are no longer symplectic.

Discrete Variational Mechanics with Forces

We define two *discrete Lagrangian forces* $f_d^+, f_d^- : Q \times Q \to T^\star Q$. In coordinates, we write them as

$$f_d^+(q_0, q_1) = (q_1, f_d^+(q_0, q_1))$$
$$f_d^-(q_0, q_1) = (q_0, f_d^-(q_0, q_1))$$

We can combine the two discrete forces into a one-form $f_d : Q \times Q \to T^\star(Q \times Q)$

$$\langle f_d(q_0, q_1), (\delta q_0, \delta q_1) \rangle = \left\langle f_d^+(q_0, q_1), \delta q_1 \right\rangle + \left\langle f_d^-(q_0, q_1), \delta q_0 \right\rangle$$

We use this to form the *discrete Lagrange-d'Alembert principle*

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) + \sum_{k=0}^{N-1} \langle f_d(q_k, q_{k+1}, (\delta q_k, \delta q_{k+1}) \rangle = 0$$

Discrete integration by parts yields the *forced discrete Euler-Lagrange equations*

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f_d^+(q_{k-1}, q_k) + f_d^-(q_k, q_{k+1}) = 0$$

Like in the continuous case, this is the (discrete) Euler-Lagrange equations with a forcing term added. This defines a *forced discrete Lagrangian map* $F_{L_d} : Q \times Q \to Q \times Q$.

Discrete Legendre Transform with Forces

In the discrete setting, we need to modify the Legendre transform to make it work with forces. We define the *forced discrete Legendre transforms*

$$\mathbb{F}^{f^+} L_d : (q_0, q_1) \mapsto (q_1, p_1) = (q_1, D_2 L_d(q_0, q_1) + f_d^+(q_0, q_1))$$
$$\mathbb{F}^{f^-} L_d : (q_0, q_1) \mapsto (q_0, p_0) = (q_0, -D_1 L_d(q_0, q_1) - f_d^-(q_0, q_1))$$

We can use the forced discrete Legendre transform to define the *forced discrete Hamiltonian map* $\tilde{F}_{L_d} = \mathbb{F}^{f^-} L_d \circ F_{L_d} \circ (\mathbb{F}^{f^-} L_d)^{-1}$. (You could also use $\mathbb{F}^{f^+}$). This map is defined by $\tilde{F}_{L_d} : (q_0, p_0) \mapsto (q_1, p_1)$ where

$$p_0 = -D_1 L_d(q_0, q_1) - f_d^-(q_0, q_1)$$
$$p_1 = D_2 L_d(q_0, q_1) + f_d^+(q_0, q_1)$$

This is the regular discrete Hamiltonian map with an extra discrete force term.

Constrained Systems

Constrained Lagrangian Systems

We will work with *holonomic constraints*. Given a constraint function $\phi : Q \to \mathbb{R}^d$, we consider dynamics on the *constraint submanifold* $N = \phi^{-1}(0)$. TN naturally embeds in TQ, so we can restrict a Lagrangian L on TQ to $L^N = L|_{TN} : TN \to \mathbb{R}$.
Let $q_0, q_T \in N$. Let $\mathcal{C}(Q)$ be the space of paths in Q from q_0 to q_T and $\mathcal{C}(N)$ be the space of paths in N from q_0 to q_T. We write $\mathcal{S}$ for the action map on $\mathcal{C}(Q)$ and $\mathcal{S}^N$ for the action map restricted to $\mathcal{C}(N)$. We define the constrait function $\Phi : \mathcal{C}(Q) \to \mathcal{C}(\mathbb{R}^d)$ by $\Phi(q)(t) = \phi(q(t))$.
TFAE:

1. $q \in \mathcal{C}(N)$ extremizes $\mathcal{S}^N$.
2. $(q, \lambda) \in \mathcal{C}(Q \times \mathbb{R}^d)$ extremizes $\bar{\mathcal{S}}(q, \lambda) = \mathcal{S}(q) - \langle \lambda, \Phi(q) \rangle$ and hence solves the Euler-Lagrange equations for the augmented Lagrangian

$$\bar{L}(q, \lambda, \dot{q}, \dot{\lambda}) = L(q, \dot{q}) - \langle \lambda, \phi(q) \rangle$$

3. $q \in \mathcal{C}(Q)$ and $\lambda \in \mathcal{C}(\mathbb{R}^d)$ satisfy the constrained Euler-Lagrange equations

$$\frac{\partial L}{\partial q^i}(q(t), \dot{q}(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}(q(t), \dot{q}(t))\right) = \left\langle \lambda(t), \frac{\partial \phi}{\partial q^i}(q(t)) \right\rangle$$
$$\phi(q(t)) = 0$$

Proof sketch: By the Lagrange multiplier theorem, (1) is equivalent to (q, λ) being an extremum of $\bar{\mathcal{S}}(q, \lambda) = \mathcal{S}(q) - \langle \lambda, \Phi(q) \rangle$ (for $\lambda \in \mathcal{C}(\mathbb{R}^d)$).

$$\begin{aligned} \bar{S}(q, \lambda) &= \mathcal{S}(q) - \langle \lambda, \Phi(q) \rangle \\ &= \int_0^T L(q(t), \dot{q}(t)) \mathrm{d}t - \int_0^T \langle \lambda(t), \Phi(q)(t) \rangle \, \mathrm{d}t \\ &= \int_0^T \left[L(q(t), \dot{q}(t)) - \langle \lambda(t), \phi(q(t)) \rangle \right] \mathrm{d}t \end{aligned}$$

This is (2).
Setting $\mathrm{d}\bar{\mathcal{S}} = 0$ and doing the standard integration by parts trick shows that (2) and (3) are equivalent.

Definitions from the Paper

Continuous Hamiltonians

$$J_H : T^\star Q \to \mathfrak{g}^\star$$
$$\langle J_H(q, p), \xi \rangle = \left\langle \Theta(q, p), \xi_{T^\star Q}(q, p) \right\rangle = \iota_{\xi_{T^\star Q}} \Theta$$

Continuous Lagrangians

$$
\begin{aligned}
(D_{EL}L)_i &= \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} \\
\Theta_L &= \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i \\
\Omega_L &= \mathrm{d}\Theta_L \\
\Phi^{TQ}(g, X) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(g, \exp(tX)) \\
\xi_Q &: Q \to TQ \\
\xi_Q(q) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(\exp(t\xi), q) \\
\xi_{TQ} &: TQ \to T(TQ) \\
\xi_{TQ}(q, v) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^{TQ}(\exp(t\xi), (q, v)) \\
J_L &: TQ \to \mathfrak{g}^\star \\
\langle J_L(q, v), \xi \rangle &= \big\langle \Theta_L(q, v), \xi_{TQ}(q, v) \big\rangle = (\iota_{\xi_{TQ}}\Theta_L)(q, v) \\
\mathrm{Ad}_g &: \mathfrak{g} \to \mathfrak{g}
\end{aligned}
$$

Discrete Lagrangians

$$
\begin{aligned}
\Theta^+_{L_d}(q_0, q_1) &= D_2 L_d(q_0, q_1)\mathrm{d}q_1 = \frac{L_d}{q_1^i}\mathrm{d}q_1^i \\
\Theta^-_{L_d}(q_0, q_1) &= -D_1 L_2(q_0, q_1)\mathrm{d}q_0 = -\frac{L_d}{q_0^i}\mathrm{d}q_0^i \\
\left\langle J^+_{L_d}, \xi \right\rangle &= \left\langle \Theta^+_{L_d}, \xi_{Q\times Q}(q_0, q_1) \right\rangle \\
\left\langle J^-_{L_d}, \xi \right\rangle &= \left\langle \Theta^-_{L_d}, \xi_{Q\times Q}(q_0, q_1) \right\rangle
\end{aligned}
$$

Legendre Transform

$$
\begin{aligned}
\mathbb{F}L &: TQ \to T^\star Q \\
\mathbb{F}L(q, \dot{q}) &= \left(q, \frac{\partial L}{\partial \dot{q}}\right)
\end{aligned}
$$

Discrete Hamiltonians

$$
\begin{aligned}
\mathbb{F}^+ L_d, \mathbb{F}^- L_d &: Q \times Q \to T^\star Q \\
\big\langle \mathbb{F}^+ L_d(q_0, q_1), \delta q_1 \big\rangle &= \langle D_2 L_d(q_0, q_1), \delta q_1 \rangle \\
\langle \mathbb{F}^- L_d(q_0, q_1), \delta q_0 \rangle &= \langle -D_1 L_d(q_0, q_1), \delta q_0 \rangle \\
\mathbb{F}^+ L_d(q_0, q_1) &= (q_1, D_2 L_d(q_0, q_1)) \\
\mathbb{F}^- L_d(q_0, q_1) &= (q_0, -D_1 L_d(q_0, q_1)) \\
p^+_{k,k+1} &= p^+(q_k, q_{k+1}) = \mathbb{F}^+ L_d(q_k, q_{k+1}) \\
p^-_{k,k+1} &= p^-(q_k, q_{k+1}) = \mathbb{F}^- L_d(q_k, q_{k+1}) \\
\tilde{F}_{L_d} &: T^\star Q \to T^\star Q \\
\tilde{F}_{L_d} &= \mathbb{F}^+ L_d \circ \mathbb{F}_{L_d} \circ \mathbb{F}^+ L_d
\end{aligned}
$$

Forced Systems

$$
\begin{aligned}
f_L &: TQ \to T^\star Q \\
f_H &: T^\star Q \to T^\star Q
\end{aligned}
$$